Proceedings of the Edinburgh Mathematical Society: page 1 of 8
doi:10.1017/S0013091524000634

LINEARITY OF COMPACT R-ANALYTIC GROUPS

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(Received 11 January 2024)

Abstract We prove that any compact R-analytic group is linear when R is a pro-p domain of characteristic zero.

Keywords: profinite groups; R-analytic groups; linear groups

2020 Mathematics subject classification: Primary: 20E18
Secondary: 03C15; 20H25

1. Introduction

The study of p-adic analytic pro-p groups has given rise to a prolific mathematical subject since Lazard [12] introduced them in 1968, see [6] and the references therein. At the core of its development lies their rich algebraic structure and the myriad of properties they satisfy: they have polynomial subgroup growth, have finite rank, they are isomorphic to a closed subgroup of a Sylow pro-p subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$, etc.

The theory of analytic profinite groups developed further by considering manifolds over more general rings. For instance, in [1, Chapter III] and [15, Part II], analytic groups over local principal ideal domains are studied. More generally, let R be a pro-p domain, namely a local Noetherian integral domain, with maximal ideal say $\mathfrak{m}$, that is complete with respect to the $\mathfrak{m}$-adic topology and whose residue field $R/\mathfrak{m}$ is finite of characteristic p (e.g. the power series rings $\mathbb{Z}_p[[t_1, \dots, t_m]]$ and $\mathbb{F}_p[[t_1, \dots, t_m]]$, where $\mathbb{Z}_p$ and $\mathbb{F}_p$ stand for the ring of p-adic integers and the finite field of p elements respectively). In [6, Chapter 13], the authors introduce and delve into the study of R-analytic groups for a pro-p domain R. These comprise a topological group equipped with an analytic manifold structure over R such that both structures are compatible in the sense that the multiplication and the inversion are analytic maps (see [6, Definition 13.3]). This notion naturally generalizes the concept of p-adic analytic group, where R is taken to be $\mathbb{Z}_p$.

Interest in more general families of analytic groups arose when it was realized that they play a rôle in combinatorial group theory or problems concerning group words. For example, Lubotzky and Shalev [13] studied R-perfect groups – a special class of R-analytic groups – proving that their subgroup growth is close to polynomial and that they satisfy the Golod–Shafarevich inequality. Subsequently, Jaikin-Zapirain and Klopsch [10] continued the investigation of the structure of analytic groups, mainly for rings of positive characteristic, and among others, proved that finitely generated $\mathbb{F}_p[[t]]$-analytic groups are verbally elliptic. Furthermore, Bradford [2] employed analytic groups to obtain polylogarithmic upper bounds for the diameters of some finite simple groups of Lie type.

The linearity of compact p-adic analytic groups is well-established [12], but whether general compact R-analytic groups are linear remains an open question (see [13, Question 2, p. 311] or [7, Question 5, p. 124]).

There are several partial results addressing the aforementioned problem, and the question has positive answer for just-infinite R-analytic groups (Jaikin-Zapirain, [8]) and perfect $\mathbb{Z}_p[[t]]$-standard groups (Camina and Du Sautoy, [3]); we refer to § 2.2 for the definition of standard group. In a broader scope, Jaikin-Zapirain [9] proves that when R is a pro-p domain of characteristic zero, finitely generated R-analytic groups are linear.

It should be noted that perfect R-analytic groups are finitely generated, and, consequently, the majority of the known cases fall into the class of finitely generated profinite groups. Nonetheless, within the setting of R-analytic groups, this condition is quite constraining, in view that every (non-discrete) finitely generated compact R-analytic group that satisfies a group identity is in fact p-adic analytic, see [10, Theorem 1.3].

In this paper, we answer this question for pro-p domains of characteristic zero. More precisely, we prove the following:

Theorem 1.1. *Let R be a pro-p domain of characteristic zero. Then every compact R-analytic group is linear.*

The main observation is that any compact R-standard group G, where R is a pro-p domain of characteristic 0, is discriminated by $\mathrm{GL}_n(\mathbb{Z}_p)$ for a suitable n, see Corollary 2.6. This result relies on the linearity of uniform pro-p groups, see § 2.3. Recall that we say that G is discriminated by H (equivalently, G is fully residually H) if, for any finite set S of elements of G, there exists a group homomorphism $h_S : G \to H$ injective on S. Accordingly, from standard model-theoretic results – they have been included for completeness in § 2.5 – we deduce that G is a subgroup of the ultrapower of $\mathrm{GL}_n(\mathbb{Z}_p)$ which is isomorphic to $\mathrm{GL}_n(\mathbb{Z}_p^{\mathcal{U}})$ where $\mathbb{Z}_p^{\mathcal{U}}$ is the ultrapower of $\mathbb{Z}_p$ and, so in particular, it is linear.

2. Proof of Theorem 1.1

2.1. Notation and conventions

For a ring Q, we denote by $\mathrm{char}\, Q$ and $\dim_{\mathrm{Krull}} Q$ the characteristic and the Krull dimension of Q. Moreover, $Q[[X_1, \ldots, X_n]]$, or simply $Q[[\mathbf{X}]]$, is the ring of formal power series with coefficients in Q, that is, power series $\sum_\alpha a_\alpha \mathbf{X}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $a_\alpha \in Q$ and $\mathbf{X}^\alpha$ is short for $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$.

Throughout the paper, p will be a prime number and R will be a pro-p domain with maximal ideal $\mathfrak{m}$.

For an ideal $\mathfrak{a}$ both the Nth power ideal and the N-Cartesian power of $\mathfrak{a}$ are used regularly in the paper. It is customary to use the notation $\mathfrak{a}^N$ to refer to both concepts. In order to distinguish them, we keep the notation $\mathfrak{a}^N$ for the N-Cartesian power of the ideal $\mathfrak{a}$, and we fix the notation $\mathfrak{a}^{*N}$ to denote the Nth power ideal of $\mathfrak{a}$.

2.2. Reduction to R-standard groups

Any compact R-analytic group contains a so-called *R-standard* group of finite index (see [6, Theorem 13.20]), which has a strong algebraic structure. By virtue of the induced linear representation, the proof of the principal theorem is going to be reduced to this subgroup, so we next recall the definition.

An R-analytic group S of dimension d is *R-standard* exist

(i) a homeomorphism $\phi\colon S \to \left(\mathfrak{m}^{*N}\right)^d$, for some $N \in \mathbb{N}$, such that $\phi(1) = \mathbf{0}$ and
(ii) some formal power series $F_j \in R[[X_1, \ldots, X_{2d}]]$, $j \in \{1, \ldots, d\}$, such that

$$\phi(xy) = (F_1(\phi(x), \phi(y)), \ldots, F_d(\phi(x), \phi(y))) \ \forall x, y \in S.$$

The integer N in (i) is the *level* of S, and the tuple of power series $\mathbf{F} = (F_1, \ldots, F_d)$ in (ii) is called the *formal group law* of S with respect to ϕ. Any R-standard group is a pro-p group, and so compact (see [6, Proposition 13.22]).

2.3. Matrix representation of uniform pro-p groups

For compact p-adic analytic groups, $\mathbb{Z}_p$-standard groups are precisely uniform pro-p groups, namely finitely generated torsion-free pro-p groups G such that $[G, G] \leq G^{\mathbf{p}}$, where $\mathbf{p} = p$ when p is odd and $\mathbf{p} = 4$ when $p = 2$ – we shall keep this definition of $\mathbf{p}$ for the rest of the section.

The linearity of uniform pro-p groups is an essential stepping stone in our work. The proof appeals to Ado's Theorem in conjunction with the Baker–Campbell–Hausdorff formula in order to link a p-adic analytic group with its Lie algebra. More precisely:

Theorem 2.1 (cf. [6, Section 7.3]). *Let G be a uniform pro-p group of dimension d. There exists a group monomorphism $m\colon G \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ where $n \leq \gamma(p, d)$ for a function $\gamma\colon \mathbb{N}^2 \to \mathbb{N}$.*

Proof. There exists a categorical isomorphism $\mathcal{L}\colon \mathfrak{UGroup} \to \mathfrak{pLie}$ between the categories $\mathfrak{UGroup}$ of d-dimensional uniform pro-p groups and the category $\mathfrak{pLie}$ of d-dimensional powerful $\mathbb{Z}_p$-lattices, namely $\mathbb{Z}_p$-Lie algebras $\mathfrak{L}$ that are free $\mathbb{Z}_p$-modules of rank d and satisfy $[\mathfrak{L}, \mathfrak{L}] \leq \mathbf{p}\mathfrak{L}$ (see [6, Theorem 9.10]). Let $\mathcal{E}$ be the inverse of $\mathcal{L}$. Then, $\mathcal{E}$ is defined by the Baker–Campbell–Hausdorff formula, and for the powerful matrix $\mathbb{Z}_p$-lattice $\mathbf{p}\,\mathrm{M}_\ell(\mathbb{Z}_p)$, $\mathcal{E}$ is nothing but the usual matrix exponentiation. In particular,

$$\mathcal{E}(\mathbf{p}\,\mathrm{M}_\ell(\mathbb{Z}_p)) = I_\ell + \mathbf{p}\,\mathrm{M}_\ell(\mathbb{Z}_p) \leq \mathrm{GL}_\ell(\mathbb{Z}_p)$$

(compare with [1, Chapter II, § 8, Proposition 4]).

Since $\mathcal{L}(G)$ is a $\mathbb{Z}_p$-lattice of dimension d, by virtue of Ado's Theorem for principal ideal domains (see [19, Theorem 1.1]), there exists a faithful Lie algebra representation $\phi\colon \mathcal{L}(G) \to \mathrm{M}_\ell(\mathbb{Z}_p)$ where $\ell \leq f(d)$ for some function $f\colon \mathbb{N} \to \mathbb{N}$.

On the one hand, $\mathcal{E}$ is a categorical isomorphism so

$$\mathcal{E}\,(\phi) : \mathcal{E}\,(\mathbf{p}\mathcal{L}(G)) \hookrightarrow \mathcal{E}(\mathbf{p}\,\mathrm{M}_\ell(\mathbb{Z}_p)) \leq \mathrm{GL}_\ell(\mathbb{Z}_p)$$

is a group monomorphism. On the other hand, by [6, Proposition 4.31(iii)], the additive cosets – as algebras – coincide with the multiplicative cosets – as groups, and so in particular,

$$|G : \mathcal{E}\,(\mathbf{p}\mathcal{L}(G))| = |\mathcal{L}(G) : \mathbf{p}\mathcal{L}(G)| = \left|\mathbb{Z}_p^d : \mathbf{p}\mathbb{Z}_p^d\right| = \mathbf{p}^d.$$

Therefore, considering the induced representation, there exists a group monomorphism

$$m : G \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p),$$

where $n \leq \gamma(p, d)$ for $\gamma(p, d) = \mathbf{p}^d f(d)$. □

Although the proof is essentially that of Lazard [12], the principal improvement comparing with [6, Theorem 7.19] and [16, Theorem C] is the bound on the degree of the linear representation in terms of the dimension d and the prime p.

The difference with the case of positive characteristic arises in this subsection, as a categorical isomorphism between standard groups and a suitable category of Lie lattices might not exist over pro-p domains of positive characteristic.

Question 2.2. Let G be an $\mathbb{F}_p[[t]]$-standard group of dimension d. Does there exist a field K and a group monomorphism $m\colon G \hookrightarrow \mathrm{GL}_n(K)$, where n is bounded only in terms of d and p?

In view of the techniques outlined in this paper, a positive answer to the question above would imply the linearity of compact R-analytic groups over a general pro-p domain R of positive characteristic.

2.4. Discrimination

If A and B are two instances of the same algebraic structure, then A is said to be *fully residually* B or A is *discriminated* by B – equivalently, B discriminates A – if for any finite subset $S \subseteq A$, there exists a homomorphism $h\colon A \to B$ in the corresponding category such that the restriction $h|_S$ on the set S is injective.

For instance, if $(R, \mathfrak{m})$ is a pro-p domain and $\mathbf{X}$ an m-tuple of variables, then $R[[\mathbf{X}]]$ is discriminated by R. Indeed, for any finite subset $S \subseteq R[[\mathbf{X}]]$, there exists $a \in \mathfrak{m}^m$ such that the continuous *evaluation epimorphism* $s_a\colon R[[\mathbf{X}]] \to R$, $F(\mathbf{X}) \mapsto F(a)$ is injective in S (see [9, Lemma 9]). Drawing from this idea, we have:

Lemma 2.3. *Let R be a pro-p domain. For each finite subset $S \subseteq R$, there exists a pro-p domain Q_S of Krull dimension one and a continuous ring epimorphism $\pi\colon R \to Q_S$ such that $\pi|_S$ is injective.*

Proof. Let $S = \{r_i\}_{i \in I} \subseteq R$ be a finite set of distinct elements, and define $r = \prod_{i \neq j}(r_i - r_j) \in R$. Notice that $r \neq 0$ since the r_i are distinct and R is a domain.

According to Cohen's Structure Theorem [5], R is a finite integral extension of $P[[t_1, \ldots, t_m]]$ where $m = \dim_{\mathrm{Krull}} R - 1$, and P is either $\mathbb{Z}_p$ or $\mathbb{Z}_p[[t]]$ depending on the characteristic of R.

For each $a \in \mathfrak{m}^m$, let $s_a \colon P[[\mathbf{t}]] \to P$, $F(\mathbf{t}) \mapsto F(a)$ be the evaluation epimorphism and $\mathfrak{p}_a = \ker s_a$. By virtue of the Going Up Theorem (see [17, Theorem V.2.3]), there exists a prime ideal $\mathfrak{q}_a \subseteq R$ such that $\mathfrak{q}_a \cap P[[\mathbf{t}]] = \mathfrak{p}_a$. Then $Q_a = R/\mathfrak{q}_a$ is a pro-p domain which is a finite integral extension of P (see [18, Remark to Lemma 4.3]), and the quotient map $\tilde{s}_a \colon R \to R/\mathfrak{q}_a = Q_a$ extends s_a. That is, the following diagram commutes with respect to the identification $\iota : P \cong P[[t]]/\mathfrak{p}_a \hookrightarrow Q_a = R/\mathfrak{q}_a$, $x + \mathfrak{p}_a \mapsto x + \mathfrak{q}_a$:

$$\begin{array}{ccc} R & \xrightarrow{\tilde{s}_a} & Q_a \\ \big\uparrow & & \big\uparrow{\scriptstyle \iota} \\ P[[t_1, \ldots, t_m]] & \xrightarrow{s_a} & P \end{array}$$

Since Q_a is an integral extension of P, then $\dim_{\mathrm{Krull}} Q_a = 1$.

By [9, Lemma 9], we have that $\cap_{a \in \mathfrak{m}^m} \ker s_a = \{0\}$, and since R is an integral extension of $P[[\mathbf{t}]]$, it follows from [17, Complement 1 to Theorem V.2.3] that $\cap_{a \in \mathfrak{m}^m} \ker \tilde{s}_a = \{0\}$. Hence, there exists $a \in \mathfrak{m}^m$ such that $\tilde{s}_a(r) \neq 0$, and thus $\tilde{s}_a(r_i) \neq \tilde{s}_a(r_j)$ for all $i \neq j \in I$. □

Remark 2.4. As mentioned in the proof of the preceding theorem, if R has characteristic zero, it is a finite integral extension of $\mathbb{Z}_p[[t_1, \ldots, t_m]]$, where $m+1 = \dim_{\mathrm{Krull}} R$. Let $\mu(R)$ be the minimum number of elements that generate R as a $\mathbb{Z}_p[[t_1, \ldots, t_m]]$-module. By construction Q_a is generated as $\mathbb{Z}_p$-module by the image of the generators of R as $\mathbb{Z}_p[[t_1, \ldots, t_m]]$-module. Hence, as $\mathbb{Z}_p$ is a principal ideal domain, Q_a is a free $\mathbb{Z}_p$-module of rank at most $\mu(R)$.

Proposition 2.5. *Let R be a pro-p domain of characteristic zero, and let G be an R-standard group. There exists an integer $n \in \mathbb{N}$, depending on R, the dimension of G and the level of G, such that G is discriminated by $\mathrm{GL}_n(\mathbb{Z}_p)$.*

Proof. Since G is an R-standard group of dimension d and level N, it can be identified with $\left(\mathfrak{m}^{*N}\right)^d$ such that the group operation is given by a formal group law with components

$$F_i(\mathbf{X}, \mathbf{Y}) = \sum_{\alpha, \beta \in \mathbb{N}_0^d} a^i_{\alpha,\beta} \mathbf{X}^\alpha \mathbf{Y}^\beta \in R[[\mathbf{X}, \mathbf{Y}]]$$

(here $\mathbf{X}$ and $\mathbf{Y}$ denote d-tuples of variables). Let $S \subseteq \left(\mathfrak{m}^{*N}\right)^d$ be a finite set of d-tuples, i.e.

$$S = \left\{(r_{i1}, \ldots, r_{id}) \in R^d\right\}_{i \in I},$$

and let $S' = \{r_{ij}\}_{\substack{i\in I \\ j=1,\dots,d}} \subseteq R$. By Lemma 2.3, there exists a pro-p domain Q of Krull dimension one with maximal ideal $\mathfrak{n}$, and a continuous ring epimorphism $\pi\colon R \to Q$ that is injective when restricted to S'.

Let $H = \left(\mathfrak{n}^{*N}\right)^d$ with the natural Q-analytic manifold structure. Then it can be endowed with a group structure, where the group operation is given by the d-dimensional formal group law with components

$$\tilde{F}_i(\mathbf{X},\mathbf{Y}) = \sum_{\alpha,\beta\in\mathbb{N}_0^d} \pi(a^i_{\alpha,\beta})\mathbf{X}^\alpha\mathbf{Y}^\beta \in Q[[\mathbf{X},\mathbf{Y}]]$$

(see [18, Corollary 3.2]). Then $\pi^d\colon G \to H$, $(r_1,\dots,r_d) \mapsto (\pi(r_1),\dots,\pi(r_d))$ is a group epimorphism that is injective when restricted to S.

Moreover, by Remark 2.4, Q is a free $\mathbb{Z}_p$-module of rank $\mu' \le \mu(R)$, and thus, by restriction of scalars (see [6, Examples 13.6(iv)]), H is a p-adic analytic group of dimension $\mu' d$. More precisely, if $\sigma\colon Q \to \mathbb{Z}_p^{\mu'}$ is a $\mathbb{Z}_p$-module isomorphism, then $p^N\mathbb{Z}_p^{\mu'} \subseteq \sigma\left(\mathfrak{m}^{*N}\right)$. Thus, H contains a uniform pro-p group of level N, say

$$U := \left(\sigma^d\right)^{-1}\left(p^N\mathbb{Z}_p^{\mu' d}\right).$$

According to Theorem 2.1, there exists a faithful linear representation $m_1\colon U \hookrightarrow \mathrm{GL}_{\gamma(p,\ \mu(R)d)}(\mathbb{Z}_p)$, for a function $\gamma\colon \mathbb{N}^2 \to \mathbb{N}$.

Furthermore, in view of [6, Proposition 4.31 (iii)], the subgroup indexes coincide with the indexes as additive groups, so

$$|H : U| \le \left|\mathbb{Z}_p^{\mu' d} : p^N\mathbb{Z}_p^{\mu' d}\right| = p^{N\mu' d} \le p^{N\mu(R)d}.$$

Let

$$n = p^{N\mu(R)d}\gamma(p,\mu(R)d) \in \mathbb{N}$$

and let $m\colon H \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be the linear representation induced from m_1. In particular, m is a group monomorphism. Lastly, $m \circ \pi^d\colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ is a group homomorphism that is injective when restricted to S. □

In other words, we have the following

Corollary 2.6. *Let R be a pro-p domain of characteristic 0. Every R-standard group is discriminated by a linear group.*

2.5. Embedding into ultrapowers

In order to conclude the proof, we use a classic result of Mal'cev [14] to deduce the following:

Theorem 2.7 *Let R be a pro-p domain of characteristic* 0. Let G be an R-standard group, then G embeds in an ultrapower of $\mathrm{GL}_n(\mathbb{Z}_p)$ for a suitable $n \in \mathbb{N}$.

Remark 2.8. The theorem follows from standard results in logic and it is well-known. Indeed, in the first-order language of groups, an *existential sentence* is a sentence that has a prenex normal form of the type

$$\exists x_1, \ldots, x_n \bigvee_{k \in K} \left(\bigwedge_{i \in I_k} u_{k,i}(x_1, \ldots, x_n) = 1 \quad \bigwedge_{j \in J_k} v_{k,j}(x_1, \ldots, x_n) \neq 1 \right),$$

where $u_{k,i}$ and $v_{k,j}$ are group words and the sets I_k, J_k and K are finite. Therefore, by Proposition 2.5, $\mathrm{GL}_n(\mathbb{Z}_p)$ satisfies the existential theory of G, and models of the existential theory of G are substructures of an ultrapower of G (cf. [4, Corollary 4.3.13]). However, we will include a direct proof for the sake of completeness.

Throughout the rest of the paper, we write $A^{\mathcal{U}}$ for the ultrapower of an algebraic structure A with respect to a nonprincipal ultrafilter $\mathcal{U}$ on I, that is, $A^{\mathcal{U}} = \prod_{i \in I} A/\mathcal{U}$.

Proof. Let $I = \mathcal{P}_{\mathrm{fin}}(G)$ be the collection of finite subsets of G. For each $S \in I$ set $A_S = \{T \in I \mid T \supseteq S\}$, then $\{A_S \mid S \in I\}$ generates a proper filter, so we take a nonprincipal ultrafilter $\mathcal{U}$ containing it, compare with the Ultrafilter Theorem [4, Proposition 4.1.3]. By Proposition 2.5, there exists an integer $n \in \mathbb{N}$ such that for each $S \in I$ there is a group homomorphism $h_S \colon G \to H_S = \mathrm{GL}_n(\mathbb{Z}_p)$ such that h_S is injective when restricted to S. Let

$$H := \prod_{S \in I} H_S/\mathcal{U} = \prod_{S \in I} \mathrm{GL}_n(\mathbb{Z}_p)/\mathcal{U} = \mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}},$$

and the group homomorphism $m \colon G \hookrightarrow H$ such that

$$m(g) = (h_S(g))_{S \in I}/\mathcal{U} \in H.$$

Finally, we will prove that m is injective. Indeed, let g be a nontrivial element in G and define

$$J_g := \{S \in I \mid h_S(g) = 1\} \subseteq I \setminus A_{\{1,g\}}.$$

Since $A_{\{1,g\}} \in \mathcal{U}$ and $\mathcal{U}$ is a filter, then $J_g \notin \mathcal{U}$, and thus $m(g) \neq 1$. □

We can deduce the main theorem from the preceding results.

Proof of Theorem 1.1. We can assume, without loss of generality, that G is R-standard. According to Theorem 2.7, G is a subgroup of the ultrapower $\mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}}$. Moreover,

$$\mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}} \cong \mathrm{GL}_n\left(\mathbb{Z}_p^{\mathcal{U}}\right) \leq \mathrm{GL}_n\left(\mathbb{Q}_p^{\mathcal{U}}\right)$$

(see [11, 1.L.6]). Thus, since $\mathbb{Q}_p^{\mathcal{U}}$ is a field, G is linear. □

Acknowledgements. The authors are grateful to Jon González-Sánchez, Arturo Rodríguez-Fanlo and the referee for their useful comments.

Funding Statement. The first author is supported by the Spanish Government, grant PID2020-117281GB-I00, partly by the European Regional Development Fund (ERDF), and the Basque Government, grant IT1483-22. The second author is supported by the Slovenian Research Agency, programs P1-0222, P1-0294 and grants J1-50001, J1-4351, N1-0216.

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