# ON THE MÖBIUS LADDERS

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Consider the graph $M_n$, where $n = 2r \geq 6$, consisting of a polygon of length $n$ and all $n/2$ chords joining opposite pairs of vertices. This graph has $2r$ vertices which we denote by $1, 2, 3, \ldots, 2r$, and the $3r$ (undirected) edges

$$(1, 2), (2, 3), \ldots, (2r-1, 2r), (2r, 1);$$

$$(1, r+1), (2, r+2), \ldots, (r, 2r).$$

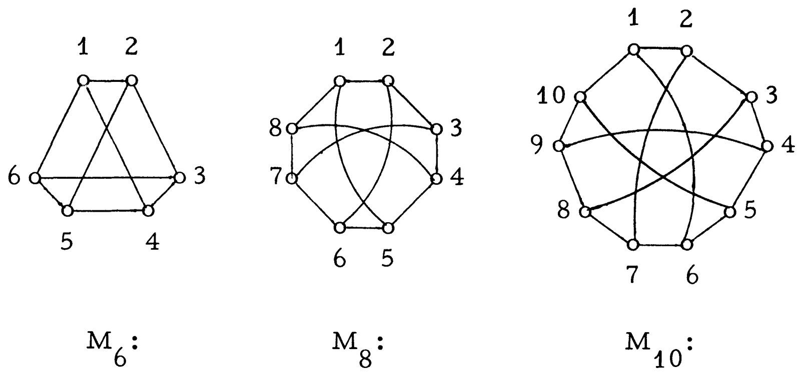


Figure 1

We call $M_n$ the <u>n-ladder</u>, defined thus far only for $n$ even. The three smallest n-ladders with $n$ even are shown in Figure 1. It is easy to see that $M_6$ is isomorphic with $K_{3,3}$,

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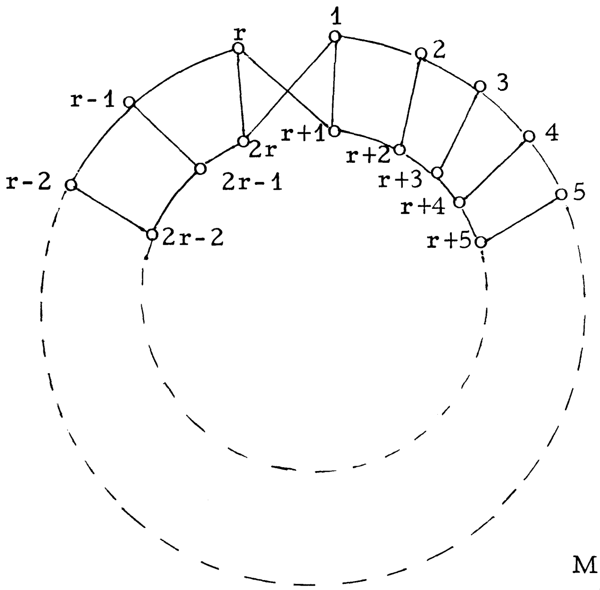
the complete bipartite graph on 2 sets of 3 vertices each, i.e., the (second) Kuratowski graph of 3 houses 1, 3, 5 and 3 utilities (electricity, gas, water) 2, 4, 6 .

The crossing number $C(G)$ of a graph $G$ is defined [3, 1] as the minimum possible number of intersections of pairs of edges when $G$ is drawn in the plane. What is the crossing number $C(M_{2r})$ of the n-ladders with n even? One might conjecture from Figure 1 that the answer is a monotonic increasing function of n. Surprisingly the answer is always $C(M_{2r}) = 1$. To prove this, we show that $C(M_{2r}) \leq 1$ and $C(M_{2r}) \geq 1$. The first of these two inequalities follows from the fact that $M_{2r}$ can be drawn with just one crossing, as in Figure 2. It was this particular representation of $M_{2r}$ (which is reminiscent of the Möbius strip) that led to the title of this note.




$M_{2r}$:

Figure 2

To prove that $C(M_{2r}) \geq 1$, we need to show that the 2r-ladder is nonplanar. Observe that the deletion of any r-3 chords from $M_{2r}$ results in a subgraph homeomorphic with $K_{3,3}$. Thus $M_{2r}$ must be nonplanar by the well-known theorem of Kuratowski [4].

One can also define the Möbius ladders $M_n$ for odd $n = 2r + 1$, $r \geq 2$, as the graph consisting of an n-gon together with two chords at each vertex joining it to the two most opposite vertices of the polygon. Obviously, $M_5$, the smallest odd n-ladder, is isomorphic to the complete graph $K_5$ with 5 vertices, also known as the first Kuratowski graph. Thus the two smallest n-ladders, $M_5$ and $M_6$, are the two Kuratowski graphs and so the family of graphs $M_n$ may be regarded as a generalization of the Kuratowski graphs.

What is the crossing number of the odd n-ladders? It is perhaps more surprising than the result for the even n-ladders that the answer is again 1 !

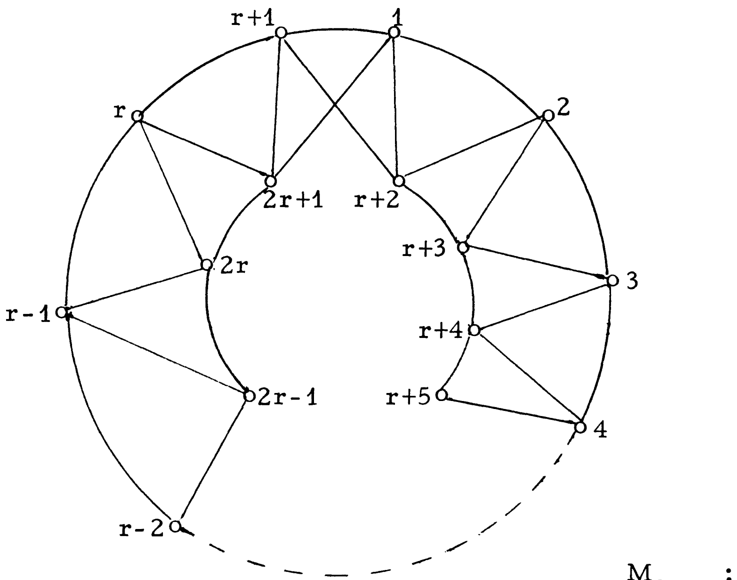



Figure 3

As for even ladders, it follows from the drawing in Figure 3 that $C(M_{2r+1}) \leq 1$ and from Kuratowski's Theorem that $C(M_{2r+1}) \geq 1$. In fact, since the odd n-ladders are regular of degree 4, the degree of each vertex is even. Hence, as noted by Zeeman [5], the parity of the number of crossings in one drawing of an odd n-ladder in the plane agrees with the parity in any other drawing. Since this parity is odd in Figure 3, it cannot be made zero.

One of us [2] defined a graph to be minimally nonplanar if its crossing number is one. Our observations can be summarized.

THEOREM. Every Möbius ladder is minimally nonplanar.

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