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FRACTIONAL POWERS OF HIGHER-ORDER VECTOR OPERATORS ON BOUNDED AND UNBOUNDED DOMAINS

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Abstract Using the H^∞-functional calculus for quaternionic operators, we show how to generate the fractional powers of some densely defined differential quaternionic operators of order $m \geq 1$, acting on the right linear quaternionic Hilbert space $L^2(\Omega, \mathbb{C} \otimes \mathbb{H})$. The operators that we consider are of the type

$$T = i^{m-1}\left(a_1(x)e_1\partial_{x_1}^m + a_2(x)e_2\partial_{x_2}^m + a_3(x)e_3\partial_{x_3}^m\right), \quad x = (x_1, x_2, x_3) \in \overline{\Omega},$$

where $\overline{\Omega}$ is the closure of either a bounded domain Ω with C^1 boundary, or an unbounded domain Ω in $\mathbb{R}^3$ with a sufficiently regular boundary, which satisfy the so-called property (R) (see Definition 1.3), $e_1, e_2, e_3 \in \mathbb{H}$ which are pairwise anticommuting imaginary units, $a_1, a_2, a_3 : \overline{\Omega} \subset \mathbb{R}^3 \to \mathbb{R}$ are the coefficients of T. In particular, it will be given sufficient conditions on the coefficients of T in order to generate the fractional powers of T, denoted by $P_\alpha(T)$ for $\alpha \in (0,1)$, when the components of T, i.e. the operators $T_l := a_l\partial_{x_l}^m$, do not commute among themselves. This kind of result is to be understood in the more general setting of the fractional diffusion problems. The method used to construct the fractional power of a quaternionic linear operator is a generalization of the method developed by Balakrishnan.

Keywords: fractional powers; higher-order vector operators; S-spectrum; S-spectrum approach

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1. Introduction

Using the S-functional calculus, in the series of papers [15, 21–24], we defined the fractional powers of a class of vector operators with non-constant coefficients. In this paper, we consider the quaternionic differential operators of the form

$$T = i^{m-1}\left(a_1(x)e_1\partial_{x_1}^m + a_2(x)e_2\partial_{x_2}^m + a_3(x)e_3\partial_{x_3}^m\right), \quad x = (x_1, x_2, x_3) \in \overline{\Omega},$$

and we prove that under suitable conditions on the coefficients, it admits well-defined fractional powers. In order to state our results, we give some details on the quaternion techniques based on the spectral theory on the S-spectrum. For a complete introduction of the S-functional calculus see the books [16], [20], here we briefly introduce the main aspects of this theory.

1.1. The S-functional calculus

An element in the quaternions $\mathbb{H}$ is of the form $s = s_0 + s_1e_1 + s_2e_2 + s_3e_3$, where s_0, s_ℓ are real numbers ($\ell = 1, 2, 3$), $\mathrm{Re}(s) := s_0$ denotes the real part of s and e_ℓ, for $\ell = 1, 2, 3$, are the imaginary units which satisfy the relations: $e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$. The modulus of s is defined as $|s| = (s_0^2 + s_1^2 + s_2^2 + s_3^2)^{1/2}$ and the conjugate is given by $\overline{s} = s_0 - s_1e_1 - s_2e_2 - s_3e_3$. In the sequel, we will denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, an element j in $\mathbb{S}$ is such that $j^2 = -1$. We consider a two-sided quaternionic Banach space V and we denote the set of closed densely defined quaternionic right linear operators on V by $\mathcal{K}(V)$. The Banach space of all bounded right linear operators on V is indicated by the symbol $\mathcal{B}(V)$ and is endowed with the natural operator norm. For $T \in \mathcal{K}(V)$, we define the operator associated with the S-spectrum as:

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}, \quad \text{for } s \in \mathbb{H} \tag{1.1}$$

where $\mathcal{Q}_s(T) : \mathcal{D}(T^2) \to V$, where $\mathcal{D}(T^2)$ is the domain of T^2. We define the S-resolvent set of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T) \text{ is invertible and } \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V)\}$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

The operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo S-resolvent operator. For $s \in \rho_S(T)$, the left S-resolvent operator is defined as

$$S_L^{-1}(s, T) := \mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1} \tag{1.2}$$

and the right S-resolvent operator is given by

$$S_R^{-1}(s, T) := -(T - \mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}. \tag{1.3}$$

The fractional powers of an operator T such that $j\mathbb{R} \subset \rho_S(T)$ for any $j \in \mathbb{S}$, are denoted by $P_\alpha(T)$ and are defined as follows. For any $j \in \mathbb{S}$, for $\alpha \in (0, 1)$ and $v \in \mathcal{D}(T)$, we set

$$P_\alpha(T)v := \frac{1}{2\pi}\int_{-j\mathbb{R}} S_L^{-1}(s, T)\, ds_j\, s^{\alpha-1}Tv, \tag{1.4}$$

or

$$P_\alpha(T)v := \frac{1}{2\pi}\int_{-j\mathbb{R}} s^{\alpha-1}\, ds_j\, S_R^{-1}(s, T)Tv, \tag{1.5}$$

where $ds_j = ds/j$. These formulas are a consequence of the quaternionic version of the H^∞-functional calculus based on the S-spectrum, again see [16] for details. For the generation of the fractional powers $P_\alpha(T)$, a crucial assumption on the S-resolvent operators

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is that, for $s \in \mathbb{H} \setminus \{0\}$ with $\mathrm{Re}(s) = 0$, the estimates

$$\left\| S_L^{-1}(s, T) \right\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|} \quad \text{and} \quad \left\| S_R^{-1}(s, T) \right\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|}, \tag{1.6}$$

hold with a constant $\Theta > 0$ that does not depend on s. It is important to observe that the conditions (1.6) assure that the integrals (1.4) and (1.5) are convergent and so the fractional powers are well defined.

For the definition of the fractional powers of the operator T, we can use equivalently the integral representation in (1.4) or the one in (1.5). Moreover, they correspond to a modified version of Balakrishnan's formula that takes only spectral points with positive real part into account.

Remark 1.1. It is clear from the definition of the S-resolvent operators that to use the S-functional calculus for the definition of the fractional powers of an operator T we have to determine if $\mathcal{Q}_s(T)$ is invertible for any $s \in \mathbb{H}$ such that $s \neq 0$ and $\mathrm{Re}(s) = 0$, and, moreover, if estimates of the type (1.6) hold, see Problem 1.4.

1.2. Operators of order one vs operators of order $m > 1$

In some of our previous papers, we have defined fractional powers of operators of first order such as

$$T := \begin{pmatrix} a_1(x)\partial_{x_1} \\ a_2(x)\partial_{x_2} \\ a_3(x)\partial_{x_3}. \end{pmatrix}$$

acting on functions $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ belonging to $H_0^1(\Omega, \mathbb{R}) \subset L^2(\Omega, \mathbb{R})$ where Ω is a (possibly) unbounded domain with C^1 boundary. To use the S-functional calculus, first we identify the gradient operator with the quaternionic gradient operator

$$\begin{pmatrix} a_1(x)\partial_{x_1} \\ a_2(x)\partial_{x_2} \\ a_3(x)\partial_{x_3} \end{pmatrix} \equiv e_1 a_1(x)\partial_{x_1} + e_2 a_2(x)\partial_{x_2} + e_3 a_3(x)\partial_{x_3}$$

and we consider the operator $\mathcal{Q}_s(T)$ defined in a weak sense over $H_0^1(\Omega, \mathbb{H}) \subset L^2(\Omega, \mathbb{H})$. It is important to observe that the above identification has some consequence on the bilinear form $b_s(u, v) = \langle \mathcal{Q}_s(T)(u), v \rangle_{\mathbb{H}}$, where $\langle a, b \rangle_{\mathbb{H}} := \overline{a}b$ for all a, $b \in \mathbb{H}$. Indeed, performing an integration by parts, we have (see in the following):

$$b_s(u, u) = \sum_{l=1}^{3} \|a_l(x)\partial_{x_l} u\|^2 + |s|^2 \|u\|^2 + \text{ other terms}$$

where the "other terms" are the scalar products of the first derivatives of u with u multiplied by the derivative of the coefficients. In the formula, we indicate that $b_s(u, u)$ contains two positive terms: the L^2-norm of u and the L^2-norm of the first derivative of u multiplied by the coefficients a_j's. This fact allows us to determine some conditions on the coefficients a_j's in order to guarantee the continuity and the coercivity of $b_s(\cdot, \cdot)$ and, moreover, the uniform estimates for the S-resolvent operator.

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In this paper, we consider vector operators of order $m > 1$ and $m \in \mathbb{N}$ of the type

$$T := \begin{pmatrix} a_1(x)\partial_{x_1}^m \\ a_2(x)\partial_{x_2}^m \\ a_3(x)\partial_{x_3}^m \end{pmatrix}.$$

If we consider the previous identification

$$T := e_1 a_1(x)\partial_{x_1}^m + e_2 a_2(x)\partial_{x_2}^m + e_3 a_3(x)\partial_{x_3}^m,$$

and we consider the operator $\mathcal{Q}_s(T)$ defined in a weak sense over $H_0^m(\Omega, \mathbb{H}) \subset L^2(\Omega, \mathbb{H})$, we have to distinguish the cases of m odd or even. If m is odd using m-times an argument of integration by parts we obtain

$$b_s(u,u) = \sum_{l=1}^{3} \|a_l(x)\partial_{x_l}^m u\|^2 + |s|^2\|u\|^2 + \text{ other terms}$$

and the bilinear form still contains two positive terms: the L^2-norm of u and the L^2-norm of the derivatives of order m. If we try to compute the bilinear form in the same way when m is even, we obtain

$$b_s(u,u) = -\sum_{l=1}^{3} \|a_l(x)\partial_{x_l}^m u\|^2 + |s|^2\|u\|^2 + \text{ other terms}$$

losing the positivity of the first term. One way to overcome this problem is to identify T with

$$e_1 a_1(x)\partial_{x_1}^m + e_2 a_2(x)\partial_{x_2}^m + e_3 a_3(x)\partial_{x_3}^m$$

if m is odd and with

$$i(e_1 a_1(x)\partial_{x_1}^m + e_2 a_2(x)\partial_{x_3}^m + e_3 a_3(x)\partial_{x_3}^m)$$

if m is even where i is the imaginary unit of $\mathbb{C}$. In other words, we have complexified the coefficients of the quaternionic gradient operator and the operator T is identified with the quaternionic gradient operator with real coefficients if m is odd or with the quaternionic gradient operator with purely imaginary coefficients if m is even. For the precise definitions of $b_s(\cdot,\cdot)$ and of the quaternionic Hilbert space $\mathbb{C} \otimes \mathbb{H}$, see § 2. In light of these considerations, we give the following definition.

Definition 1.2. Let Ω be a C^1-domain in $\mathbb{R}^3$, bounded or unbounded, and let $a_l : \overline{\Omega} \to \mathbb{R}$ for $l = 1, 2, 3$ be $C^m(\overline{\Omega})$ functions. We define in a classical way over $C^m(\overline{\Omega}, \mathbb{C} \otimes \mathbb{H})$ the operator

$$T := i^{m-1}\left(a_1(x)e_1\partial_{x_1}^m + a_2(x)e_2\partial_{x_2}^m + a_3(x)e_3\partial_{x_3}^m\right), \quad x = (x_1, x_2, x_3) \in \overline{\Omega}.$$

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1.3. Ω bounded vs Ω unbounded

We will treat separately the cases of Ω bounded and of Ω unbounded. The unbounded case is more complicated as explained at the end of this section and needs some more constraints on the shape of Ω that we now introduce. For the following definition, we shall utilize n-dimensional spherical coordinates (r, ω) where $r \geq 0$ is the distance from the origin, $\omega \in S_{n-1}$, and S_{n-1} denotes the sphere in $\mathbb{R}^n$ (see [26]; in our case, however, $n = 3$).

Definition 1.3. An open set $\Omega \subset \mathbb{R}^3$ is said to have the property (R) if there exists $P \in \mathbb{R}^3 \setminus \overline{\Omega}$ such that every ray through P has the intersection with Ω which is either empty or an infinite interval. More precisely, for each $\omega \in S_2$ set

$$\begin{cases} f(\omega) := \inf\{r \geq 0 : \ P + r\omega \in \Omega\} & \text{if } \{P + r\omega : r \geq 0\} \cap \Omega \neq \emptyset \\ f(\omega) := \infty & \text{if } \{P + r\omega : r \geq 0\} \cap \Omega = \emptyset. \end{cases}$$

We are assuming that if $f(\omega) \neq \infty$, then $P + r\omega \in \Omega$ for all $r \in (f(\omega), \infty)$.

Examples of unbounded domains, which satisfy the property (R) are: $\Omega := \{x \in \mathbb{R}^3 : |x - P| > M\}$ and $\Omega := \{x \in \mathbb{R}^3 : \langle P - x, v\rangle > 0\}$ where $v \in \mathbb{R}^3$ is a vector, $P \in \mathbb{R}^3$ is a point and $M > 0$ is a positive constant (here $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{R}^3$). We are ready to formulate in the precise way the problems that we have to solve.

Problem 1.4. Let $\Omega \subset \mathbb{R}^3$ be with C^1 boundary, which is either bounded or unbounded and satisfying the property (R). Let $F : \Omega \to \mathbb{C} \otimes \mathbb{H}$ be a given L^2-function and denote by $u : \Omega \to \mathbb{C} \otimes \mathbb{H}$ the unknown function satisfying the boundary value problem:

$$\begin{cases} \mathcal{Q}_s(T)(u) = F \\ \partial^{\mathbf{b}} u(x) = 0 \qquad \forall\, \mathbf{b} \in \mathbb{N}_0^3 \text{ such that } |\mathbf{b}| \leq m-1 \,\text{and}\, x \in \partial\Omega, \end{cases} \tag{1.7}$$

where $\mathbf{b} = (b_1, b_2, b_3)$ and $\partial^{\mathbf{b}} = \partial_{x_1}^{b_1} \partial_{x_2}^{b_2} \partial_{x_3}^{b_3}$. Determine the conditions on the coefficients $a_1, a_2, a_3 : \Omega \to \mathbb{R}$ such that the boundary value problem has a unique solution in a suitable function space and, moreover, the L^2-estimates (1.6) for the S-resolvent operators hold.

Remark 1.5. If Ω is bounded and its boundary is of class C^m, the previous boundary value problem is equivalent to

$$\begin{cases} \mathcal{Q}_s(T)(u) = F \\ \partial_\nu^j u(x) = 0 \qquad \text{for } 0 \leq j \leq m-1 \text{ and } x \in \partial\Omega, \end{cases} \tag{1.8}$$

where, ν is the normal vector field pointing outside $\partial\Omega$ (see Theorem 7.41 in [33]).

We are going to solve the previous problem by the use of the Lax–Milgram Lemma. In particular, to solve the boundary value problem (1.7), we want to find some conditions on the coefficients a_j's in a such way that the continuity and the coercivity of the quadratic form $b_s(u, v)$ associated with $\mathcal{Q}_s(T)$ hold (see Definition 2.3). The need of proving also the estimates (1.6) makes the assumptions on the coefficients a_l's stronger than the

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usual one that we have to require for the coercivity of $b_s(\cdot,\cdot)$, since we can not rely on the term $|s|^2\|u\|^2$ of $b_s(\cdot,\cdot)$. For this reason, the other positive term in $b_s(u,u)$, that is $\sum_{l=1}^{3}\|a_l\partial_{x_l}^m u\|$, has to control the L^2-norm of u and the L^2-norms of all the partial derivatives of u up to order $m-1$. In § 3, when Ω is a bounded domain of $\mathbb{R}^3$, through an iterated use of Poincaré's inequality, we will show that the conditions:

- $|a_j| \gg \max(C_\Omega^m, C_\Omega)$;
- $|a_j| \gg |\partial^\beta a_j|$ for any $|\beta| < m$,

are sufficient to solve the Problem 1.4. Here and in what follows, C_Ω denotes the Poincaré constant of Ω. When Ω is an unbounded domain of $\mathbb{R}^3$ and $m=1$, the role of the Poincaré inequalities is replaced by the Gagliardo–Nirenberg estimates and a condition of integrability on the first derivatives of the coefficients is sufficient to get the coercivity of $b_s(\cdot,\cdot)$ (see [24]).

When Ω is an unbounded domain of $\mathbb{R}^3$ and the order of the operator T is greater than 1, the Gagliardo–Nirenberg estimates can not be used in an iterated way as for Poincaré's inequality. In § 2, we propose one way to overcome this problem by the use of a weighted Poincaré's inequality on some unbounded domains under an exponential decay condition at infinity of the coefficients a_j's (see Theorem 4.3).

In this paper, the definition of the fractional quaternionic operators is based on the S-spectrum approach to fractional diffusion problems see [2, 13, 14]. This method is a generalization of the Balakrishnan's method to construct the fractional power of a real operator, see [5]. There are also several nonlinear models that involve the fractional power of scalar elliptic operators, see for example [11, 34].

2. The weak formulation of Problem 1.4

The boundary $\partial\Omega$ of Ω is assumed to be of class C^1 even though for some lemmas in the sequel the conditions on the open set Ω can be weakened. We consider the right quaternionic Hilbert space $\mathbb{C}\otimes\mathbb{H}$ endowed with the scalar product

$$\langle u, v\rangle := \overline{q}_1 w_1 + \overline{q}_2 w_2 \quad \forall\, u,\, v \in \mathbb{C}\otimes\mathbb{H}$$

where i is the imaginary unit of $\mathbb{C}$, $u = q_1 + iq_2$, $v = w_1 + iw_2$ and $q_1,\, q_2,\, w_1,\, w_2 \in \mathbb{H}$. As usual, we define the modulus of $u \in \mathbb{C}\otimes\mathbb{H}$ as

$$|u| := \sqrt{\langle u, u\rangle}.$$

We observe that a function $u : \Omega \to \mathbb{C}\otimes\mathbb{H}$ is determined by eight real functions $u_{j,l} : \Omega \to \mathbb{R}$ where $j = 0,\, 1,\, 2,\, 3$ and $l = 1,\, 2$. We call these functions the components of u. We will use the following notation

$$\begin{aligned} u(x) &= (u_{0,1}(x) + u_{1,1}(x)e_1 + u_{2,1}(x)e_2 + u_{3,1}(x)e_3) + i(u_{0,2}(x) + u_{1,2}(x)e_1 \\ &\quad + u_{2,2}(x)e_2 + u_{3,2}(x)e_3) \\ &= u_1(x) + iu_2(x). \end{aligned}$$

where $u_1(x) := u_{0,1}(x) + u_{1,1}(x)e_1 + u_{2,1}(x)e_2 + u_{3,1}(x)e_3$ and $u_2(x) := u_{0,2}(x) + u_{1,2}(x)e_1 + u_{2,2}(x)e_2 + u_{3,2}(x)e_3$. We can consider the space of L^p-integrable functions

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from a domain $\Omega \subset \mathbb{R}^3$ to $\mathbb{C}\otimes\mathbb{H}$

$$L^p := L^p(\Omega, \mathbb{C}\otimes\mathbb{H}) := \left\{ u : \Omega \to \mathbb{C}\otimes\mathbb{H} : \int_\Omega |u(x)|^p \, dx < +\infty \right\}.$$

The space L^2 with the scalar product:

$$\langle u, v\rangle_{L^2} := \langle u, v\rangle_{L^2(\Omega,\mathbb{C}\otimes\mathbb{H})} := \int_\Omega \langle u(x), v(x)\rangle \, dx \quad \forall u,\, v \in L^2(\Omega, \mathbb{C}\otimes\mathbb{H}),$$

is a right linear quaternionic Hilbert space. We furthermore introduce the quaternionic Sobolev space of order m

$$\begin{aligned} H^m &:= H^m(\Omega, \mathbb{C}\otimes\mathbb{H}) \\ &:= \left\{ u \in L^2(\Omega, \mathbb{C}\otimes\mathbb{H}) : \, u_{j,l} \in H^m(\Omega, \mathbb{R}) \quad j = 0, 1, 2, 3 \text{ and } l = 1, 2 \right\}, \end{aligned}$$

where the space $H^m(\Omega, \mathbb{R})$ is the Sobolev space of order m defined as in [25] Chapter 5. We have that $H^m(\Omega, \mathbb{C}\otimes\mathbb{H})$ endowed with the quaternionic scalar product

$$\langle u, v\rangle_{H^m} := \langle u, v\rangle_{H^m(\Omega,\mathbb{H})} := \langle u, v\rangle_{L^2} + \sum_{1\le|\mathbf{b}|\le m}^{3} \left\langle \partial^{\mathbf{b}} u, \partial^{\mathbf{b}} v\right\rangle_{L^2},$$

where $\mathbf{b} \in \mathbb{N}^3$, becomes a right linear quaternionic Hilbert space. As usual the space $H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H})$ is the closure of the space $C_0^\infty(\Omega, \mathbb{C}\otimes\mathbb{H})$ in $H^m(\Omega, \mathbb{C}\otimes\mathbb{H})$ with respect to the norm $\|\cdot\|_{H^m}$. This space can be characterized as the set of all functions $u \in H^m$ such that $\mathrm{Tr}(\partial^{\mathbf{b}} u) = 0$ for any multiindex $\mathbf{b} \in \mathbb{N}^3$ with $|\mathbf{b}| \le m - 1$ (here the trace operator $\mathrm{Tr}(\cdot)$ is defined as in [25] Chapter 5). Now we give to the problem (1.4) the weak formulation in order to apply the Lax–Milgram lemma in the space $H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H})$.

Remark 2.1. When Ω is bounded, we can endow $H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H})$ with the scalar product

$$\langle u, v\rangle_{D^m} := \sum_{l=1}^{3} \langle \partial_{x_l}^m u, \partial_{x_l}^m v\rangle_{L^2}.$$

The H^m-norm is equivalent to the norm

$$\|u\|_{D^m}^2 := \langle u, u\rangle_{D^m} = \sum_{l=1}^{3} \|\partial_{x_l}^m u\|_{L^2}^2.$$

This is a consequence of the following estimates

$$\|u\|_{D^m} \le \|u\|_{H^m} \le K(m) K_\Omega \sum_{|\mathbf{b}|=m} \|\partial^{\mathbf{b}} u\|_{L^2} \le K(m) K K_\Omega \|u\|_{D^m},$$

where the second inequality is obtained by Poincaré's inequality applied repeatedly to the term $\|\partial^{\mathbf{b}} u\|_{L^2}$ for $|\mathbf{b}| < m$ and

$$K_\Omega := \sup(C_\Omega, C_\Omega^m).$$

The constant $K(m)$ represents the maximum number of times that $\|\partial^{\mathbf{b}} u\|$ for some $|\mathbf{b}| = m$ appear on the left-hand side of the second inequality after the use of the

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Poincaré's inequality. The last estimate follows using the Fourier transform on the terms $\partial^{\mathbf{b}}u$ when $|\mathbf{b}|=m$ and since there exists a positive constant $K>0$ such that

$$\sum_{|\mathbf{b}|=m}|\xi^{2\mathbf{b}}|\leq K\sum_{l=1}^{3}|\xi_l|^{2m}$$

(we can use the Fourier transform since any $u\in H_0^m(\Omega,\mathbb{C}\otimes\mathbb{H})$ can be extended by 0 outside Ω preserving the H^m-regularity in $\mathbb{R}^3$). We define the constant $K(m,\Omega):=KK(m)K_\Omega$ (we will use several times this constant in the sequel and according to our necessity it could be rescaled by other constants which depends on m). When Ω is unbounded $\|\cdot\|_{D^m}$ is not a norm, still we will use several times the estimate

$$\sum_{|\beta|=m}\|\partial^{\mathbf{b}}u\|\leq K\|u\|_{D^m}.$$

Remark 2.2. We will use several times a classical argument of integration by parts that we describe now. Let $f_1,\ f_2\in H_0^k(\Omega,\mathbb{R})$ and $\partial^{\mathbf{b}}f_3,\ \partial^{\mathbf{b}}f_4\in L^2(\Omega)\cap L^\infty(\Omega)$ for any $\mathbf{b}\in\mathbb{N}^3$ with $|\mathbf{b}|\leq k$ then, integrating by parts k-times and recalling that the traces at the boundary of $\partial^{\mathbf{b}}f_2$ for all $|\mathbf{b}|<k$ are zero, we have that for any $i=1,\ 2,\ 3$

$$\begin{aligned}\int_\Omega\partial_{x_i}^k(f_1)f_2f_3f_4\,dx&=(-1)^k\int_\Omega f_1\partial_{x_i}^k(f_2f_3f_4)\,dx\\&=(-1)^k\int_\Omega f_1\partial_{x_i}^k(f_2)f_3f_4\,dx+(-1)^k\sum_{|\mathbf{t}|=k\,\wedge\,t_1\leq k-1}\binom{k}{\mathbf{t}}\int_\Omega f_1\partial_{x_i}^{t_1}(f_2)\partial_{x_i}^{t_2}(f_3)\partial_{x_i}^{t_3}(f_4)\,dx\\&=(-1)^k\sum_{|\mathbf{t}|=k}\binom{k}{\mathbf{t}}\int_\Omega f_1\partial_{x_i}^{t_1}(f_2)\partial_{x_i}^{t_2}(f_3)\partial_{x_i}^{t_3}(f_4)\,dx,\end{aligned}$$

where $\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{N}^3$ and $\binom{k}{\mathbf{t}}=\frac{k!}{t_1!t_2!t_3!}$.

From Definition 1.2, we have

$$\begin{aligned}\mathcal{Q}_s(T)&=T^2-2s_0T+|s|^2\mathcal{I}\\&=(-1)^m\Bigg[\sum_{l=1}^{3}a_l^2(x)\partial_{x_l}^{2m}+\sum_{l=1}^{3}\sum_{k=1}^{m}\binom{m}{k}a_l(x)\partial_{x_l}^k(a_l(x))\partial_{x_l}^{2m-k}\\&\quad+\sum_{l<j}e_le_j\left(\sum_{k=1}^{m}\binom{m}{k}\left(a_l(x)\partial_{x_l}^k(a_j(x))\partial_{x_l}^{m-k}\partial_{x_j}^m-a_j(x)\partial_{x_j}^k(a_l(x))\partial_{x_j}^{m-k}\partial_{x_l}^m\right)\right)\Bigg]\\&\quad-2s_0T+|s|^2\mathcal{I},\end{aligned}$$

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where $\begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{k!(m-k)!}$, and the scalar part of $\mathcal{Q}_s(T)$ is

$$\mathrm{Scal}(\mathcal{Q}_s(T)) := (-1)^m \left(\sum_{l=1}^{3} a_l^2(x) \partial_{x_l}^{2m} + \sum_{l=1}^{3} \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} a_l(x) \partial_{x_l}^k (a_l(x)) \partial_{x_l}^{2m-k} \right) + |s|^2 \mathcal{I},$$

while the vectorial part is

$$\begin{aligned}\mathrm{Vect}(\mathcal{Q}_s(T)) := \sum_{l<j} e_l e_j \left(\sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \left(a_l(x) \partial_{x_l}^k (a_j(x)) \partial_{x_l}^{m-k} \partial_{x_j}^m \right.\right. \\ \left.\left. - a_j(x) \partial_{x_j}^k (a_l(x)) \partial_{x_j}^{m-k} \partial_{x_l}^m \right) \right) - 2 s_0 T.\end{aligned}$$

We consider the bilinear form

$$\langle \mathcal{Q}_s(T) u, v \rangle_{L^2} = \int_\Omega \langle \mathcal{Q}_s(T) u(x), v(x) \rangle \, dx$$

for functions $u,\, v \in C_0^{2m}(\overline{\Omega}, \mathbb{C} \otimes \mathbb{H})$. Note that

$$\langle \mathcal{Q}_s(T) u, v \rangle_{L^2} = \langle \mathrm{Scal}(\mathcal{Q}_s(T)) u, v \rangle_{L^2} + \langle \mathrm{Vect}(\mathcal{Q}_s(T)) u, v \rangle_{L^2}.$$

Using Remark 2.2, we have that

$$\begin{aligned}
&\langle \mathrm{Scal}(\mathcal{Q}_s(T)) u, v \rangle_{L^2} \\
&\quad = (-1)^m \left(\sum_{l=1}^{3} \int_\Omega \langle a_l^2(x) \partial_{x_l}^{2m}(u), v \rangle \, dx \right. \\
&\qquad \left. + \sum_{l=1}^{3} \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} \int_\Omega \langle a_l(x) \partial_{x_l}^k (a_l(x)) \partial_{x_l}^{2m-k}(u), v \rangle \, dx \right) \\
&\qquad + |s|^2 \int_\Omega \overline{u} v \, dx = \sum_{l=1}^{3} \int_\Omega \langle a_l^2(x) \partial_{x_l}^m (u), \partial_{x_l}^m (v) \rangle \, dx + |s|^2 \int_\Omega \langle u, v \rangle \, dx \\
&\qquad + \sum_{l=1}^{3} \sum_{|\mathbf{t}'| = m \,\wedge\, t_2' \leq m-1} \begin{pmatrix} m \\ \mathbf{t}' \end{pmatrix} \int_\Omega \langle \partial_{x_l}^{t_1'} (a_l^2(x)) \partial_{x_l}^m (u), \partial_{x_l}^{t_2'} (v) \rangle \, dx \\
&\qquad + \sum_{l=1}^{3} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}| = m-k} \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} m-k \\ \mathbf{t} \end{pmatrix} \int_\Omega \langle \partial_{x_l}^{t_1} (a_l(x)) \partial_{x_l}^{t_2+k} (a_l(x)) \partial_{x_l}^m (u), \partial_{x_l}^{t_3} (v) \rangle \, dx,
\end{aligned}$$

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where $\mathbf{t}' = (t'_1, t'_2) \in \mathbb{N}^2$ and also

$$
\begin{aligned}
&\langle \mathrm{Vect}(\mathcal{Q}_s(T))u, v\rangle_{L^2} \\
&\quad = (-1)^m \sum_{l<j} \sum_{k=1}^{m} \binom{m}{k} \int_\Omega \langle e_l e_j \Big(a_l(x) \partial^k_{x_l}(a_j(x)) \partial^{m-k}_{x_l} \partial^m_{x_j}(u) \\
&\qquad - a_j(x) \partial^k_{x_j}(a_l(x)) \partial^{m-k}_{x_j} \partial^m_{x_l}(u) \Big), v\rangle \, dx \\
&\qquad - 2s_0 \langle T(u), v\rangle_{L^2} \\
&\quad = \sum_{l<j} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \Big(\int_\Omega \langle e_l e_j \partial^{t_1}_{x_l}(a_l(x)) \partial^{t_2+k}_{x_l}(a_j(x)) \partial^m_{x_j}(u), \partial^{t_3}_{x_l} v\rangle \, dx \\
&\qquad - \int_\Omega \langle e_l e_j \partial^{t_1}_{x_j}(a_j(x)) \partial^{t_2+k}_{x_j}(a_l(x)) \partial^m_{x_l}(u), \partial^{t_3}_{x_j} v\rangle \, dx - 2s_0 \langle T(u), v\rangle_{L^2} \Big).
\end{aligned}
$$

Relying on the above considerations we can give the following two definitions.

Definition 2.3. Let Ω be a bounded domain (or an unbounded domain) in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class C^1, let a_1, a_2, $a_3 \in C^m(\overline{\Omega}, \mathbb{R})$ (or in the case of the unbounded domains a_1, a_2, $a_3 \in C^m(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega)$ such that $\partial^{\mathbf{b}} a_j \in L^\infty(\Omega)$ for any $j = 1, 2, 3$ and for any $\mathbf{b} \in \mathbb{N}^3$ such that $|\mathbf{b}| \leq m$). We define the bilinear form:

$$
\begin{aligned}
b_s(u, v) :={}& \sum_{l=1}^{3} \int_\Omega \langle a_l^2(x) \partial^m_{x_l}(u), \partial^m_{x_l}(v)\rangle \, dx + |s|^2 \int_\Omega \langle u, v\rangle \, dx \\
&+ \sum_{l=1}^{3} \sum_{|\mathbf{t}'|=m \,\wedge\, t_2 \leq m-1} \binom{m}{\mathbf{t}'} \int_\Omega \langle \partial^{t_1}_{x_l}(a_l^2(x)) \partial^m_{x_l}(u), \partial^{t_2}_{x_l}(v)\rangle \, dx \\
&+ \sum_{l=1}^{3} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \\
&\times \int_\Omega \langle \partial^{t_1}_{x_l}(a_l(x)) \partial^{t_2+k}_{x_l}(a_l(x)) \partial^m_{x_l}(u), \partial^{t_3}_{x_l}(v)\rangle \, dx \\
&+ \sum_{l<j} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \\
&\times \Big(\int_\Omega \langle e_l e_j \partial^{t_1}_{x_l}(a_l(x)) \partial^{t_2+k}_{x_l}(a_j(x)) \partial^m_{x_j}(u), \partial^{t_3}_{x_l} v\rangle \, dx \\
&- \int_\Omega \langle e_l e_j \partial^{t_1}_{x_j}(a_j(x)) \partial^{t_2+k}_{x_j}(a_l(x)) \partial^m_{x_l}(u), \partial^{t_3}_{x_j} v\rangle \, dx - 2s_0 \langle T(u), v\rangle_{L^2} \Big),
\end{aligned}
\tag{2.1}
$$

for all functions $u, v \in H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H})$.

Definition 2.4. Let Ω be a bounded domain in $\mathbb{R}^3$ with C^1 boundary (or an unbounded domain which satisfy property (R)). We say that $u \in H_0^m$ is the weak solution of the

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existence problem in 1.4 for some $s \in \mathbb{H}$ and a given $F \in L^2(\Omega, \mathbb{C} \otimes \mathbb{H})$, if we have

$$b_s(u,v) = \langle F, v \rangle_{L^2}, \text{ for all } v \in H_0^m,$$

where b_s is the bilinear form defined in Definition 2.3.

3. Weak solution of Problem 1.4 when Ω is bounded

To prove the existence and uniqueness of the weak solutions of the Problem 1.4 in the case Ω is bounded, it will be sufficient to show that the bilinear forms $b_s(\cdot,\cdot)$, in Definition 2.3, are continuous and coercive in $H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H})$.

First, we prove the continuity in § 3.1. The coercivity will be proved in § 3.2 when $s = js_1$ for $j \in \mathbb{S}$ and $s_1 \in \mathbb{R}$ with $s_1 \neq 0$. As a direct consequence of the coercivity, we will prove an L^2 estimate for the weak solution u that belongs to $H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H})$ and also we will prove an L^2 estimate for the term $T(u)$.

3.1. The continuity of the bilinear form $b_s(\cdot,\cdot)$

The bilinear form

$$b_s(\cdot,\cdot) : H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H}) \times H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H}) \to \mathbb{H},$$

for some $s \in \mathbb{H}$, is continuous if there exists a positive constant $C(s)$ such that

$$|b_s(u,v)| \leq C(s)\|u\|_{D^m}\|v\|_{D^m}, \text{ for all } u, v \in H^m(\Omega, \mathbb{C} \otimes \mathbb{H}).$$

We note that the constant $C(s)$ depends on $s \in \mathbb{H}$ but does not depend on u and $v \in H^m(\Omega, \mathbb{C} \otimes \mathbb{H})$.

Proposition 3.1 (Continuity of b_s). *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^1. Assume that a_1, a_2, $a_3 \in C^m(\overline{\Omega}, \mathbb{R})$ then we have*

$$\left| \sum_{l=1}^{3} \int_\Omega \langle a_l^2(x) \partial_{x_l}^m(u), \partial_{x_l}^m(v) \rangle \, dx \right| \leq C_{1,m,a_j,\Omega} \|u\|_{D^m} \|v\|_{D^m} \tag{3.1}$$

$$\left| \sum_{l=1}^{3} \sum_{|\mathbf{t}'|=m \,\wedge\, t_2 \leq m-1} \binom{m}{\mathbf{t}'} \int_\Omega \langle \partial_{x_l}^{t_1'}(a_l^2(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_2'}(v) \rangle \, dx \right| \leq C_{2,m,a_j,\Omega} \|u\|_{D^m} \|v\|_{D^m} \tag{3.2}$$

$$\left| \sum_{l=1}^{3} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \int_\Omega \langle \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_3}(v) \rangle \, dx \right|$$
$$\leq C_{3,m,a_j,\Omega} \|u\|_{D^m} \|v\|_{D^m} \tag{3.3}$$

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$$\Bigg| \sum_{l<j} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \bigg(\int_\Omega \langle e_l e_j \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_j(x)) \partial_{x_j}^{m}(u), \partial_{x_l}^{t_3} v \rangle \, dx$$

$$- \int_\Omega \langle e_l e_j \partial_{x_j}^{t_1}(a_j(x)) \partial_{x_j}^{t_2+k}(a_l(x)) \partial_{x_l}^{m}(u), \partial_{x_j}^{t_3} v \rangle \, dx \bigg) - 2s_0 \langle T(u), v\rangle_{L^2} \Bigg|$$

$$\leq C_{4,\,m,\,a_j,\,\Omega,\,s} \|u\|_{D^m} \|v\|_{D^m}. \tag{3.4}$$

The previous constants can be estimated as follows

$$C_{1,\,m,\,a_j,\,\Omega} \leq \max_{l=1,2,3} \sup_{x\in\Omega} (a_l^2(x)), \quad C_{2,\,m,\,a_j,\,\Omega} \leq CK(m,\Omega) \max_{\substack{l=1,2,3,\\ t=1,\dots,m}} \sup_{x\in\Omega} (|\partial_{x_l}^t (a_l^2(x))|)$$

$$C_{3,\,m,\,a_j,\,\Omega} \leq CK(m,\Omega) \left(\max_{\substack{l=1,2,3,\\ t=0,\dots,m}} \sup_{x\in\Omega} (|\partial_{x_l}^t (a_l(x))|) \right)^2$$

$$C_{4,\,m,\,a_j,\,\Omega,\,s} \leq CK(m,\Omega) \left[\left(\max_{\substack{l=1,2,3,\\ t=0,\dots,m}} \sup_{x\in\Omega} (|\partial_{x_l}^t (a_j(x))|) \right)^2 + |s_0| \max_{l=1,2,3} \sup_{x\in\Omega} (|a_l(x)|) \right],$$

where the constant C is the sum of the maximum of the integrals in the inequalities (3.2), (3.3) and (3.4).

Moreover, for any $s \in \mathbb{H}$ there exists a positive constant $C(s)$ such that for any $(u, v) \in H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H}) \times H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H})$, the bilinear form $b_s(\cdot,\cdot)$ satisfies the estimate

$$|b_s(u, v)| \leq C(s) \|u\|_{D^m} \|v\|_{D^m}, \tag{3.5}$$

that is, $b_s(\cdot,\cdot)$ is a continuous bilinear form.

Proof. The inequality (3.1) is a direct consequence of the boundedness of the coefficients a_l's and the Hölder inequality. The estimates (3.2), (3.3) and (3.4) can be proved in a similar way and so we briefly explain how to prove the inequality (3.2). First, to each integral that is on the left-hand side of the equation, we repeatedly apply Hölder's and Poincaré's inequalities. In this way, each integral can be estimated by a constant depending on the sup norm of $\partial_{x_l}^s a_t$'s times $\|u\|_{D^m} \|v\|_{D^m}$. Second, we sum up term by term. Finally, the continuity of $b_s(\cdot,\cdot)$ is a direct consequence of the previous estimates. □

3.2. Weak solution of the Problem 1.4

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^1. Let T be the operator defined in (1.2) with coefficients $a_1, a_2, a_3 \in C^m(\overline{\Omega}, \mathbb{R})$ and set*

$$M := K(m,\Omega)^2 \left(\sup_{\substack{t=1,\dots,m\\ l=1,2,3\\ x\in\Omega}} |\partial_{x_l}^t (a_l(x))| \right)^2.$$

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Suppose

$$C_T := \min_{l=1,2,3} \inf_{x\in\Omega} (a_l^2(x)) > 0 \quad \frac{C_T}{2} - M > 0. \tag{3.6}$$

then:

(I) *The boundary value Problem (1.4) has a unique weak solution* $u \in H_0^m(\Omega, \mathbb{H})$, *for* $s \in \mathbb{H} \setminus \{0\}$ *with* $\mathrm{Re}(s) = 0$, *and*

$$\|u\|_{L^2}^2 \le \frac{1}{|s|^2} \mathrm{Re}(b_s(u,u)). \tag{3.7}$$

(II) *The following estimate holds*

$$\|T(u)\|_{L^2}^2 \le C_1^{-1} \mathrm{Re}(b_s(u,u)), \tag{3.8}$$

for every $u \in H_0^1(\Omega, \mathbb{H})$, *and* $s \in \mathbb{H} \setminus \{0\}$ *with* $\mathrm{Re}(s) = 0$, *where*

$$C_1 := \frac{C_T - 2M}{6C_T}.$$

Observe that we assume that the minimum of each $a_l^2(x)$ for $x \in \overline{\Omega}$ is strictly positive and in fact greater than $2M$. This fact can always be achieved by modifying T by adding a suitable constant to each coefficients a_l of T, $l = 1, 2, 3$.

Proof. In order to use the Lax–Milgram Lemma to prove the existence and the uniqueness of the solution for the weak formulation of the problem, it is sufficient to prove the coercivity of the bilinear form $b_s(\cdot,\cdot)$ in Definition 2.1 since the continuity is proved in Proposition 3.1. First, we write explicitly $\mathrm{Re}\, b_{js_1}(u,u)$, where we have set $s = js_1$, for $s_1 \in \mathbb{R}$ and $j \in \mathbb{S}$:

$$\begin{aligned}
\mathrm{Re}\, b_{js_1}(u,u) = {} & s_1^2 \|u\|_{L^2}^2 + \sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2 \\
& + \mathrm{Re}\Big(\sum_{l=1}^{3} \sum_{|\mathbf{t}'|=m \,\wedge\, t_2 \le m-1} \binom{m}{\mathbf{t}'} \int_\Omega \langle \partial_{x_l}^{t_1'}(a_l^2(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_2'}(u)\rangle \, dx \\
& + \sum_{l=1}^{3} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \\
& \times \int_\Omega \langle \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_3}(u)\rangle \, dx \\
& + \sum_{l<j} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \\
& \times \Big(\int_\Omega \langle e_l e_j \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_j(x)) \partial_{x_j}^m(u), \partial_{x_l}^{t_3} u\rangle \, dx \\
& - \int_\Omega \langle e_l e_j \partial_{x_j}^{t_1}(a_j(x)) \partial_{x_j}^{t_2+k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_j}^{t_3} u\rangle \, dx \Big).
\end{aligned} \tag{3.9}$$

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We see that the first two terms are positive. What remain to estimate are the three summations of the integrals in (3.9). Since they can be treated in the same way, we explain in detail the estimate for the second. By Hölder's inequality and the repeatedly use of Poincaré's inequality, we have for $t_1 = 0$

$$\left|\int_\Omega \langle a_l(x)\partial_{x_l}^{t_2+k}(a_l(x))\partial_{x_l}^m(u), \partial_{x_l}^{t_3}(v)\rangle\, dx\right| \leq \epsilon\|a_l\partial_{x_l}^m u\|^2 + \frac{1}{\epsilon}\|\partial_{x_l}^{t_2+k}(a_l)\partial_{x_l}^{t_3}(u)\|^2$$
$$\leq \epsilon\|a_l\partial_{x_l}^m u\|^2 + \frac{1}{\epsilon}\left(\sup_{x\in\Omega}(\partial_{x_l}^{t_2+k}(a_l))\right)^2 \|\partial_{x_l}^{t_3}(u)\|^2$$
$$\leq \epsilon\|a_l\partial_{x_l}^m u\|^2 + \frac{1}{\epsilon}\left(\sup_{x\in\Omega}(\partial_{x_l}^{t_2+k}(a_l))\,(C_\Omega)^{m-t_3}\right)^2 \sum_{|\beta|=m}\|\partial^\beta(u)\|^2$$
$$\leq \epsilon\|a_l\partial_{x_l}^m u\|^2 + \frac{1}{\epsilon}\left(\sup_{x\in\Omega}(\partial_{x_l}^{t_2+k}(a_l))KK(m)\,(C_\Omega)^{m-t_3}\right)^2 \|u\|_{D^m}^2.$$

In the case $t_1 \neq 0$, since $k > 0$, we have

$$\left|\int_\Omega \langle \partial_{x_l}^{t_1}(a_l(x))\partial_{x_l}^{t_2+k}(a_l(x))\partial_{x_l}^m(u), \partial_{x_l}^{t_3}(v)\rangle\, dx\right| \leq \frac{1}{2}\|\partial_{x_l}^{t_1}(a_l)\partial_{x_l}^m u\|^2 + \frac{1}{2}\|\partial_{x_l}^{t_2+k}(a_l)\partial_{x_l}^{t_3}(u)\|^2$$
$$\leq \frac{1}{2}\sup_{x\in\Omega}\left(\left|\partial_{x_l}^{t_1}a_l\right|^2\right)\|\partial_{x_l}^m u\|^2 + \frac{1}{2}\sup_{x\in\Omega}\left|\partial_{x_l}^{t_2+k}a_l\right|^2\left(KK(m)\,(C_\Omega)^{m-t_3}\right)^2\|u\|_{D^m}^2.$$

Summing up all the previous estimates of the terms in the second summation and rescaling the constants $K(m,\Omega)$ multiplying it by a constant which depends only on m, we obtain

$$\left|\sum_{l=1}^3 \sum_{|\mathbf{t}'|=m\,\wedge\, t_2\leq m-1} \sum_{k=0}^{t_1} \binom{m}{\mathbf{t}'}\binom{t_1}{k}\int_\Omega \langle \partial_{x_l}^k(a_l(x))\partial_{x_l}^{t_1-k}(a_l(x))\partial_{x_l}^m(u), \partial_{x_l}^{t_2}(u)\rangle\, dx\right|$$
$$\leq \epsilon\sum_{l=1}^3\|a_l\partial_{x_l}^m u\|^2 + \left(\frac{K(m,\Omega)^2}{\epsilon}\max_{\substack{j,\,l=1,2,3\\ t=1,\ldots,\,m}}\sup_{x\in\Omega}\left(\left|\partial_{x_j}^t a_l\right|^2\right)\right)\|u\|_{D^m}^2.$$

Analogous estimates also hold for the other summation of integrals. Eventually, summing up all the estimates, choosing ϵ in such a way that $\sum_{l=1}^3\|a_l\partial_{x_l}^m u\|^2 - C\epsilon\sum_{l=1}^3\|a_l\partial_{x_l}^m u\|^2 \geq \frac{1}{2}\sum_{l=1}^3\|a_l\partial_{x_l}^m u\|^2$ (where C depends only on m) and uniformazing the constants $K(m,\Omega)$, we obtain

$$\begin{aligned}\operatorname{Re} b_{\mathbf{j}_{s_1}}(u,u) &\geq s_1^2\|u\|_{L^2}^2 + \frac{1}{2}\sum_{l=1}^3\|a_l\partial_{x_l}^m u\| - M\|u\|_{D^m}^2\\ &\geq s_1^2\|u\|_{L^2}^2 + \left(\frac{1}{2}C_T - M\right)\|u\|_{D^m}^2.\end{aligned} \tag{3.10}$$

and, moreover,

$$\frac{1}{2}C_T - M > 0.$$

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Thus, the quadratic form $b_{js_1}(\cdot,\cdot)$ is coercive for every $s_1 \in \mathbb{R}$. In particular, we have

$$\operatorname{Re} b_{js_1}(u,u) \geq s_1^2 \|u\|_{L^2}^2 \quad \text{and} \quad \operatorname{Re} b_{js_1}(u,u) \geq \left(\frac{1}{2}C_T - M\right)\|u\|_{D^m}^2. \tag{3.11}$$

By the Lax–Milgram Lemma, we have that for any $w \in L^2(\Omega, \mathbb{C}\otimes\mathbb{H})$ there exists $u_w \in H_0^m(\Omega,\mathbb{H})$, for $s_1 \in \mathbb{R}\setminus\{0\}$ and $j \in \mathbb{S}$, such that

$$b_{js_1}(u_w, v) = \langle w, v\rangle_{L^2}, \quad \text{for all } v \in H_0^m(\Omega,\mathbb{H}).$$

What remains to prove is the inequality (3.8). Applying the first of the inequalities in (3.10) and observing that

$$\sum_{\ell=1}^{3} \|\partial_{x_\ell}^m u\|_{L^2}^2 \leq \frac{1}{C_T}\sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2$$

we have:

$$\begin{aligned}
\operatorname{Re} b_{js_1}(u,u) &\geq \frac{1}{2}\sum_{l=1}^{3} \|a_l \partial_{x_l}^m u\| - M\|u\|_{D^m}^2 \\
&\geq \frac{1}{2}\sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2 - \frac{M}{C_T}\sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2 \\
&\geq \frac{C_T - 2M}{2C_T}\sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2 \geq C_1\|Tu\|_{L^2}^2,
\end{aligned}$$

where the second inequality is due to the fact that $C_T^{-1}a_\ell^2(x) \geq 1$ for any $x \in \Omega$, the fourth inequality is due to the fact that $\|Tu\|_{L^2}^2 \leq 3\sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2$ and we have set

$$C_1 := \frac{C_T - 2M}{6C_T}.$$

This concludes the proof. □

4. Weak solution of the Problem 1.4 for unbounded Ω

To prove the existence and uniqueness of the weak solutions of the Problem 1.4 in the case of Ω unbounded, we proceed as in the previous section.

First, we prove the continuity in § 4.1. The coercivity will be proved in § 4.2 when $s = js_1$ for $j \in \mathbb{S}$ and $s_1 \in \mathbb{R}$ with $s_1 \neq 0$. As a direct consequence of the coercivity, we will prove an L^2 estimate for the weak solution u that belongs to $H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H})$ and also we will prove an L^2 estimate for the term $T(u)$.

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4.1. The continuity of the bilinear form $b_s(\cdot,\cdot)$

The bilinear form

$$b_s(\cdot,\cdot) : H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H}) \times H_0^m(\Omega, \mathbb{C}\otimes\mathbb{H}) \to \mathbb{H},$$

for some $s \in \mathbb{H}$, is continuous if there exists a positive constant $C(s)$ such that

$$|b_s(u,v)| \le C(s)\|u\|_{H^m}\|v\|_{H^m}, \quad \text{for all } u, v \in H^m(\Omega, \mathbb{C}\otimes\mathbb{H}).$$

We note that the constant $C(s)$ depends on $s \in \mathbb{H}$ but does not depend on u and $v \in H^m(\Omega, \mathbb{C}\otimes\mathbb{H})$.

Proposition 4.1 (Continuity of b_s)**.** *Let Ω be an unbounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^1. Assume that a_1, a_2, $a_3 \in C^m(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega)$ and that $\partial^\beta a_j \in L^\infty(\Omega)$ for any $j = 1, 2, 3$ and for any $\beta \in \mathbb{N}^3$, $|\beta| \le m$. Then we have*

$$\left|\sum_{l=1}^{3}\int_\Omega \langle a_l^2(x)\partial_{x_l}^m(u), \partial_{x_l}^m(v)\rangle\, dx\right| \le C_{1,\,m,\,a_j,\,\Omega}\|u\|_{H^m}\|v\|_{H^m} \tag{4.1}$$

$$\left|\sum_{l=1}^{3}\sum_{|\mathbf{t}'|=m\,\wedge\, t_2\le m-1}\binom{m}{\mathbf{t}'}\int_\Omega \langle \partial_{x_l}^{t_1'}(a_l^2(x))\partial_{x_l}^m(u), \partial_{x_l}^{t_2'}(v)\rangle\, dx\right| \le C_{2,\,m,\,a_j,\,\Omega}\|u\|_{H^m}\|v\|_{H^m} \tag{4.2}$$

$$\left|\sum_{l=1}^{3}\sum_{k=1}^{m}(-1)^k\sum_{|\mathbf{t}|=m-k}\binom{m}{k}\binom{m-k}{\mathbf{t}}\int_\Omega \langle \partial_{x_l}^{t_1}(a_l(x))\partial_{x_l}^{t_2+k}(a_l(x))\partial_{x_l}^m(u), \partial_{x_l}^{t_3}(v)\rangle\, dx\right|$$
$$\le C_{3,\,m,\,a_j,\,\Omega}\|u\|_{H^m}\|v\|_{H^m} \tag{4.3}$$

$$\Bigg|\sum_{l<j}\sum_{k=1}^{m}(-1)^k\sum_{|\mathbf{t}|=m-k}\binom{m}{k}\binom{m-k}{\mathbf{t}}\int_\Omega \langle e_l e_j\partial_{x_l}^{t_1}(a_l(x))\partial_{x_l}^{t_2+k}(a_j(x))\partial_{x_j}^m(u), \partial_{x_l}^{t_3}v\rangle\, dx$$
$$- \int_\Omega \langle e_l e_j\partial_{x_j}^{t_1}(a_j(x))\partial_{x_j}^{t_2+k}(a_l(x))\partial_{x_l}^m(u), \partial_{x_j}^{t_3}v\rangle\, dx - 2s_0\langle T(u), v\rangle_{L^2}\Bigg|$$
$$\le C_{4,\,m,\,a_j,\,\Omega,\,s}\|u\|_{H^m}\|v\|_{H^m} \tag{4.4}$$

where the previous constants can be estimated as follows

$$C_{1,\,m,\,a_j,\,\Omega} \le \max_{l=1,2,3}\sup_{x\in\Omega}(a_l^2(x)), \quad C_{2,\,m,\,a_j,\,\Omega} \le CK(m)\max_{\substack{l=1,2,3\\ t=1,\dots,m}}\sup_{x\in\Omega}(|\partial_{x_l}^t(a_l^2(x))|)$$

$$C_{3,\,m,\,a_j,\,\Omega} \le CK(m)\left(\max_{\substack{l=1,2,3\\ t=0,\dots,m}}\sup_{x\in\Omega}(|\partial_{x_l}^t(a_l(x))|)\right)^2$$

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$$C_{4,\,m,\,a_j,\,\Omega,\,s} \le CK(m)\left[\left(\max_{\substack{l=1,2,3\\ t=0,\ldots,m}}\sup_{x\in\Omega}(|\partial^t_{x_l}(a_j(x))|)\right)^2 + |s_0|\max_{l=1,2,3}\sup_{x\in\Omega}(|a_l(x)|)\right]$$

where the constant C is the sum of the maximum of the integrals in the inequalities (3.2), (3.3) and (3.4).

Moreover, for any $s\in\mathbb{H}$ there exists a positive constant $C(s)$ such that for any $(u,v)\in H^m_0(\Omega,\mathbb{C}\otimes\mathbb{H})\times H^m_0(\Omega,\mathbb{C}\otimes\mathbb{H})$, the bilinear form $b_s(\cdot,\cdot)$ satisfies the estimate

$$|b_s(u,v)| \le C(s)\|u\|_{H^m}\|v\|_{H^m}, \tag{4.5}$$

i.e., $b_s(\cdot,\cdot)$ is a continuous bilinear form.

Proof. The inequality (4.1) is a direct consequence of the boundedness of the coefficients a_l's and the Hölder inequality. The estimates (4.2), (4.3) and (4.4) can be proved in a similar way and so we briefly explain how to prove the inequality (4.2). First we apply the Hölder inequality to each integral that belongs to the left-hand side. Thus, each integral can be estimated by a constant (that depends on the sup norm of $\partial^s_{x_l}a_t$'s) times $\|\partial^m_{x_l}u\|\|\partial^\beta v\|$ for $|\beta|\le m$ and $l=1,2,3$. Since $\|\partial^m_{x_l}u\|\|\partial^\beta v\|\le\|u\|_{H^m}\|v\|_{H^m}$ for any $|\beta|\le m$ and $j=1,2,3$, we sum up term by term to get the desired estimate. Eventually, the continuity of $b_s(\cdot,\cdot)$ is a direct consequence of the previous estimates. □

4.2. Weak solution of the Problem 1.4

For the coercivity, we need the following results (see [26]), which are the corresponding of Poincaré's inequality in the case of the unbounded domains.

Theorem 4.2. *Let Ω be a domain in $\mathbb{R}^3$ which satisfies the property (R). Let $\phi\in C^1(\overline{\Omega})$ be a bounded positive function for which there exists a positive constant λ such that*

$$\frac{d}{dr}[r^2\phi(r\omega)] \le -\lambda r^2\phi(r\omega) \tag{4.6}$$

for all $\omega\in\mathbb{S}^2$. Then for each $u\in C^\infty_0(\Omega,\mathbb{R})$ and any $(1\le p<\infty)$, we have

$$\int_\Omega |u|^p\phi\,dx \le \left(\frac{p}{\lambda}\right)^p\int_\Omega|\nabla u|^p\phi\,dx.$$

We call ϕ a weight function for Ω with a rate of exponentially decaying λ. As explained in [26], we can observe that the graph of ϕ has the shape of an exponentially decaying hill. An example of a weight function for all the domains which satisfy the property (R) is

$$\phi(x) = \frac{e^{-\lambda|x-P|}}{|x-P|^2} \quad \text{when } P\notin\Omega$$

If the domain is contained in a half-space, after an affine changing of variables, we can suppose that it is contained in $\{x\in\mathbb{R}^3 : x_1>0\}$. The previous theorem holds also requiring the weight function ϕ satisfies the assumption

$$\partial_{x_1}\phi(x) \leq -\lambda\phi(x).$$

instead of the assumption (4.6). In this case, the graph of ϕ has the shape of an exponentially decaying ridge and an example of this kind of function is

$$\phi = e^{-\lambda x_1}$$

Theorem 4.3. *Let Ω be an unbounded domain in $\mathbb{R}^3$ such that it satisfies the property (R) and with its boundary $\partial\Omega$ of class C^1. Let ϕ be a bounded weight function for Ω with the rate of exponential decay λ. Let T be the operator defined in (1.2) with coefficients $a_1,\, a_2,\, a_3 \in C^m(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega)$ such that $|\partial^t_{x_l} a_j(x)|^2 \leq C\phi$ for some positive constant C and for all $t = 1, \ldots, m$ and $j,\, l = 1,\, 2,\, 3$. We set*

$$M := K(m) \max_{\substack{l=1,2,3\\ t=1,\ldots,m}} \sup_{x\in\Omega} \left(\left|\partial^t_{x_l} a_l\right|^2\right) + K(m, \phi, \lambda),$$

where $K(m)$ and $K(m, \phi, \lambda)$ are positive constants that depend on the order m of the operator, on ϕ and on λ. We suppose that

$$C_T := \min_{l=1,2,3} \inf_{x\in\Omega} (a_l^2(x)) > 0, \quad \frac{1}{2}C_T - M > 0. \tag{4.7}$$

Then:

(I) *The boundary value Problem (1.4) has a unique weak solution $u \in H^m_0(\Omega, \mathbb{H} \otimes \mathbb{C})$, for $s \in \mathbb{H} \setminus \{0\}$ with $\mathrm{Re}(s) = 0$, and*

$$\|u\|^2_{L^2} \leq \frac{1}{|s|^2} \mathrm{Re}(b_s(u, u)). \tag{4.8}$$

(II) *Moreover, we have the following estimate*

$$\|T(u)\|^2_{L^2} \leq C_1^{-1} \mathrm{Re}(b_s(u, u)), \tag{4.9}$$

for every $u \in H^m_0(\Omega, \mathbb{H})$, and $s \in \mathbb{H} \setminus \{0\}$ with $\mathrm{Re}(s) = 0$, where

$$C_1 := \frac{C_T - 2M}{6C_T}.$$

Proof. In order to use the Lax–Milgram Lemma to prove the existence and the uniqueness of the solution for the weak formulation of the problem, it is sufficient to prove the coercivity of the bilinear form $b_s(\cdot, \cdot)$ in Definition 2.1 since the continuity is proved in Proposition 4.1. First, we write explicitly $\mathrm{Re}\, b_{\mathbf{j}s_1}(u, u)$, where we have set $s = \mathbf{j}s_1$, for

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$s_1 \in \mathbb{R}$ and $\mathbf{j} \in \mathbb{S}$:

$$
\begin{aligned}
&\operatorname{Re} b_{\mathbf{j}s_1}(u,u)\\
&\quad = s_1^2\|u\|_{L^2}^2 + \sum_{\ell=1}^{3} \|a_\ell \partial_{x_\ell}^m u\|_{L^2}^2\\
&\qquad + \operatorname{Re}\left(\sum_{l=1}^{3} \sum_{|\mathbf{t}'|=m \wedge t_2 \le m-1} \sum_{k=0}^{t_1} \binom{m}{\mathbf{t}'} \binom{t_1}{k} \int_\Omega \langle \partial_{x_l}^k(a_l(x)) \partial_{x_l}^{t_1-k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_2}(u)\rangle dx\right.\\
&\qquad + \sum_{l=1}^{3} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}} \int_\Omega \langle \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_3}(u)\rangle\, dx\\
&\qquad + \sum_{l<j} \sum_{k=1}^{m} (-1)^k \sum_{|\mathbf{t}|=m-k} \binom{m}{k} \binom{m-k}{\mathbf{t}}\\
&\qquad \times \left(\int_\Omega \langle e_l e_j \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^{t_2+k}(a_j(x)) \partial_{x_j}^m(u), \partial_{x_l}^{t_3} u\rangle\, dx\right.\\
&\qquad \left.\left. - \int_\Omega \langle e_l e_j \partial_{x_j}^{t_1}(a_j(x)) \partial_{x_j}^{t_2+k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_j}^{t_3} u\rangle\, dx\right)\right). \qquad (4.10)
\end{aligned}
$$

We see that the first two terms in (4.10) are positive. The other terms, that we have collected in four summations, can be estimated all in the same way and for this reason we explain how to estimate only the integrals in the first summation. By Hölder's inequality and the repeated application of Theorem 4.2, we have for $k = 0$ or $k = t_1$

$$
\begin{aligned}
\left|\int_\Omega \langle a_l(x) \partial_{x_l}^{t_1}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_2}(u)\rangle\, dx\right| &\le \epsilon\|a_l \partial_{x_l}^m u\|^2 + \frac{1}{\epsilon}\|\partial_{x_l}^{t_1}(a_l)\partial_{x_l}^{t_2}(u)\|^2\\
&\le \epsilon\|a_l \partial_{x_l}^m u\|^2 + C\frac{1}{\epsilon}\|\phi^{\frac{1}{2}}\partial_{x_l}^{t_2}(u)\|^2\\
&\le \epsilon\|a_l \partial_{x_l}^m u\|^2 + C'\frac{1}{\epsilon}\left(\frac{2}{\lambda}\right)^{m-t_2} \sum_{|\beta|=m} \|\partial^\beta(u)\|^2\\
&\le \epsilon\|a_l \partial_{x_l}^m u\|^2 + C''\frac{1}{\epsilon}\left(\frac{2}{\lambda}\right)^{m-t_2} \|u\|_{D^m}^2
\end{aligned}
$$

where the constants C, C' and C'' depend on ϕ, the derivatives of a_j's and m. In the case $0 < k < t_1$, we have

$$
\begin{aligned}
&\left|\int_\Omega \langle \partial_{x_l}^k(a_l(x)) \partial_{x_l}^{t_1-k}(a_l(x)) \partial_{x_l}^m(u), \partial_{x_l}^{t_2}(u)\rangle\, dx\right|\\
&\quad \le \frac{1}{2}\|\partial_{x_l}^k(a_l)\partial_{x_l}^m u\|^2 + \frac{1}{2}\|\partial_{x_l}^{t_1-k}(a_l)\partial_{x_l}^{t_2}(u)\|^2\\
&\quad \le \frac{1}{2}\sup_{x\in\Omega}\left(\left|\partial_{x_l}^k a_l\right|^2\right)\|\partial_{x_l}^m u\|^2 + C''\frac{1}{2}\left(\frac{2}{\lambda}\right)^{2(m-t_2)} \|u\|_{D^m}^2.
\end{aligned}
$$

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Summing up all the previous inequalities and choosing the constants $k(m)$ and $k(m,\phi,\lambda)$ in a suitable way, we obtain

$$\left|\sum_{l=1}^{3}\sum_{|\mathbf{t}'|=m\,\wedge\, t_2\leq m-1}\sum_{k=0}^{t_1}\binom{m}{\mathbf{t}'}\binom{t_1}{k}\int_\Omega \langle \partial^k_{x_l}(a_l(x))\partial^{t_1-k}_{x_l}(a_l(x))\partial^m_{x_l}(u), \partial^{t_2}_{x_l}(u)\rangle\, dx\right|$$
$$\leq \epsilon\sum_{l=1}^{3}\|a_l\partial^m_{x_l}u\| + \left(K(m)\max_{\substack{j,\,l=1,2,3\\ t=1,\ldots,m}}\sup_{x\in\Omega}\left(\left|\partial^t_{x_j}a_l\right|^2\right) + \frac{K(m,\phi,\lambda)}{\epsilon}\right)\|u\|^2_{D^m}.$$

We note that $K(m,\phi,\lambda)$ depends on λ through the multiplicative constant $\max((\frac{2}{\lambda})^{2m}, (\frac{2}{\lambda})^2)$. Analogous estimates also holds for the other summation of integrals (here the role of k is played by t_1 and thus, we have to distinguish the cases of $t_1=0$ and $t_1\neq 0$) with possibly different constants $K(m)$ and $K(m,\phi)$. Summing up all the estimates and choosing in the right way ϵ, we obtain

$$\begin{aligned}\operatorname{Re} b_{\mathbf{j}s_1}(u,u) &\geq s_1^2\|u\|^2_{L^2} + \frac{1}{2}\sum_{l=1}^{3}\|a_l\partial^m_{x_l}u\| - M\|u\|^2_{D^m}\\ &\geq s_1^2\|u\|^2_{L^2} + \left(\frac{1}{2}C_T - M\right)\|u\|^2_{D^m}.\end{aligned} \tag{4.11}$$

By hypothesis (4.7), we know that

$$\frac{1}{2}C_T - M > 0$$

thus, the quadratic form $b_{\mathbf{j}s_1}(\cdot,\cdot)$ is coercive for every $s_1\in\mathbb{R}$. In particular, we have

$$\operatorname{Re} b_{\mathbf{j}s_1}(u,u)\geq s_1^2\|u\|^2_{L^2} \quad \text{and} \quad \operatorname{Re} b_{\mathbf{j}s_1}(u,u)\geq (C_T-M)\|u\|^2_{D^m}. \tag{4.12}$$

By the Lax–Milgram Lemma, we have that for any $w\in L^2(\Omega,\mathbb{C}\otimes\mathbb{H})$, there exists $u_w\in H^m_0(\Omega,\mathbb{C}\otimes\mathbb{H})$, for $s_1\in\mathbb{R}\setminus\{0\}$ and $\mathbf{j}\in\mathbb{S}$, such that

$$b_{\mathbf{j}s_1}(u_w,v) = \langle w,v\rangle_{L^2}, \quad \text{for all } v\in H^m_0(\Omega,\mathbb{C}\otimes\mathbb{H}).$$

What remains to prove is the inequality (4.9). Applying the first of the inequalities in (4.11) and observing that

$$\|u\|_{D^m} = \sum_{\ell=1}^{3}\|\partial^m_{x_\ell}u\|^2_{L^2} \leq \frac{1}{C_T}\sum_{\ell=1}^{3}\|a_\ell\partial^m_{x_\ell}u\|^2_{L^2}$$

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we have:

$$\begin{aligned}\operatorname{Re} b_{js_1}(u,u) &\geq \frac{1}{2}\sum_{l=1}^{3}\|a_l\partial_{x_l}^m u\| - M\|u\|_{D^m}^2\\ &\geq \frac{1}{2}\sum_{\ell=1}^{3}\|a_\ell\partial_{x_\ell}^m u\|_{L^2}^2 - \frac{M}{C_T}\sum_{\ell=1}^{3}\|a_\ell\partial_{x_\ell}^m u\|_{L^2}^2\\ &\geq \frac{C_T-2M}{2C_T}\sum_{\ell=1}^{3}\|a_\ell\partial_{x_\ell}^m u\|_{L^2}^2 \geq C_1\|Tu\|_{L^2}^2,\end{aligned}$$

where the second inequality is due to the fact that $C_T^{-1}a_\ell^2(x) \geq 1$ for any $x \in \Omega$, the fourth inequality is due to the fact that $\|Tu\|_{L^2}^2 \leq 3\sum_{\ell=1}^3 \|a_\ell\partial_{x_\ell}^m u\|_{L^2}^2$ and we have set

$$C_1 := \frac{C_T-2M}{6C_T}.$$

This conclude the proof. □

Remark 4.4. We provide here two examples of domains and coefficients a_l's that satisfy the hypothesis of Theorem 4.3.

- Let $\Omega := \{x \in \mathbb{R}^3 : |x-P| > M\}$ and $\phi(x) := \frac{e^{-\lambda|x-P|}}{|x-P|^2}$, we define $a_l(x) := K_l + e^{-\lambda|x-P|}s_l(x)$ where $s_l \in \mathcal{S}(\mathbb{R}^3)$ and K_l is a positive constant large enough for $l = 1, 2, 3$ ($\mathcal{S}(\mathbb{R}^3)$ is the space of the Schwartz functions). By the properties of the Schwartz functions we have that

$$\sup_{x\in\Omega}\left(\left|\frac{\partial^{\mathbf{b}}(a_l(x))}{\phi(x)}\right|\right) \leq C_{\mathbf{b},s_l}, \quad \forall\, \mathbf{b} \in \mathbb{N}^3 \quad \text{and} \quad |\mathbf{b}| > 0.$$

- Let $\Omega := \{x \in \mathbb{R}^3 : \langle x-P, v\rangle > 0\}$ and $\phi(x) := e^{-\lambda\langle x-P,v\rangle}$, we define $a_l(x) := K + e^{-\lambda\langle x-P,v\rangle}$ where K is a positive constant large enough.

5. The estimates for the S-resolvent operators and the fractional powers of T

Using the estimates in Theorem 4.3, we can now show that the S-resolvent operator of T decays fast enough along the set of purely imaginary quaternions.

Theorem 5.1. *Under the hypotheses of Theorem 3.2 or of Theorem 4.3, the operator $\mathcal{Q}_s(T)$ is invertible for any $s = \mathbf{j}s_1$, for $s_1 \in \mathbb{R}\setminus\{0\}$ and $\mathbf{j} \in \mathbb{S}$ and the following estimate*

$$\|\mathcal{Q}_s(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s_1^2} \tag{5.1}$$

holds. Moreover, the $\mathcal{S}$-resolvent operators satisfy the estimates

$$\|\mathcal{S}_L^{-1}(s,T)\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s|} \quad \textit{and} \quad \|\mathcal{S}_R^{-1}(s,T)\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s|}, \tag{5.2}$$

for any $s = \mathbf{j}s_1$, for $s_1 \in \mathbb{R}\setminus\{0\}$ and $\mathbf{j} \in \mathbb{S}$, with a constant Θ that does not depend on s.

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Proof. We saw in Theorem 3.2 or in Theorem 4.3 that for all $w \in L^2(\Omega, \mathbb{C} \otimes \mathbb{H})$ there exists $u_w \in H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H}))$, for $s_1 \in \mathbb{R} \setminus \{0\}$ and $\mathbf{j} \in \mathbb{S}$, such that

$$b_{\mathbf{j}s_1}(u_w, v) = \langle w, v \rangle_{L^2}, \quad \text{for all } v \in H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H}).$$

Thus, we can define the inverse operator $\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}(w) := u_w$ for any $w \in L^2(\Omega, \mathbb{C} \otimes \mathbb{H})$ (we note that the range of $\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}$ is in $H_0^m(\Omega, \mathbb{C} \otimes \mathbb{H})$). The first inequality (3.11) (or the first inequality in (4.12) in the case Ω unbounded), applied to $u := \mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}(w)$, implies:

$$\begin{aligned} s_1^2 \|\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}(w)\|_{L^2}^2 &\leq \operatorname{Re} b_{\mathbf{j}s_1}(Q_{js_1}(T)^{-1}(w), Q_{\mathbf{j}s_1}(T)^{-1}(w)) \\ &\leq |b_{\mathbf{j}s_1}(Q_{\mathbf{j}s_1}(T)^{-1}(w), Q_{\mathbf{j}s_1}(T)^{-1}(w))| \\ &\leq |\langle w, Q_{\mathbf{j}s_1}(T)^{-1}(w)\rangle_{L^2}| \\ &\leq \|w\|_{L^2} \|Q_{\mathbf{j}s_1}(T)^{-1}(w)\|_{L^2}, \end{aligned} \tag{5.3}$$

for any $w \in L^2(\Omega, \mathbb{C} \otimes \mathbb{H})$. Thus, we have

$$\|\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s_1^2}, \quad \text{for} \quad s_1 \in \mathbb{R} \setminus \{0\} \quad \text{and } \mathbf{j} \in \mathbb{S}.$$

The estimates (5.2) follow from the estimate (3.8) (or the estimate (4.9) in the case of Ω unbounded). Indeed, we have

$$\begin{aligned} C_1 \|T u_w\|^2 &\leq \operatorname{Re}(b_{\mathbf{j}s_1}(u_w, u_w)) \\ &\leq |b_{\mathbf{j}s_1}(u_w, u_w)| \\ &\leq |\langle w, u_w \rangle_{L^2}| \\ &\leq \|w\|_{L^2} \|u_w\|_{L^2} \\ &\overset{(5.1)}{\leq} \frac{1}{s_1^2} \|w\|_{L^2}^2, \end{aligned}$$

for $s_1 \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{S}$. This estimate implies

$$\left\|T\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1} w\right\|_{L^2} = \|T u_w\|_{L^2} \leq \frac{1}{\sqrt{C_1}|s_1|} \|w\|_{L^2},$$

so that we obtain

$$\left\|T\mathcal{Q}_{\mathbf{j}s_1}(T)^{-1}\right\|_{\mathcal{B}(L^2)} \leq \frac{1}{\sqrt{C_1}|s_1|}. \tag{5.4}$$

In conclusion, if we set

$$\Theta := 2 \max \left\{1, \frac{1}{\sqrt{C_1}}\right\},$$

estimates (5.4) and (5.1) yield

$$\begin{aligned} \left\|S_R^{-1}(s, T)\right\|_{\mathcal{B}(L^2)} &= \left\|(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}\right\|_{\mathcal{B}(L^2)} \\ &\leq \left\|T\mathcal{Q}_s(T)^{-1}\right\|_{\mathcal{B}(L^2)} + \left\|\overline{s}\mathcal{Q}_s(T)^{-1}\right\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s_1|} \end{aligned} \tag{5.5}$$

and

$$\left\|S_L^{-1}(s,T)\right\|_{\mathcal{B}(L^2)} = \left\|T\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T)^{-1}\overline{s}\right\|_{\mathcal{B}(L^2)}$$
$$\leq \left\|T\mathcal{Q}_s(T)^{-1}\right\|_{\mathcal{B}(L^2)} + \left\|\mathcal{Q}_s(T)^{-1}\overline{s}\right\|_{\mathcal{B}(L^2)} \leq \frac{\Theta}{|s_1|},$$

for any $s = \mathbf{j}s_1 \in \mathbb{H} \setminus \{0\}$. □

Thanks to the above results, we are now ready to establish our main statement.

Theorem 5.2. *Under the hypotheses of Theorem 3.2 or of Theorem 4.3, for any* $\alpha \in (0,1)$ *and* $v \in \mathcal{D}(T)$*, the integral*

$$P_\alpha(T)v := \frac{1}{2\pi}\int_{-j\mathbb{R}} s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv$$

converges absolutely in L^2.

Proof. The right S-resolvent equation implies

$$S_R^{-1}(s,T)Tv = sS_R^{-1}(s,T)v - v, \quad \forall v \in \mathcal{D}(T)$$

and so

$$\frac{1}{2\pi}\int_{-j\mathbb{R}} \left\|s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv\right\|_{L^2} \leq \frac{1}{2\pi}\int_{-\infty}^{-1} |t|^{\alpha-1}\left\|S_R^{-1}(-jt,T)\right\|_{\mathcal{B}(L^2)}\|Tv\|_{L^2}\, dt$$
$$+ \frac{1}{2\pi}\int_{-1}^{1} |t|^{\alpha-1}\left\|(-jt)S_R^{-1}(-jt,T)v - v\right\|_{L^2}\, dt$$
$$+ \frac{1}{2\pi}\int_{1}^{+\infty} t^{\alpha-1}\left\|S_R^{-1}(jt,T)\right\|_{\mathcal{B}(L^2)}\|Tv\|_{L^2}\, dt.$$

As $\alpha \in (0,1)$, the estimate (5.2) now yields

$$\frac{1}{2\pi}\int_{-j\mathbb{R}} \left\|s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv\right\|_{L^2}$$
$$\leq \frac{1}{2\pi}\int_1^{+\infty} t^{\alpha-1}\frac{\Theta}{t}\|Tv\|_{L^2}\, dt + \frac{1}{2\pi}\int_{-1}^{1}|t|^{\alpha-1}\left(|t|\frac{\Theta}{|t|}+1\right)\|v\|_{L^2}\, dt$$
$$+ \frac{1}{2\pi}\int_1^{+\infty} t^{\alpha-1}\frac{\Theta}{t}\|Tv\|_{L^2}\, dt$$
$$< +\infty.$$

□

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6. Concluding remarks

We list in the following some references associated with the spectral theory on the S-spectrum and some research directions in order to orientate the interested reader in this field. Moreover, we give some references for classical fractional problems for scalar operators.

(I) In the literature, there are several nonlinear models that involve the fractional Laplacian and even the fractional powers of more general elliptic operators, see for example, the books [11, 34].

(II) The S-spectrum approach to fractional diffusion problems used in this paper is a generalization of the method developed by Balakrishnan, see [5], to define the fractional powers of a real operator A. In the paper [13], following the book of M. Haase, see [29], has developed the theory on fractional powers of quaternionic linear operators, see also [2, 14].

(III) The spectral theorem on the S-spectrum is also an other tool to define the fractional powers of vector operators, see [1] and for perturbation results, see [12].

(IV) An historical note on the discovery of the S-resolvent operators and of the S-spectrum can be found in the introduction of the book [20].
The most important results in quaternionic operators theory based on the S-spectrum and the associated theory of slice hyperholomorphic functions are contained in the books [3, 4, 16, 18–20, 27, 28], for the case on n-tuples of operators see [17].

(V) Our future research directions will consider the development of ideas from one and several complex variables, such as in [6–9, 30–32] to the quaternionic setting.

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