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Group Actions on Quasi-Baer Rings

Dedicated to Professor J. W. Fisher on his sixty-fifth birthday

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Abstract. A ring R is called *quasi-Baer* if the right annihilator of every right ideal of R is generated by an idempotent as a right ideal. We investigate the quasi-Baer property of skew group rings and fixed rings under a finite group action on a semiprime ring and their applications to C^*-algebras. Various examples to illustrate and delimit our results are provided.

Introduction

Throughout this paper all rings are associative with identity unless indicated otherwise, and R denotes such a ring. Recall from [34] and [25] that a ring R is called (*quasi-*) *Baer* if the right annihilator of every (right ideal) nonempty subset of R is generated by an idempotent as a right ideal.

The quasi-Baer condition was used to characterize a finite dimensional algebra over an algebraically closed field as a twisted matrix units semigroup algebra [25]. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer. Thus prime rings with nonzero right singular ideal [22, 36] are quasi-Baer but not Baer. The n-by-n ($n > 1$) matrix ring over a commutative non-Prüfer domain is a quasi-Baer ring, but it is not Baer. Also the n-by-n ($n > 1$) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not Baer [43] and [34, p.16]. Polynomial extensions of Baer rings are quasi-Baer, but in general they are not Baer. From [11], a ring R is called *right FI-extending* if for any two-sided ideal I of R there exists an idempotent $e \in R$ such that I is right essential in eR. In [18] it is shown that if R is a semiprime ring then R is right FI-extending if and only if R is quasi-Baer.

For another interesting class of quasi-Baer rings, it is shown in [21] that the symmetric normed algebra of quotients $Q_b(A)$ of a C^*-algebra A is quasi-Baer. Also it is shown in [4] that the local multiplier algebra $M_{\mathrm{loc}}(A)$ of a C^*-algebra A is quasi-Baer. For studying a wider class of C^*-algebras, quasi-AW^*-algebras are defined. Recall from [21] that a unital C^*-algebra is called a *quasi-AW^*-algebra* if it is a quasi-Baer $*$-ring (*i.e.,* the right annihilator of every right ideal of A is generated by a projection as a right ideal). It is shown in [21] that a unital C^*-algebra is a quasi-AW^*-algebra

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if and only if A is quasi-Baer. Thereby $M_{\mathrm{loc}}(A)$ and every prime unital C^*-algebra are quasi-AW^*-algebras. In [20] and [21], the concept of ring hulls is introduced and it is also shown that every semiprime ring R (hence every unital C^*-algebra) has a quasi-Baer absolute to $Q(R)$ right ring hulls $\widehat{Q}_{\mathfrak{qB}}(R)$, where $Q(R)$ is a maximal right ring of quotients of R. It is proved in [21] that for a unital C^*-algebra A with only finitely many minimal prime ideals, $\widehat{Q}_{\mathfrak{qB}}(A)$ is a quasi-AW^*-algebra. Moreover, examples of quasi-AW^*-algebras which are not AW^*-algebras are provided in [21].

Many authors have studied the Baer and quasi-Baer properties of rings as well as the transference of the Baer and quasi-Baer properties between a ring R and various extensions of R including polynomial type extensions and group ring extensions (see [7, 10–21, 23, 30, 31, 33, 37, 45, 46]). Moreover in [6] and [44], the skew group ring $A * G$ and the fixed ring A^G of a C^*-algebra A with a group G of ring automorphisms of A have been investigated. In [27] the skew group ring $A * G$ of a locally compact group G over a C^*-algebra A has also been studied.

Motivated by the above considerations, we are mainly concerned here with the quasi-Baer property of skew group rings and fixed rings under a finite group action on a given semiprime ring and their applications to C^*-algebras. Assume that R is a semiprime ring with a finite group G of X-outer ring automorphisms of R. Then we show that $R * G$ is quasi-Baer if and only if R^G is quasi-Baer if and only if R is G-quasi-Baer (Theorem 1.10).

As applications of our results, for a unital C^*-algebra A with a finite group G of X-outer $*$-automorphisms of A, we obtain that $A * G$ is a quasi-AW^*-algebra if and only if A^G is a quasi-AW^*-algebra if and only if A is G-quasi-Baer (Theorem 3.3). Thereby, if G is a finite group of X-outer $*$-automorphisms of a boundedly centrally closed unital C^*-algebra A, then $A * G$ and A^G are quasi-AW^*-algebras. In particular, if G is a finite group of X-outer $*$-automorphisms of $M_{\mathrm{loc}}(A)$, then $M_{\mathrm{loc}}(A) * G$ and $M_{\mathrm{loc}}(A)^G$ are quasi-AW^*-algebras. Moreover, for a unital C^*-algebra A, if G is a finite group of X-outer $*$-automorphisms of A, then $Q_b(A) * G$ and $Q_b(A)^G$ are quasi-Baer rings (Corollary 3.4).

Also as applications, assume that A is a unital C^*-algebra and G is a finite group of X-outer $*$-automorphisms of A. We show that if $|\operatorname{Minspec}(A)| < \infty$, then

$$|\operatorname{Minspec}(A^G)| = |\operatorname{Minspec}(A * G)| \leq |\operatorname{Minspec}(A)| \leq |\operatorname{Minspec}(A * G)| \cdot |G|,$$

where $\operatorname{Minspec}(-)$ is the set of all minimal prime ideals of a ring and $|\cdot|$ denotes the cardinality of a set (Theorem 3.5). In addition, if A is a quasi-AW^*-algebra, we prove that $\operatorname{Tdim}(A^G) = \operatorname{Tdim}(A * G) \leq \operatorname{Tdim}(A) \leq \operatorname{Tdim}(A * G) \cdot |G|$, where $\operatorname{Tdim}(-)$ is the triangulating dimension of a ring (Theorem 3.6). Thereby, our results provide answers to Open Problems (3) and (4) in [19] for triangulating dimension of certain skew group ring extensions. Furthermore, we show that if A is a finite direct sum of n prime unital C^*-algebras, then for any finite group G of X-outer $*$-automorphisms of A, there exists $k \leq n$ such that both $A * G$ and A^G are direct sums of k prime unital C^*-algebras (Theorem 3.6).

Various examples are provided to illustrate and delimit our results. In fact, we show that the following exist:

(i) a semiprime quasi-Baer ring R with a finite group G of ring automorphisms of R such that R has no nonzero $|G|$-torsion, however $R * G$ is not quasi-Baer (Example 2.1);

(ii) a semiprime ring R with a finite group G of X-outer ring automorphisms of R such that R has no nonzero $|G|$-torsion, $R * G$ is quasi-Baer, and R is G-quasi-Baer, but R is not quasi-Baer. Moreover $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R{*}G) \ncong \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R){*}G$ (Example 2.2);

(iii) a semiprime Baer ring R with a finite group G of X-outer ring automorphisms of R such that R has no nonzero $|G|$-torsion, but $R{*}G$ is not Baer (Example 2.3);

(iv) a commutative domain R (hence right extending or equivalently, right CS) with a finite group G of X-outer ring automorphisms of R such that R has no nonzero $|G|$-torsion, however $R * G$ is not right extending (Example2.4); and

(v) a quasi-AW^*-algebra A with a finite group G of X-outer $*$-automorphisms of A such that $\mathrm{Tdim}(A * G) \lneq \mathrm{Tdim}(A)$ (Example 3.7).

For a ring R, we use $Q(R)$ and $\mathbf{B}(R)$ to denote a maximal right ring of quotients of R and the set of all central idempotents of R, respectively. According to [9] an idempotent e of a ring R is called *left* (resp., *right*) *semicentral* if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. Equivalently, an idempotent e is left (resp., right) semicentral if and only if eR (resp., Re) is a two-sided ideal of R. For a ring R, we let $\mathcal{S}_\ell(R)$ (resp., $\mathcal{S}_r(R)$) denote the set of all left (resp., right) semicentral idempotents. An idempotent e of a ring R is called *semicentral reduced* if $\mathcal{S}_\ell(eRe) = \{0, e\}$. It can be seen that $\mathcal{S}_\ell(eRe) = \{0, e\}$ if and only if $\mathcal{S}_r(eRe) = \{0, e\}$. Note that $\mathbf{B}(R) = \mathcal{S}_\ell(R) \cap \mathcal{S}_r(R)$. Recall from [12] that a ring R is called *semicentral reduced* if $\mathcal{S}_\ell(R) = \{0, 1\}$, *i.e.,* 1 is a semicentral reduced idempotent of R.

For a nonempty subset X of a ring R, we let $r_R(X)$ and $\ell_R(X)$ denote the right annihilator and the left annihilator of X in R, respectively. If R is semiprime and I is a two-sided ideal of R, then $r_R(I) = \ell_R(I)$. In this case, we use $\mathrm{Ann}_R(I)$ to denote $r_R(I)$ (or $\ell_R(I)$). For a right R-module M_R and a submodule N_R of M_R, we use $N_R \leq^{\mathrm{ess}} M_R$ to denote that N_R is essential in M_R.

In a sequel to this paper, we will investigate the p.q.-Baer property of skew group rings and fixed rings under a finite group action.

1 Results

In this section, we establish the quasi-Baer property of certain skew group rings and fixed rings under a finite group action. Also we provide various examples to illustrate and delimit our results.

For a ring R, we let $\mathrm{Aut}(R)$ denote the *group of ring automorphisms* of R. Let G be a subgroup of $\mathrm{Aut}(R)$. For $r \in R$ and $g \in G$ we let r^g denote the image of r under g. We use R^G to denote the *fixed ring* of R under G, *i.e.,*

$$R^G = \{r \in R \mid r^g = r \text{ for every } g \in G\}.$$

The *skew group ring*, $R * G$, is defined to be $R * G = \bigoplus \sum_{g \in G} Rg$ with addition given componentwise and multiplication given as follows: if $a, b \in R$ and $g, h \in G$, then $(ag)(bh) = ab^{g^{-1}}gh \in Rgh$.

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Skew group ring $R * G$ is an important tool for Galois theory because it is related to the fixed ring R^G. The skew group ring $R * G$ and the fixed ring R^G have been extensively studied in [8, 26, 29, 39, 41] when G is X-outer or R has no $|G|$-torsion.

We begin with the following example.

Example 1.1 **(See also [46, Examples 3.2 and 3.3])** There exist a ring R and a finite group G of ring automorphisms of R such that R is Baer but neither $R * G$ nor R^G is quasi-Baer. Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field of characteristic 2. Then note that R is Baer. Let $g \in \mathrm{Aut}(R)$ be the conjugation by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $g^2 = 1$ since the characteristic of F is 2. Let $G = \{1, g\}$.

First, we show that $R*G$ is not quasi-Baer. Suppose $r_{R*G}((1+g)(R*G)) = e(R*G)$ for some $e = e^2 \in R * G$. Note that the idempotents of $R * G$ are $0, 1, \left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) g, \left(\begin{smallmatrix} 0 & a \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) g$ with $a, b \in F$. Since $e \in r_{R*G}((1 + g)(R * G))$, the only possible choice for e is 0, hence $r_{R*G}((1 + g)(R * G)) = 0$. This is a contradiction because

$$r_{R*G}((1+g)(R*G)) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x & x+y \\ 0 & 0 \end{pmatrix} g \,\middle|\, x, y \in F \right\}.$$

Therefore $R * G$ is not quasi-Baer. Now the fixed ring under G is

$$R^G = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in R \,\middle|\, x, y \in F \right\}.$$

We see that the only idempotents of R^G are 0 and 1, thus R^G is semicentral reduced. So if R^G is quasi-Baer, then R^G is a prime ring by [12, Lemma 4.2], a contradiction. Thus R^G is not quasi-Baer.

Definition 1.2 Let R be a semiprime ring. For $g \in \mathrm{Aut}(R)$, let $\phi_g = \{x \in Q^m(R) \mid xr^g = rx \text{ for each } r \in R\}$, where $Q^m(R)$ is the Martindale right ring of quotients of R (see [1] for more on $Q^m(R)$). We say that g is *X-outer* if $\phi_g = 0$. A subgroup G of $\mathrm{Aut}(R)$ is called *X-outer* on R if every $1 \neq g \in G$ is X-outer.

Assume that R is a semiprime ring. For $g \in \mathrm{Aut}(R)$, let $\Phi_g = \{x \in Q(R) \mid xr^g = rx \text{ for each } r \in R\}$. Then, for $g \in \mathrm{Aut}(R)$, clearly $\phi_g \subseteq \Phi_g$. Conversely, if $x \in \Phi_g$, then x is an R-normalizing element (*i.e.,* $xR = Rx$) in $Q(R)$. Therefore $x \in Q^s(R)$ by [35, Theorem 14.30, p. 395], where $Q^s(R)$ is the symmetric Martindale ring of quotients of R. Since $Q^s(R) \subseteq Q^m(R)$, it follows that $x \in Q^m(R)$. Hence $x \in \phi_g$. Therefore $\Phi_g = \phi_g$. So if G is X-outer on R, then G is X-outer on any right ring of quotients of R. For more details on X-outer ring automorphisms of a ring, see [29] and [35, p.396].

We say that a ring R has *no nonzero n-torsion* (n is a positive integer) if $na = 0$ with $a \in R$ implies $a = 0$. The following lemma follows from [8, Proposition 2.3], [26, Theorem 1.21], [29, Theorem 7 and Corollary 3], and [39, Theorem 2.1 and Theorem 3.1].

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Lemma 1.3 *Let R be a semiprime ring and G a group of ring automorphisms of R.*

(i) *If G is X-outer, then every nonzero two-sided ideal of $R*G$ intersects R nontrivially. Hence $R*G$ and R^G are semiprime.*

(ii) *If G is finite and R has no nonzero $|G|$-torsion, then $R*G$ and R^G are semiprime.*

In [21], it is shown that a semiprime ring R has a smallest quasi-Baer right ring of quotients $\widehat{Q}_{\mathfrak{qB}}(R)$, called the *quasi-Baer absolute to $Q(R)$ right ring hull* of R. This satisfies the following: (1) $\widehat{Q}_{\mathfrak{qB}}(R)$ is a quasi-Baer ring. (2) If T is a quasi-Baer right ring of quotients of R, then $\widehat{Q}_{\mathfrak{qB}}(R)$ is a subring of T. It is proved in [21] that $\widehat{Q}_{\mathfrak{qB}}(R)$ is the subring, $R\mathbf{B}(Q(R))$, of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$. For more details on ring hulls, see [20, 21].

From these results, we have the following.

Lemma 1.4 **([21, Theorem 3.3])** *Let R be a semiprime ring. Then R is quasi-Baer if and only if $\mathbf{B}(Q(R)) \subseteq R$. Thereby, a right ring of quotients S of R is quasi-Baer if and only if $\widehat{Q}_{\mathfrak{qB}}(R) \subseteq S$.*

For a ring R, we let $\mathrm{Cen}(R)$ denote the center of R.

Lemma 1.5 *For a semiprime ring R, let G be a group of X-outer ring automorphisms of R. Then $\mathrm{Cen}(R*G) = \mathrm{Cen}(R)^G$.*

Proof Let $\alpha = a_1 1 + a_2 g_2 + \cdots + a_n g_n \in \mathrm{Cen}(R*G)$ with $a_i \in R$, 1 the identity of G, and $g_i \in G$. Then $(a_1 1 + a_2 g_2 + \cdots + a_n g_n)b = b(a_1 1 + a_2 g_2 + \cdots + a_n g_n)$ for all $b \in R$. So $a_1 b = ba_1, a_2 b^{g_2^{-1}} = ba_2, \ldots,$ and $a_n b^{g_n^{-1}} = ba_n$ for all $b \in R$. Since G is X-outer, it follows that $a_2 = \cdots = a_n = 0$. Hence $\alpha = a_1 1 = a_1 \in R$. Also since $\alpha b = b\alpha$ for all $b \in R$, we have that $a_1 \in \mathrm{Cen}(R)$. Note that for all $g \in G$, $a_1 g = g a_1 = a_1^{g^{-1}} g$ implies $a_1 = a_1^{g^{-1}}$. Hence $\alpha = a_1 \in \mathrm{Cen}(R)^G$. So $\mathrm{Cen}(R*G) \subseteq \mathrm{Cen}(R)^G$. Conversely, $\mathrm{Cen}(R)^G \subseteq \mathrm{Cen}(R*G)$ is clear. Therefore $\mathrm{Cen}(R*G) = \mathrm{Cen}(R)^G$. ∎

Lemma 1.6 **([38, Theorem 5] and [41, Theorem])** *Let R be a ring and G a finite group of ring automorphisms of R. Then $Q(R*G) \cong Q(R)*G$.*

Assume that G is a finite group of ring automorphisms of a ring R. Then for $a \in R$, let $\mathrm{tr}(a) = \sum_{g\in G} a^g$, which is called the *trace* of a. Also for a right ideal I of R, the right ideal $\mathrm{tr}(I) = \{\mathrm{tr}(a) \mid a \in I\}$ of R^G is called the *trace* of I. Say $G = \{g_1, \ldots, g_n\}$. We put

$$t = g_1 + \cdots + g_n \in R*G.$$

For $r \in R$ and $\alpha = a_1 g_1 + \cdots + a_n g_n \in R*G$ with $a_i \in R$, define

$$r \cdot \alpha = r^{g_1} a_1^{g_1} + \cdots + r^{g_n} a_n^{g_n}.$$

Then R is a right $R*G$-module. Moreover, we see that ${}_{R^G}R_{R*G}$ is an $(R^G, R*G)$-bimodule. Consider the following pairings

$$(\cdot\,,\cdot)\colon Rt \otimes_{R^G} R \to R*G \text{ and } [\cdot\,,\cdot]\colon R \otimes_{R*G} Rt \to R^G$$

defined by $(at, b) = atb$ and $[a, bt] = \mathrm{tr}(ab)$. By [26], $(R*G,\ {}_{R^G}R_{R*G},\ {}_{R*G}Rt_{R^G},\ R^G)$ is a Morita context with the pairings.

The following lemma is of interest in its own right. As a byproduct, under the same assumption as in Lemma 1.7, we see that the fixed ring of the extended centroid is isomorphic to the extended centroid of the fixed ring by Lemmas 1.7 and 1.9(ii).

Lemma 1.7 *Assume that R is a semiprime ring and G is a finite group of X-outer ring automorphisms of R. Then* $\operatorname{Cen}(Q(R)^G) = [\operatorname{Cen}(Q(R))]^G$.

Proof Note that $Q(R)$ is semiprime and G is also X-outer on $Q(R)$. So we may assume that $R = Q(R)$ and it is enough to show that $\operatorname{Cen}(R^G) = \operatorname{Cen}(R)^G$.

Define $\theta\colon R * G \to \operatorname{Hom}(Rt_{R^G}, Rt_{R^G})$ by $\theta(x)(rt) = xrt$ for $x \in R * G$ and $r \in R$. We claim that θ is a ring isomorphism.

First θ is a ring homomorphism because Rt is a left ideal of $R * G$. Now $\operatorname{Ker}(\theta) = \operatorname{Ann}_{R*G}(RtR)$. Since $\operatorname{Ann}_{R*G}(RtR) \cap R = 0$, it follows that $\operatorname{Ann}_{R*G}(RtR) = 0$ by Lemma 1.3(i). Therefore θ is one-to-one.

Next, to show that θ is onto, take $f \in \operatorname{Hom}(Rt_{R^G}, Rt_{R^G})$. Define $\lambda\colon Rt \times R \to R*G$ by $\lambda(at, b) = f(at)b$ with $a, b \in R$. Then λ is biadditive. Moreover, for $r \in R^G$, we have that $\lambda(atr, b) = \lambda(art, b) = f(art)b = f(atr)b = f(at)rb = \lambda(at, rb)$. Therefore there exists an additive group homomorphism

$$\alpha\colon Rt \otimes_{R^G} R \to R * G$$

such that $\alpha(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k) = f(a_1t)b_1 + \cdots + f(a_kt)b_k$. In this case, we prove that

$$\alpha \in \operatorname{Hom}(Rt \otimes_{R^G} R_{R*G}, R * G_{R*G}).$$

For this, first note that $\alpha[(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k)r] = \alpha(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k)r$ for $r \in R$. Take $g \in G$. Then

$$\begin{aligned}\alpha[(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k) \cdot g] &= \alpha[a_1t \otimes (b_1 \cdot g) + \cdots + a_kt \otimes (b_k \cdot g)]\\ &= \alpha(a_1t \otimes b_1^g + \cdots + a_kt \otimes b_k^g)\\ &= f(a_1t)b_1^g + \cdots + f(a_kt)b_k^g.\end{aligned}$$

On the other hand, there exist $r_1, \ldots, r_k \in R$ such that

$$f(a_1t) = r_1t, \ldots, f(a_kt) = r_kt.$$

So it follows that

$$\begin{aligned}\alpha(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k)g &= (f(a_1t)b_1 + \cdots + f(a_kt)b_k)g\\ &= (r_1tb_1 + \cdots + r_ktb_k)g = r_1tb_1g + \cdots + r_ktb_kg\\ &= r_1tgb_1^g + \cdots + r_ktgb_k^g\\ &= r_1tb_1^g + \cdots + r_ktb_k^g = f(a_1t)b_1^g + \cdots + f(a_kt)b_k^g.\end{aligned}$$

Thus

$$\alpha[(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k) \cdot g] = \alpha(a_1t \otimes b_1 + \cdots + a_kt \otimes b_k)g$$

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for $g \in G$. Therefore $\alpha \in \mathrm{Hom}(Rt \otimes_{R^G} R_{R*G}, R * G_{R*G})$.

Define $\overline{f}\colon RtR \to R * G$ by

$$\overline{f}(a_1 t b_1 + \cdots + a_k t b_k) = f(a_1 t) b_1 + \cdots + f(a_k t) b_k$$

for $a_1, b_i \in R$, $i = 1, \ldots, k$. To see that $\overline{f}$ is well-defined, suppose that $c_1 t d_1 + \cdots + c_m t d_m = u_1 t v_1 + \cdots + u_n t v_n$ with $c_i, d_i \in R$ and $u_j, v_j \in R$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then $c_1 t d_1 + \cdots + c_m t d_m + (-u_1) t v_1 + \cdots + (-u_n) t v_n = 0$. Then by an argument in the proof of [40, Theorem 3]

$$\alpha(c_1 t \otimes d_1 + \cdots + c_m t \otimes d_m + (-u_1) t \otimes v_1 + \cdots + (-u_n) t \otimes v_n)(RtR) = 0$$

because $\alpha \in \mathrm{Hom}(Rt \otimes_{R^G} R_{R*G}, R * G_{R*G})$. Note that $\mathrm{Ann}_{R*G}(RtR) = 0$. So

$$\begin{aligned} 0 &= \alpha(c_1 t \otimes d_1 + \cdots + c_1 t \otimes d_m + (-u_1) t \otimes v_1 + \cdots + (-u_n) \otimes v_n) \\ &= f(c_1 t) d_1 + \cdots + f(c_m t) d_m + f(-u_1 t) v_1 + \cdots + f(-u_n t) v_n. \end{aligned}$$

Hence $f(c_1 t) d_1 + \cdots + f(c_m t) d_m = f(u_1 t) v_1 + \cdots + f(u_n t) v_n$. Therefore $\overline{f}$ is well-defined. Obviously $\overline{f}$ is additive. Also, for $r \in R$, $\overline{f}[(a_1 t b_1 + \cdots + a_k t b_k) r] = [\overline{f}(a_1 t b_1 + \cdots + a_k t b_k)] r$. Take $g \in G$. Then similarly as above

$$\begin{aligned} \overline{f}[(a_1 t b_1 + \cdots + a_k t b_k) g] &= \overline{f}(a_1 t b_1 g + \cdots + a_k t b_k g) = \overline{f}(a_1 t g b_1^g + \cdots + a_k t g b_k^g) \\ &= \overline{f}(a_1 t b_1^g + \cdots + a_k t b_k^g) = f(a_1 t) b_1^g + \cdots + f(a_k t) b_k^g. \end{aligned}$$

Also note that $f(a_1 t) = r_1 t, \ldots, f(a_k t) = r_k t$ for some $r_1, \ldots, r_k \in R$. Hence

$$\begin{aligned} [\overline{f}(a_1 t b_1 + \cdots + a_k t b_k)] g &= [f(a_1 t) b_1 + \cdots + f(a_k t) b_k] g = (r_1 t b_1 + \cdots + r_k t b_k) g \\ &= r_1 t b_1 g + \cdots + r_k t b_k g = r_1 t g b_1^g + \cdots + r_k t g b_k^g \\ &= r_1 t b_1^g + \cdots + r_k t b_k^g = f(a_1 t) b_1^g + \cdots + f(a_k t) b_k^g. \end{aligned}$$

Thus $\overline{f}[(a_1 t b_1 + \cdots + a_k t b_k) g] = [\overline{f}(a_1 t b_1 + \cdots + a_k t b_k)] g$. So $\overline{f} \in \mathrm{Hom}(RtR_{R*G}, R * G_{R*G})$. Since $R = Q(R)$, so is $R * G$ by Lemma 1.6. Also we see that RtR is a dense right ideal of $R * G$ because RtR is a two-sided ideal of $R * G$ with $\mathrm{Ann}_{R*G}(RtR) = 0$. Hence, from [35, Proposition 13.20, p.369], there exists $q \in Q(R * G) = R * G$ such that $\overline{f} = q_\ell|_{RtR}$, where q_ℓ is the left multiplication by q. Now

$$\theta(q)(rt) = qrt = q_\ell(rt) = \overline{f}(rt) = f(rt)$$

for $r \in R$. Thus $\theta(q) = f$, so θ is onto. Therefore θ is a ring isomorphism.

It is obvious that $\mathrm{Cen}(R)^G \subseteq \mathrm{Cen}(R^G)$. Let $a \in \mathrm{Cen}(R^G)$. Define $f_a\colon Rt \to Rt$ by $f_a(rt) = rat$ for $r \in R$. Note that for $b \in R^G$,

$$f_a(rtb) = f_a(rbt) = rbat = rabt = ratb = f_a(rt)b.$$

So $f_a \in \operatorname{Hom}(Rt_{R^G}, Rt_{R^G})$, since f_a is additive. To see $f_a \in \operatorname{Cen}(\operatorname{Hom}(Rt_{R^G}, Rt_{R^G}))$, take $f \in \operatorname{Hom}(Rt_{R^G}, Rt_{R^G})$. For $rt \in Rt$ with $r \in R$, let $f(rt) = s_r t$ with $s_r \in R$. Then $(f \circ f_a)(rt) = f(f_a(rt)) = f(rat) = f(rta) = f(rt)a = s_r at$. Also $(f_a \circ f)(rt) = f_a(f(rt)) = f_a(s_r t) = s_r at$. Thus $f \circ f_a = f_a \circ f$. Hence $f_a \in \operatorname{Cen}(\operatorname{Hom}(Rt_{R^G}, Rt_{R^G}))$. Since $\operatorname{Cen}(R * G) \cong \operatorname{Cen}(\operatorname{Hom}(Rt_{R^G}, Rt_{R^G}))$ via θ, there exists $q \in \operatorname{Cen}(R * G)$ such that $\theta(q) = f_a$. So $\theta(q)(rt) = f_a(rt)$, thus $qrt = rat$ for $r \in R$. By Lemma 1.5, $\operatorname{Cen}(R * G) = \operatorname{Cen}(R)^G$. Hence $q \in \operatorname{Cen}(R)^G$. Taking $r = 1$ in $qrt = rat$, we have that $qt = at$. Therefore $a = q$ because $q \in \operatorname{Cen}(R)^G \subseteq R$. Thus $a \in \operatorname{Cen}(R)^G$. Consequently, $\operatorname{Cen}(R^G) = \operatorname{Cen}(R)^G$. ■

Lemma 1.8 *Assume that R is a semiprime ring and $e \in \mathbf{B}(Q(R))$. Let I be a two-sided ideal of R such that $I_R \leq^{\text{ess}} eR_R$ and $\operatorname{Ann}_R(I) = fR$ with $f \in \mathbf{B}(R)$. Then $e = 1 - f$.*

Proof If $I = 0$, then we are done. So we may assume that $I \neq 0$. Note that $I \subseteq \operatorname{Ann}_R[\operatorname{Ann}_R(I)]$. Hence $I \subseteq (1 - f)R(1 - f)$. Now

$$\operatorname{Ann}_{(1-f)R(1-f)}(I) = \operatorname{Ann}_R(I) \cap (1 - f)R(1 - f) = fR \cap (1 - f)R = 0.$$

Since $1-f \neq 0$ and $(1-f)R(1-f)$ is a semiprime ring, $I_{(1-f)R(1-f)} \leq^{\text{ess}} (1-f)R(1-f)_{(1-f)R(1-f)}$. So we can see that $I_R \leq^{\text{ess}} (1-f)R_R$. Hence $I_R \leq^{\text{ess}} (1-f)Q(R)_R$. Also we have $I_R \leq^{\text{ess}} eQ(R)_R$. Therefore $I_R \leq^{\text{ess}} [(1-f)Q(R) \cap eQ(R)]_R = (1-f)eQ(R)_R$ since e and $1 - f$ are central in $Q(R)$. So $(1 - f)eQ(R)_R \leq^{\text{ess}} (1 - f)Q(R)_R$ and $(1 - f)eQ(R)_R \leq^{\text{ess}} eQ(R)_R$. By using the modular law, we get $(1 - f)Q(R) = (1 - f)eQ(R) = eQ(R)$. Thus $e = 1 - f \in R$. ■

Lemma 1.9 **([41, Theorem 2])** *Let R be a semiprime ring and G a finite group of X-outer ring automorphisms of R. Then we have the following.*

(i) *For $q \in Q(R^G)$, let J be a dense right ideal of R^G such that $qJ \subseteq R^G$. Then JR is a dense right ideal of R and the map $\widetilde{q}\colon JR \to R$ defined by $\widetilde{q}\,(\sum a_i r_i) = \sum q(a_i)r_i$, with $a_i \in J$ and $r_i \in R$, is a right R-homomorphism. Moreover $\widetilde{q} \in Q(R)^G$.*
(ii) *The map $\sigma\colon Q(R^G) \to Q(R)^G$ defined by $\sigma(q) = \widetilde{q}$ is a ring isomorphism.*
(iii) *For a right ideal I of R, I_R is dense in R_R if and only if $\operatorname{tr}(I)$ is a dense right ideal of R^G.*
(iv) *Let $\widetilde{q} \in Q(R)^G$ and let K be a dense right ideal of R such that $\widetilde{q}K \subseteq R$. Then $K \cap R^G$ is a dense right ideal of R^G and $\widetilde{q}|_{K\cap R^G}(K \cap R^G) \subseteq R^G$, where $\widetilde{q}|_{K\cap R^G}$ is the restriction of $\widetilde{q}$ to $K \cap R^G$. Thus $\widetilde{q}|_{K\cap R^G} \in Q(R^G)$.*

For a ring R with a group G of ring automorphisms of R, we say that a right ideal I of R is *G-invariant* if $I^g \subseteq I$ for every $g \in G$, where $I^g = \{a^g \mid a \in I\}$. Also, we say that R is *G-quasi-Baer* if the right annihilator of every G-invariant two-sided ideal is generated by an idempotent as a right ideal. The condition G-quasi-Baer is right-left symmetric. In fact, suppose that R is G-quasi-Baer. Say I is a G-invariant two-sided ideal of R. Then $\ell_R(I)$ is also a G-invariant two-sided ideal, thus $r_R(\ell_R(I)) = eR$ for some $e = e^2 \in R$. So $\ell_R(I) = \ell_R[r_R(\ell_R(I))] = \ell_R(eR) = R(1 - e)$. Obviously if R is quasi-Baer, then R is G-quasi-Baer. But in Example 2.1, there exists a ring and a finite group G of X-outer automorphisms of R such that R is G-quasi-Baer, but not quasi-Baer.

With these preparations, in spite of Example 1.1, we have the following result for the quasi-Baer condition of $R * G$ and R^G for the case when R is semiprime, and G is finite and X-outer.

Theorem 1.10 *Assume that R is a semiprime ring and G is a finite group of X-outer ring automorphisms of R. Then the following are equivalent.*

(i) *$R * G$ is quasi-Baer.*
(ii) *R is G-quasi-Baer.*
(iii) *R^G is quasi-Baer.*

Proof By Lemma 1.3(i), $R * G$ is semiprime. Note that $Q(R) * G$ is a maximal right ring of quotients of $R * G$ by Lemma 1.6.

(i) $\Rightarrow$ (ii): Suppose that $R * G$ is quasi-Baer. Let I be a G-invariant two-sided ideal of R. Then $I * G$ is a two-sided ideal of $R * G$. Since $R * G$ is semiprime quasi-Baer, there exists $e \in \mathbf{B}(R * G)$ such that

$$(I * G)_{R*G} \leq^{\text{ess}} e(R * G)_{R*G}$$

by [18, Theorem 4.7]. From Lemma 1.5, note that $e \in \operatorname{Cen}(R)^G$. First, we show that $I_R \leq^{\text{ess}} eR_R$. To see this, take $0 \neq er \in eR$ with $r \in R$. Since $(I*G)_{R*G} \leq^{\text{ess}} e(R*G)_{R*G}$, it follows that there exists $\beta \in R * G$ such that $0 \neq er\beta \in I * G$. Say

$$\beta = b_1g_1 + b_2g_2 + \cdots + b_ng_n$$

with $b_i \in R$ and $g_i \in G$ for $i = 1, \ldots, n$. Then

$$er\beta = (erb_1)g_1 + (erb_2)g_2 + \cdots + (erb_n)g_n \in I * G.$$

Hence there exists j such that $0 \neq erb_j \in I$. Thus $I_R \leq^{\text{ess}} eR_R$. Note that $e \in \operatorname{Cen}(R)^G$ and $e = e^2$, hence $I \subseteq eRe$. So we can see that $I_{eRe} \leq^{\text{ess}} eRe_{eRe}$.

Now we show that $\operatorname{Ann}_R(I) = (1 - e)R$. If $e = 0$, then $I = 0$. Thus $\operatorname{Ann}_R(I) = R$. So we may assume that $e \neq 0$. Note that eRe is semiprime. Thus $\operatorname{Ann}_{eRe}(I) = 0$ because $I_{eRe} \leq^{\text{ess}} eRe_{eRe}$. Hence $0 = eR \cap \operatorname{Ann}_R(I) = eRe \cap \operatorname{Ann}_R(I)$. Since $I \subseteq eR$, we have that $(1 - e)R \subseteq \operatorname{Ann}_R(I)$. From the modular law, it follows that $\operatorname{Ann}_R(I) = (1 - e)R \oplus (eR \cap \operatorname{Ann}_R(I))$. But since $eR \cap \operatorname{Ann}_R(I) = 0$, $\operatorname{Ann}_R(I) = (1 - e)R$. Therefore R is G-quasi-Baer.

(ii) $\Rightarrow$ (iii): Assume that R is G-quasi-Baer. By Lemma 1.3(i), since R^G is semiprime, it is enough to see that $\mathbf{B}(Q(R^G)) \subseteq R^G$ to prove that R^G is quasi-Baer by Lemma 1.4. Let $e \in \mathbf{B}(Q(R^G))$. Then $\widetilde{e} \in \mathbf{B}(Q(R)^G)$ since $Q(R^G) \cong Q(R)^G$ by Lemma 1.9(ii). Also $[\operatorname{Cen}(Q(R))]^G = \operatorname{Cen}(Q(R)^G)$ from Lemma 1.7. Thus

$$\widetilde{e} \in \mathbf{B}(Q(R)^G) \subseteq \operatorname{Cen}(Q(R)^G) = [\operatorname{Cen}(Q(R))]^G \subseteq \operatorname{Cen}(Q(R)).$$

Let $I = R \cap \widetilde{e}R$. Then $I_R \leq^{\text{ess}} \widetilde{e}R_R$ since $\widetilde{e} \in Q(R)$ and $R_R \leq^{\text{ess}} Q(R)_R$. If $I = 0$, then $\widetilde{e} = 0$, hence $\widetilde{e} \in R^G$. Thus we may assume that $I \neq 0$ or equivalently, $\widetilde{e} \neq 0$. Note that I is a G-invariant two-sided ideal of R. So there exists $f \in \mathcal{S}_\ell(R)$ such that $\operatorname{Ann}_R(I) = fR$. Since R is semiprime, $f \in \mathbf{B}(R)$ by [9]. Note that $\widetilde{e} \in \operatorname{Cen}(Q(R))$ and

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$\widetilde{e} = \widetilde{e}^2$, so $\widetilde{e} \in \mathbf{B}(Q(R))$. Thus from Lemma 1.8, $\widetilde{e} = 1 - f \in R$. Since $\widetilde{e} \in \mathbf{B}(Q(R)^G)$, we have that $\widetilde{e} \in R^G \subseteq R$. Therefore $\widetilde{e}R \subseteq R$. Hence $eR^G = e(R \cap R^G) \subseteq R^G$ by Lemma 1.9(iv), so $e \in R^G$. Hence $\mathbf{B}(Q(R^G)) \subseteq R^G$. Therefore R^G is a quasi-Baer ring by Lemma 1.4.

(iii)$\Rightarrow$(i): Assume that R^G is quasi-Baer. Let $e \in \mathbf{B}(Q(R)*G)$. Then by Lemma 1.5, $e \in [\mathbf{B}(Q(R)]^G$ since G is X-outer on $Q(R)$. Thus $R \cap eR$ is a G-invariant two-sided ideal of R. Therefore $\mathrm{Ann}_R(R \cap eR)$ is also a G-invariant two-sided ideal of A. So $\mathrm{tr}(R \cap eR) \subseteq R \cap eR$ and $\mathrm{tr}(\mathrm{Ann}_R(R \cap eR)) \subseteq \mathrm{Ann}_R(R \cap eR)$. Now $(R \cap eR) \oplus \mathrm{Ann}_R(R \cap eR)$ is a dense right ideal of R. We prove that $e[\mathrm{Ann}_R(R \cap eR)] = 0$. To see this, let $x \in \mathrm{Ann}_R(R \cap eR)$. Then $ex(R \cap eR) = 0$. If $ex \neq 0$, then there exists $b \in R$ such that $0 \neq exb \in R \cap eR$ because $(R \cap eR)_R \leq^{\mathrm{ess}} eR_R$. So

$$exb(R \cap eR) \subseteq ex(R \cap eR) = 0,$$

hence $0 \neq exb \in (R \cap eR) \cap \mathrm{Ann}_R(R \cap eR) = 0$, which is a contradiction. Therefore $e[\mathrm{Ann}_R(R \cap eR)] = 0$. Thus $e[(R \cap eR) \oplus \mathrm{Ann}_R(R \cap eR)] = e(R \cap eR) = R \cap eR \subseteq R$.

By Lemma 1.9(iii),

$$\mathrm{tr}[(R \cap eR) \oplus \mathrm{Ann}_R(R \cap eR)] = \mathrm{tr}(R \cap eR) \oplus \mathrm{tr}(\mathrm{Ann}_R(R \cap eR))$$

is a dense right ideal of R^G. Now let e_0 be the restriction of e to

$$\mathrm{tr}[(R \cap eR) \oplus \mathrm{Ann}_R(R \cap eR)] = \mathrm{tr}(R \cap eR) \oplus \mathrm{tr}(\mathrm{Ann}_R(R \cap eR)).$$

Then $e = \widetilde{e_0}$ and $e_0 \in \mathbf{B}(Q(R^G))$ by Lemma 1.9(ii) and (iv). Since $e \in [\mathbf{B}(Q(R)]^G$, it follows that $\mathrm{tr}(R \cap eR) \subseteq eR^G$. Thus $\mathrm{tr}(R \cap eR) \subseteq e_0R^G$.

We claim that $\mathrm{tr}(R \cap eR)_{R^G} \leq^{\mathrm{ess}} e_0R^G_{R^G}$. For this, take $0 \neq e_0a \in e_0R^G$ such that $a \in R^G$. Note that $R^G_{R^G} \leq^{\mathrm{ess}} Q(R^G)_{R^G}$. Thus there exists $c \in R^G$ such that $0 \neq e_0ac \in R^G$. Since $[\mathrm{tr}(R \cap eR) \oplus \mathrm{tr}(\mathrm{Ann}_R(R \cap eR))]_{R^G} \leq^{\mathrm{ess}} R^G_{R^G}$, there is $r \in R^G$ with $0 \neq e_0acr \in [\mathrm{tr}(R \cap eR) \oplus \mathrm{tr}(\mathrm{Ann}_R(R \cap eR))]$. By noting that

$$e_0[\mathrm{tr}(\mathrm{Ann}_R(R \cap eR))] = e[\mathrm{tr}(\mathrm{Ann}_R(R \cap eR))] \subseteq e[\mathrm{Ann}_R(R \cap eR)] = 0,$$

we have $0 \neq e_0acr \in \mathrm{tr}(R \cap eR)$ and $cr \in R^G$. Hence $\mathrm{tr}(R \cap eR)_{R^G} \leq^{\mathrm{ess}} e_0R^G_{R^G}$.

Note that R^G is semiprime from Lemma 1.3(i), $\mathrm{tr}(R \cap eR)$ is a two-sided ideal of R^G, and e_0 is a central idempotent in $Q(R^G)$. Since R^G is quasi-Baer, there exists $f \in \mathcal{S}_\ell(R^G)$ such that $\mathrm{Ann}_{R^G}(\mathrm{tr}(R \cap eR)) = fR^G$. From [9] $f \in \mathbf{B}(R^G)$ since R^G is semiprime. Thus from Lemma 1.8, $e_0 = 1 - f$ because $e_0 \in \mathbf{B}(Q(R^G))$. Thus $e_0 \in R^G$, so $e_0R^G \subseteq R^G$. By Lemma 1.9(i), $eR = \widetilde{e_0}(R^GR) = (e_0R^G)R \subseteq R^GR = R$, hence $e \in R$. So $\mathbf{B}(Q(R) * G) \subseteq R \subseteq R * G$. Note that $R * G$ is semiprime by Lemma 1.3(i). Therefore $R * G$ is quasi-Baer by Lemma 1.4. ∎

For a semiprime ring R and a group G of X-outer ring automorphisms of R, we note that G also acts as X-outer ring automorphisms on the semiprime ring $\widehat{Q}_{\mathfrak{qB}}(R)$. Thus we get the following corollary immediately from Theorem 1.10.

Corollary 1.11 *Let R be a semiprime ring and G a finite group of X-outer ring automorphisms of R. Then both $\widehat{Q}_{\mathfrak{qB}}(R) * G$ and $\widehat{Q}_{\mathfrak{qB}}(R)^G$ are quasi-Baer.*

A ring R is called *reduced* if R has no nonzero nilpotent element. By observing that a reduced quasi-Baer ring is Baer, we get the following immediately.

Corollary 1.12 *Let R be a reduced ring with a finite group G of X-outer ring automorphisms of R. Then R is G-quasi-Baer if and only if R^G is Baer.*

2 Examples

In this section, we provide various examples which illustrate and delimit the results in Section 1. From Lemma 1.3 and Theorem 1.10, we may raise the following question:

> Assume that a ring R is semiprime quasi-Baer and G is a finite group of ring automorphisms of R such that R has no nonzero $|G|$-torsion. Then is $R * G$ quasi-Baer?

The following example answers the question in the negative.

Example 2.1 For a commutative domain A with no nonzero 2-torsion, let $R = A \oplus A \oplus \mathbb{Z}$ and $g \in \operatorname{Aut}(R)$ defined by $g[(a, b, n)] = (b, a, n)$ for $a,\ b \in A$ and $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. Put $G = \{1,\ g\}$, $S = A \oplus A$, and $h = g|_S$. Then $h \in \operatorname{Aut}(S)$. Let $H = \{1, h\}$. In this case, $R * G \cong (S * H) \oplus \mathbb{Z}[G]$, where $\mathbb{Z}[G]$ is the group ring of G over $\mathbb{Z}$. Since H is X-outer on S, the ring $S * H$ is quasi-Baer by Theorem 1.10. Thus if $R * G$ is quasi-Baer, then $\mathbb{Z}[G]$ is quasi-Baer. But this is a contradiction by [19, Example 1.11]. Thus R is a semiprime quasi-Baer with no nonzero $|G|$-torsion; however the ring $R * G$ is not quasi-Baer.

In Example 2.2, there is a semiprime ring R with a finite group G of X-outer ring automorphisms such that R has no $|G|$-torsion and R is G-quasi-Baer; however R is not quasi-Baer. We see that, for a semiprime ring R, if G is a group of ring automorphisms of R, then it can be checked that G induces group actions on $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$. Thus we also have the skew group ring $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) * G$. If G is a finite group of X-outer ring automorphisms of R, then G is also X-outer on $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$. Thus $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) * G$ is semiprime and quasi-Baer by Lemma 1.3(i) and Theorem 1.10 because $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ is semiprime quasi-Baer. Hence by Lemma 1.4, $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R * G) \subseteq \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) * G$. Thus one might expect that $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R * G) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) * G$ when R is a semiprime ring and G is a finite group of X-outer ring automorphisms of R. But Example 2.2 eliminates the possibility. Moreover, we can see that $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R * G) \ncong \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) * G$. Therefore in Lemma 1.6, "$Q(\,\cdot\,)$" cannot be replaced by "$\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\,\cdot\,)$".

Example 2.2 Assume that A is a commutative domain which is not a field and A has no 2-torsion. Take a nonzero proper ideal I of A. Let $R = \{(a, b) \in A \oplus A \mid a - b \in I\}$, which is a subring of $A \oplus A$. Note that $Q(R) = K \oplus K$, where K is the field of fractions of A. Define $g \in \operatorname{Aut}(R)$ by $g[(a, b)] = (b, a)$ for $(a, b) \in R$. Let $G = \{1, g\}$, where $g^2 = 1$. By noting that $I \neq 0$ it can be checked that G is X-outer. Since R is semiprime, so is $R * G$ by Lemma 1.3(i). Now

$$\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R * G) = (R * G)\mathbf{B}(Q(R) * G) = (R * G)[\mathbf{B}(Q(R))]^G$$

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by Lemma 1.5 since G is X-outer on $Q(R)$. Note that $[\mathbf{B}(Q(R)]^G = \{(0,0), (1,1)\}$, hence we have that $\widehat{Q}_{\mathfrak{qB}}(R * G) = R * G$. Thus $R * G$ is quasi-Baer. In this case, R is not quasi-Baer. In fact, assume to the contrary that R is quasi-Baer. Since R is semicentral reduced, R is a prime ring by [12, Lemma 4.2], which is a contradiction. Thus R is G-quasi-Baer by Theorem 1.10, but R is not quasi-Baer.

Next note that $(1,0) \in \widehat{Q}_{\mathfrak{qB}}(R) = R\mathbf{B}(Q(R))$, but $(1,0) \notin R$. Therefore $\widehat{Q}_{\mathfrak{qB}}(R * G) = R * G \subsetneq \widehat{Q}_{\mathfrak{qB}}(R) * G$. Also, note that R has no nonzero $|G|$-torsion because A has no 2-torsion. Take $A = \mathbb{Z}$ and $I = 2\mathbb{Z}$. Then $R * G = \widehat{Q}_{\mathfrak{qB}}(R * G)$ has only trivial idempotents 0 and $1 = (1,1) + (0,0)g$. Now $\widehat{Q}_{\mathfrak{qB}}(R) = \mathbb{Z} \oplus \mathbb{Z}$, hence $\widehat{Q}_{\mathfrak{qB}}(R) * G$ has a nontrivial idempotent, for example $(0,1) \in \widehat{Q}_{\mathfrak{qB}}(R)$. Thus $\widehat{Q}_{\mathfrak{qB}}(R * G) \ncong \widehat{Q}_{\mathfrak{qB}}(R) * G$.

The following example shows that Theorem 1.10 does not hold if "quasi-Baer" is replaced by "Baer".

Example 2.3 Let A be a commutative domain in which 2 is not invertible and let $R = A[x,y]$, the ordinary polynomial ring. Define $g \in \mathrm{Aut}(R)$ by $g[a(x,y)] = a(y,x)$ for $a(x,y) \in R$. Then $g^2 = 1$. Let $G = \{1, g\}$. Let K be the field of fractions of A. Then note that $Q(R) = K(x,y)$, where $K(x,y)$ is the field of fractions of $K[x,y]$.

First we check that $\Phi_g = 0$, *i.e.,* g is X-outer. For this, take $\alpha(x,y) \in \Phi_g$. Then $\alpha(x,y)a(x,y)^g = a(x,y)\alpha(x,y)$ for every $a(x,y) \in R$. Thus $\alpha(x,y)x^g = x\alpha(x,y)$ and so $\alpha(x,y)y = x\alpha(x,y)$. Hence $\alpha(x,y)(y-x) = 0$. Therefore $\alpha(x,y) = 0$, *i.e.,* g is an X-outer ring automorphism of R. Thus the group G is X-outer, so $R * G$ is quasi-Baer by Theorem 1.10. Now we show that $R * G$ is not Baer. Say $e = a(x,y) + b(x,y)g \in R * G$ is an idempotent. Then

$$e = [a(x,y)+b(x,y)g]^2 = a(x,y)^2+b(x,y)b(y,x)+[a(x,y)b(x,y)+b(x,y)a(y,x)]g.$$

So we have that

$$a(x,y) = a(x,y)^2 + b(x,y)b(y,x),$$
$$b(x,y) = a(x,y)b(x,y) + a(y,x)b(x,y).$$

Thus, from the second equation, $b(x,y)[1 - a(x,y) - a(y,x)] = 0$, hence either $b(x,y) = 0$ or $1 = a(x,y) + a(y,x)$. Assume $1 = a(x,y) + a(y,x)$. Then $2a_0 = 1$, where $a_0 \in A$ is the constant term of $a(x,y)$. This is a contradiction. Hence $b(x,y) = 0$, so $e = a(x,y)$. Therefore $e = 0$ or $e = 1$. Consider the right annihilator $r_{R*G}(x + yg)$. If $R * G$ is Baer, then $r_{R*G}(x + yg) = 0$ or $r_{R*G}(x + yg) = R * G$. But since $x + yg \neq 0$, it follows that $r_{R*G}(x + yg) = 0$. Now note that $y + (-y)g \in r_{R*G}(x + yg)$, a contradiction. Therefore the skew group ring $R * G$ is not Baer.

Moreover, if $A = \mathbb{Z}$, then G is X-outer and R has no nonzero $|G|$-torsion. But the skew group ring $R * G$ is not Baer.

According to [24] and [32], a ring R is said to be *right extending* (or *right CS*) if for every right ideal I of R there exists $e = e^2 \in R$ such that $I_R \leq^{\text{ess}} eR_R$. Note that if R is a semiprime ring, then R is right FI-extending if and only if R is quasi-Baer by [18, Theorem 4.7]. Thus by Theorem 1.10, if a ring R is semiprime right FI-extending

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with a finite group G of X-outer automorphisms of R, then $R*G$ is right FI-extending. So we may raise the following question:

> Assume that R is a semiprime right extending ring and G is a finite group of X-outer ring automorphisms of R. Then is $R * G$ right extending?

The following example gives a negative answer to the question.

Example 2.4 Let $A = \mathbb{Z}$ and let $G = \{1, g\}$ as in Example 2.3. Then $R = \mathbb{Z}[x, y]$ is right extending, G is X-outer and moreover R has no nonzero $|G|$-torsion. We claim that the skew group ring $R * G$ is not right extending. Assume to the contrary that $R * G$ is right extending. Note that $R * G$ has only trivial idempotents as is shown in Example 2.3. Hence $R * G$ is right uniform. Now $Q(R) * G = \mathbb{Q}(x, y) * G$ is a maximal right ring of quotients of $R * G$ by Lemma 1.6, where $\mathbb{Q}$ is the field of rational numbers and $\mathbb{Q}(x, y)$ is the field of fractions of $\mathbb{Q}[x, y]$. Thus $\mathbb{Q}(x, y) * G$ is also right uniform. But $e = 1/2 + (1/2)g$ is a nontrivial idempotent in $\mathbb{Q}(x, y) * G$. This is a contradiction. Therefore $R * G$ is not right extending.

3 Applications

In this section, C^*-algebras are assumed to be nonunital unless indicated otherwise. We apply our results from Section 1 to quasi-AW^*-algebras (for example, the local multiplier C^*-algebra $M_{\mathrm{loc}}(A)$ of C^*-algebra A). Also as applications, for a unital C^*-algebra A and a finite group G of X-outer $*$-automorphisms of A, the relationship between $|\operatorname{Minspec}(A)|$, $|\operatorname{Minspec}(A * G)|$, and $|\operatorname{Minspec}(A^G)|$ is investigated. Using this relationship, we study the triangulating dimension of $A * G$ and A^G for a unital C^*-algebra A which gives answers to Open Problems (3) and (4) in [19] for certain skew group ring extensions. As a byproduct we obtain that if a quasi-AW^*-algebra A is a direct sum of n prime C^*-algebras, then for a finite group G of X-outer $*$-automorphisms of A there exists k with $k \le n$ such that both $A * G$ and A^G are direct sums of k prime C^*-algebras.

If A is a C^*-algebra, then the set $\mathcal{F}$ of all norm closed essential two-sided ideals forms a filter directed downward by inclusion. The ring $Q_b(A)$ denotes the algebraic inductive limit of $\{M(I)\}_{I\in\mathcal{F}}$, where $M(I)$ is the C^*-algebra multipliers of I. In [3], the ring $Q_b(A)$ is called the *symmetric normed algebra of quotients* of A. The norm completion of $Q_b(A)$, *i.e.*, the C^*-algebra inductive limit $M_{\mathrm{loc}}(A)$ of $\{M(I)\}_{I\in\mathcal{F}}$, is called the *local multiplier algebra* of A which was used to solve operator equations on A (see [28] and [42]). In [2–5], $Q_b(A)$ and $M_{\mathrm{loc}}(A)$ of a C^*-algebra A have been extensively studied. For more details on local multiplier algebras, see [6].

According to [6, Definition 3.2.1, p.73], for a C^*-algebra A, the C^*-subalgebra $\overline{AC_b(A)}$ (the norm closure of $AC_b(A)$ in $M_{\mathrm{loc}}(A)$) of $M_{\mathrm{loc}}(A)$ is called the *bounded central closure* of A and denoted by ${}^cA$, where $C_b(A)$ is $\operatorname{Cen}(Q_b(A))$. If $A = {}^cA$, then A is called *boundedly centrally closed*. It was shown in [6, Theorem 3.2.8 and Corollary 3.2.9, pp.75-76] that the local multiplier algebra and the bounded central closure of a C^*-algebra are boundedly centrally closed.

Recall from [34] that an AW^*-algebra is a C^*-algebra which is also a Baer $*$-ring. From [13, Proposition 1.5] and [34, p.10], a reduced quasi-AW^*-algebra is a commutative AW^*-algebra. In [21], it is proved that the center of a quasi-AW^*-algebra is an AW^*-algebra.

By [6] a $*$-preserving algebra automorphism of a C^*-algebra is called a $*$-*automorphism*. When A is a C^*-algebra with a finite group G of X-outer $*$-automorphisms of A, it was shown in [6, Section 4.4, pp.139–141] that $A * G$ and A^G are C^*-algebras.

For a C^*-algebra A, let $A^1 = \{a + \lambda 1_{M(A)} \mid a \in A \text{ and } \lambda \in \mathbb{C}\}$, where $M(A)$ is the multiplier algebra of A, $1_{M(A)}$ is the identity of $M(A)$, and $\mathbb{C}$ is the field of complex numbers. Note that quasi-AW^*-algebras are unital. For the class of quasi-AW^*-algebras, we have the following.

Theorem 3.1 **([21, Theorem 4.15 and Corollary 4.17])** *Let A be a C^*-algebra and n a positive integer. Then the following are equivalent.*

(i) $|\operatorname{Minspec}(A)| = n$.
(ii) *The extended centroid of A is isomorphic to $\mathbb{C}^n$.*
(iii) $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(A^1)$ *is a direct sum of n prime C^*-algebras.*
(iv) $Q(A)$ *is a direct product n prime rings.*
(v) *Some boundedly centrally closed intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$ is a direct sum of n prime C^*-algebras.*
(vi) *Every boundedly centrally closed intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$ is a direct sum of n prime C^*-algebras.*

In this case, $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(A^1)$ is a quasi-AW^*-algebra.

Proposition 3.2 **([21, Lemma 4.12(i)])** *A unital C^*-algebra is boundedly centrally closed if and only if it is quasi-AW^*-algebra. In particular, the local multiplier algebra $M_{\mathrm{loc}}(A)$ of a C^*-algebra A is a quasi-AW^*-algebra* [4, Lemma 3].

Noting that C^*-algebras are semiprime, we have the following immediately from Theorem 1.10 and [6, Section 4.4, pp. 139–141].

Theorem 3.3 *Assume that A is a unital C^*-algebra and G is a finite group of X-outer $*$-automorphisms of A. Then the following are equivalent.*

(i) *$A * G$ is a quasi-AW^*-algebra.*
(ii) *A is G-quasi-Baer.*
(iii) *A^G is a quasi-AW^*-algebra.*

Observe that if G is a finite group of X-outer $*$-automorphisms of A, then G also acts as X-outer $*$-automorphisms on $Q_b(A)$. It is shown in [21, Lemma 4.9] that $Q_b(A)$ is quasi-Baer. The following corollary follows immediately from Theorems 1.10, 3.1, and 3.3 and Proposition 3.2.

Corollary 3.4 (i) *Assume that A is a unital C^*-algebra and G is a finite group of X-outer $*$-automorphisms of A. Then $Q_b(A) * G$ and $Q_b(A)^G$ are quasi-Baer.*

(ii) *If G is a finite group of X-outer $*$-automorphisms of the local multiplier algebra $M_{\mathrm{loc}}(A)$ of a C^*-algebra A, then $M_{\mathrm{loc}}(A) * G$ and $M_{\mathrm{loc}}(A)^G$ are quasi-AW^*-algebras.*

(iii) *If A is a unital C^*-algebra with only finitely many minimal prime ideals and G is a finite group of X-outer $*$-automorphisms of A, then $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(A) * G$ is a quasi-AW^*-algebra.*

Theorem 3.5 *Assume that A is a unital C^*-algebra and G is a finite group of X-outer $*$-automorphisms of A. Then the following are equivalent.*

(i) $|\operatorname{Minspec}(A)| < \infty$.
(ii) $|\operatorname{Minspec}(A^G)| < \infty$.
(iii) $|\operatorname{Minspec}(A * G)| < \infty$.

In this case, $|\operatorname{Minspec}(A * G)| \leq |\operatorname{Minspec}(A)| \leq |\operatorname{Minspec}(A * G)| \cdot |G|$ *and* $|\operatorname{Minspec}(A * G)| = |\operatorname{Minspec}(A^G)|$.

Proof Since G is X-outer on $Q(A)$, we have that

$$\begin{aligned}\operatorname{Cen}(Q(A * G)) &\cong \operatorname{Cen}(Q(A) * G) = [\operatorname{Cen}(Q(A))]^G \\ &= \operatorname{Cen}(Q(A)^G) \cong \operatorname{Cen}(Q(A^G))\end{aligned}$$

by Lemmas 1.5, 1.6, 1.7, and 1.9(ii).

(i)⇒(ii), (i)⇒(iii): Suppose that $|\operatorname{Minspec}(A)| = n < \infty$. Then $\operatorname{Cen}(Q(A)) \cong \mathbb{C}^n$ by Theorem 3.1 because $\operatorname{Cen}(Q(A)) = \operatorname{Cen}(Q^m(A))$. Hence $\operatorname{u.dim}[\operatorname{Cen}(Q(A))] = n$. Note that G induces a group H of ring automorphisms of $\operatorname{Cen}(Q(A))$ and H is an epimorphic image of G. In this case $|H|$ is invertible in $\operatorname{Cen}(Q(A))$. Thus $\operatorname{Cen}(Q(A)) * H$ is semiprime by Lemma 1.3(ii). Hence by [39, Theorem 2.1 and Proposition 2.2(1)],

$$\operatorname{u.dim}[\operatorname{Cen}(Q(A))]^H \leq \operatorname{u.dim}[\operatorname{Cen}(Q(A))],$$

where $\operatorname{u.dim}(-)$ is the right uniform dimension of a ring. Let

$$k = \operatorname{u.dim}[\operatorname{Cen}(Q(A))]^H.$$

Then $k \leq n$.

By noting that $[\operatorname{Cen}(Q(A))]^H = [\operatorname{Cen}(Q(A))]^G$, we have that $[\operatorname{Cen}(Q(A))]^H = \operatorname{Cen}(Q(A^G))$ by Lemma 1.7. Also note that by Lemma 1.3(ii) A^G is semiprime. Thus $\operatorname{Cen}(Q(A^G))$ is von Neumann regular by [1, Theorem 5]. Thus $\operatorname{Cen}(Q(A^G))$ is a direct sum of k fields because $\operatorname{u.dim}[\operatorname{Cen}(Q(A^G))] = k < \infty$. Therefore $|\operatorname{Minspec}(A^G)| = k$ by [1, Theorem 10]. Since A^G is a C^*-algebra from [6, Section 4.4, pp. 139–141], $\operatorname{Cen}(Q(A^G)) \cong \mathbb{C}^k$ from Theorem 3.1. Hence $\operatorname{Cen}(Q(A * G)) \cong \operatorname{Cen}(Q(A^G)) \cong \mathbb{C}^k$. Since $A * G$ is a C^*-algebra by [6, Section 4.4, pp. 139–141], $|\operatorname{Minspec}(A * G)| = |\operatorname{Minspec}(A^G)| = k \leq n$ by Theorem 3.1.

(ii)⇔(iii) It follows immediately from the fact that $\operatorname{Cen}(Q(A * G)) \cong \operatorname{Cen}(Q(A^G))$ and Theorem 3.1 because A^G and $A * G$ are C^*-algebras from [6, Section 4.4, pp. 139–141].

(ii)⇒(i): Let $|\operatorname{Minspec}(A^G)| = k < \infty$. Since A^G is a C^*-algebra from [6, Section 4.4, pp. 139–141], $\operatorname{Cen}(Q(A^G)) \cong \mathbb{C}^k$ by Theorem 3.1. Note that, as above, G induces a group H of ring automorphisms of $\mathrm{Cen}(Q(A))$ induced by G and H is an epimorphic image of G. Note that $[\mathrm{Cen}(Q(A))]^H = [\mathrm{Cen}(Q(A))]^G \cong \mathrm{Cen}(Q(A^G)) \cong \mathbb{C}^k$, so $\mathrm{u.dim}[\mathrm{Cen}(Q(A))]^H = k$. Since $|H|$ is invertible, $\mathrm{Cen}(Q(A)) * H$ is semiprime by Lemma 1.3(ii). Hence

$$\mathrm{u.dim}[\mathrm{Cen}(Q(A))]^H \leq \mathrm{u.dim}[\mathrm{Cen}(Q(A))] \leq \mathrm{u.dim}[\mathrm{Cen}(Q(A)]^H \cdot |H|$$

from [39, Theorem 2.1 and Proposition 2.2(1)]. Now

$$\mathrm{u.dim}[\mathrm{Cen}(Q(A)]^G \leq \mathrm{u.dim}[\mathrm{Cen}(Q(A))] \leq \mathrm{u.dim}[\mathrm{Cen}(Q(A)]^G \cdot |G|$$

because $|H| \leq |G|$. Note that $\mathrm{u.dim}[\mathrm{Cen}(Q(A)]^G = k$. Thus

$$n := \mathrm{u.dim}[\mathrm{Cen}(Q(A))] \leq k \cdot |G|.$$

Since $\mathrm{Cen}(Q(A))$ is von Neumann regular by [1, Theorem 5], $\mathrm{Cen}(Q(A))$ is a finite direct sum of n fields. Therefore $|\mathrm{Minspec}(A)| = n$ by [1, Theorem 10]. ■

Recall from [12] that an ordered set $\{b_1, \dots, b_n\}$ of nonzero distinct idempotents in a ring R is called a set of *left triangulating idempotents* of R if all the following hold:

(i) $1 = b_1 + \cdots + b_n$;
(ii) $b_1 \in \mathcal{S}_\ell(R)$;
(iii) $b_{k+1} \in \mathcal{S}_\ell(c_k R c_k)$, where $c_k = 1 - (b_1 + \cdots + b_k)$, for $1 \leq k \leq n-1$.

Similarly we define a *set of right triangulating idempotents* of R using (i), $b_1 \in \mathcal{S}_r(R)$, and $b_{k+1} \in \mathcal{S}_r(c_k R c_k)$. From part (iii) of the above definition, a set of left (right) triangulating idempotents is a set of pairwise orthogonal idempotents. A set $\{b_1, \dots, b_n\}$ of left (right) triangulating idempotents is said to be *complete* if each b_i is also semicentral reduced.

Observe from [12, Corollary 1.7 and Theorem 2.10] that the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set) and this is also the number of elements in any complete set of right triangulating idempotents of R. This motivates the following definition: R has *triangulating dimension* n, written $\mathrm{Tdim}(R) = n$, if R has a complete set of left triangulating idempotents with exactly n elements. Note that R is semicentral reduced if and only if $\mathrm{Tdim}(R) = 1$. If R has no complete set of left triangulating idempotents, then we say R has *infinite triangulating dimension*, denoted $\mathrm{Tdim}(R) = \infty$. In [12, Theorem 4.4], a structure theorem for a quasi-Baer ring with finite triangulating dimension is given. Also in [19, Theorem 3.4], for a quasi-Baer ring R, it is shown that $\mathrm{Tdim}(R) = n < \infty$ if and only if R has exactly n minimal prime ideals. Also in [19] the equality of triangulating dimension of a ring R and its various ring extensions of R has been investigated.

The following theorem gives answers to Open Problems (3) and (4) in [19] for triangulating dimension of certain skew group ring extensions.

Theorem 3.6 *Let A be a quasi-AW^*-algebra and G a finite group of X-outer $*$-automorphisms of A. Then we have the following.*

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(i) $\mathrm{Tdim}(A * G) = \mathrm{Tdim}(A^G)$.
(ii) $\mathrm{Tdim}(A * G) \leq \mathrm{Tdim}(A) \leq \mathrm{Tdim}(A * G) \cdot |G|$.
(iii) *If* $\mathrm{Tdim}(A) = n < \infty$, *then there exists a positive integer* $k \leq n$ *such that both* $A * G$ *and* A^G *are direct sums of* k *prime* C^**-algebras.*

Proof (i) From Theorem 3.3, $A*G$ and A^G are quasi-AW^*-algebras. If $\mathrm{Tdim}(A*G) = n < \infty$, then by [19, Theorem 3.4] and Theorem 3.5, $n = |\operatorname{Minspec}(A * G)| = |\operatorname{Minspec}(A^G)| = \mathrm{Tdim}(A^G)$. Next if $\mathrm{Tdim}(A*G) = \infty$, then $|\operatorname{Minspec}(A*G)| = \infty$ by [19, Theorem 3.4], since $A * G$ is a quasi-AW^*-algebra. Thus from Theorem 3.5, $|\operatorname{Minspec}(A^G)| = \infty$. Hence $\mathrm{Tdim}(A^G) = \infty$ by [19, Theorem 3.4] because A^G is a quasi-AW^*-algebra. Consequently, $\mathrm{Tdim}(A * G) = \mathrm{Tdim}(A^G)$.

(ii) From Theorem 3.5 and [19, Theorem 3.4], if one of $\mathrm{Tdim}(A)$, $\mathrm{Tdim}(A * G)$, and $\mathrm{Tdim}(A^G)$ is finite, then all are finite and also $\mathrm{Tdim}(A * G) \leq \mathrm{Tdim}(A) \leq \mathrm{Tdim}(A * G) \cdot |G|$. Next, if one of $\mathrm{Tdim}(A)$, $\mathrm{Tdim}(A * G)$, and $\mathrm{Tdim}(A^G)$ is infinite, then we are also done by Theorem 3.5 and [19, Theorem 3.4].

(iii) Suppose that $\mathrm{Tdim}(A) = n < \infty$ (note that by [19, Theorem 3.4] and Theorem 3.1, it is equivalent to the fact that A is a direct sum of n prime C^*-algebras). Then by Theorem 3.3, Theorem 3.5, and [19, Theorem 3.4], $\mathrm{Tdim}(A * G) = \mathrm{Tdim}(A^G) = k \leq n$ for some k. Therefore $A * G$ and A^G are direct sums of k prime C^*-algebras by Theorem 3.3 and [12, Theorem 4.4]. ∎

In the following example, there exist a quasi-AW^*-algebra A and a finite group G of X-outer $*$-automorphisms of A such that $\mathrm{Tdim}(A * G) \lneq \mathrm{Tdim}(A)$.

Example 3.7 Let $A = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (n-times with $n \geq 2$) and $*$ be the componentwise conjugate involution. Define $g \in \mathrm{Aut}(A)$ such that $g[(a_1, a_2, \ldots, a_n)] = (a_2, a_3, \ldots, a_n, a_1)$ for $(a_1, a_2, \ldots, a_n) \in A$. Then g is an X-outer $*$-automorphism and $g^n = 1$. Let G be the cyclic group generated by g. Then G is X-outer. By Lemma 1.3(i), A^G is semiprime. Thus $\mathcal{S}_\ell(A^G) = \mathbf{B}(A^G)$ by [9]. Now $\mathbf{B}(A^G) = \{0, 1\}$. Therefore A^G is semicentral reduced. Hence by Theorem3.6, $\mathrm{Tdim}(A * G) = \mathrm{Tdim}(A^G) = 1$, but $\mathrm{Tdim}(A) = n \geq 2$.

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