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ENDOMORPHISM RINGS AND GABRIEL TOPOLOGIES

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Introduction. A basic tool in the usual presentation of the Morita theorems is the correspondence theorem for projective modules. Let ${}_RM$ be a left R-module and $B = \mathrm{Hom}_R\,(M, M)$. When M is a progenerator, there is a close connection (in fact a lattice isomorphism) between left R-submodules of M and left ideals of B, which can be applied to the solution of problems such as characterizing when the endomorphism ring of a finitely generated projective faithful module is simple or right Noetherian. More generally, Faith proved that this connection can be retained in suitably modified form when M is just a generator in R-mod ([**4**], [**2**], [**3**]). In this form the correspondence theorem can be applied to show, e.g., that, when ${}_RM$ is a generator, then (a): ${}_RM$ is finite-dimensional if and only if B is a left finite-dimensional ring and in this case $d({}_RM) = d({}_BB)$, and (b): If ${}_RM$ is nonsingular then B is a left nonsingular ring ([**6**]).

Notation. If U is a submodule of M, let

$$I_B(U) = \{b \in B : Mb \subseteq U\}.$$

A generator is, in particular, a self-generator, in the sense that, for any submodule, U, of M, we have $MI_B(U) = U$. Assume next that M is a module with a closure operator, $\varphi(U) = U^c$, defined on its lattice, $\mathscr{L}(M)$, of submodules, and that M, while not necessarily a generator, is a c-self-generator, in the sense that, for any c-closed submodule, U, we have

$$[MI_B(U)]^c = U.$$

Then, it is shown in [**7**] that, for certain closure operators and under suitable conditions on M, there is a lattice isomorphism between the lattice $\mathscr{L}^c(M)$ of c-closed submodules of M and the lattice of annihilator left ideals of B. This lattice isomorphism can be used to find out, for example, what conditions on M are necessary and sufficient to ensure that B is a Baer ring or a left Utumi ring ([**7**]).

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In the preceding examples, properties of M are deduced from properties of B or, conversely, by making use of some lattice isomorphism or correspondence theorem between a lattice of c-closed submodules of M and a lattice of c'-closed left ideals of B, where c and c' are specific closure operators. One way of treating such questions in a more general framework is by using kernel functors or Gabriel topologies, which have naturally associated closure operators, in combination with Morita contexts, which are very useful in manipulating endomorphism rings of modules.

Thus, suppose σ is a kernel functor on the category of all left R-modules, ${}_R\mathfrak{M}$, and let (R, M, M^*, B) be the standard Morita context for M; let $\mathscr{L}^\sigma(M)$ be the lattice of σ-closed submodules of M. Then two key questions arise naturally in this situation:

First, is there, on the category ${}_B\mathfrak{M}$ of left B-modules, a kernel functor, $\bar{\sigma}$, which is associated with, or most natural for, the kernel functor σ on ${}_R\mathfrak{M}$?

And, secondly, given σ on ${}_R\mathfrak{M}$ and $\bar{\sigma}$ on ${}_B\mathfrak{M}$, is there a lattice isomorphism between $\mathscr{L}^\sigma(M)$ and $\mathscr{L}^{\bar{\sigma}}(B)$ which will enable us to link together properties of M and B?

With regard to the first question, it is shown in [**8**] that, if σ is an idempotent kernel functor on ${}_R\mathfrak{M}$ such that $(M, M^*) \subset {}^\sigma R$ and if the standard Morita context for M satisfies the non-degeneracy condition: $[M^*, m] \neq 0$ whenever $0 \neq m \in M$, then there is an idempotent kernel functor $\bar{\sigma}$ on ${}_B\mathfrak{M}$ which is naturally derived from σ. After making the necessary definitions in Section 2, we shall see in Section 3 that, if ${}_RM$ is a generator, then the above conditions are satisfied for any kernel functor, and we shall give examples of σ and M satisfying these conditions, with M not necessarily a generator. Thus, given σ on ${}_R\mathfrak{M}$ and $M \in {}_R\mathfrak{M}$ satisfying the above hypothesis, we have a naturally derived $\bar{\sigma}$ on ${}_B\mathfrak{M}$, and we show, in Theorem 3.3, that there is a lattice isomorphism between $\mathscr{L}^\sigma(M)$ and $\mathscr{L}^{\bar{\sigma}}(B)$. Some related results occur in [**9**], where it is shown that the lattices of closed submodules are isomorphic whenever the kernel functor is the one determined by the trace ideal; the method of proof there differs from the one in this article, however, the emphasis there being on the equivalence between certain quotient categories. As an application in Section 4, we consider the important special case of the Goldie kernel functor, and in Theorem 4.3, we use the lattice isomorphism to show that:

(a) If M is nonsingular, then B is a left nonsingular ring with maximal left quotient ring $Q^l_{\max} \cong \tilde{B} \cong \mathrm{Hom}_R(\tilde{M}, \tilde{M})$, where $M(B)$ is the injective hull of $M(B)$;

(b) B is a left CS ring if and only if M is a CS module (i.e., every complement submodule of M is a direct summand of M, cf. [**1**]);

(c) $d({}_RM) < \infty$ if and only if $d({}_BB) < \infty$ (here d indicates Goldie dimension), and in this case $d({}_RM) = d({}_BB)$ and $\widetilde{B}$ is semisimple artinian.

2. Definitions and preliminaries. Throughout this paper, R will denote an associative ring with identity, ${}_R\mathfrak{M}$ the category of left R-modules and σ a *kernel functor* on ${}_R\mathfrak{M}$, that is, a functor $\sigma:{}_R\mathfrak{M} \to {}_R\mathfrak{M}$ satisfying:

K1. For each left R-module M, $\sigma(M)$ is a submodule of M.
K2. If M, M' are left R-modules and $f \in \mathrm{Hom}_R(M, M')$, then

$$[\sigma(M)]f \subseteq \sigma(M')$$

and $\sigma(f)$ is the restriction of f to $\sigma(M)$.
K3. If M, M' are left R-modules such that $M \subseteq M'$, then

$$\sigma(M) = M \cap \sigma(M').$$

A kernel functor σ on ${}_R\mathfrak{M}$ is said to be *idempotent* if it satisfies the additional property:

K4. $\sigma(M/\sigma(M)) = 0$, for each left R-module M.

Assume henceforth that σ is an idempotent kernel functor on ${}_R\mathfrak{M}$.

A left R-module M is said to be σ*-torsion free* if $\sigma(M) = 0$ and σ*-torsion* if $\sigma(M) = M$. An R-submodule N of M is said to be a σ*-submodule* of M or σ*-dense* in M, written $N \subset {}^{\sigma}M$, if M/N is σ-torsion; N is σ*-closed* in M if M/N is σ-torsion free. The σ-closed submodule, N^{σ}, of M defined by $N^{\sigma}/N = \sigma(M/N)$ is called the σ*-closure* of N. The mapping $N \to N^{\sigma}$ is a *closure operator* on the lattice $\mathscr{L}(M)$ of submodules of M, i.e., it satisfies:

C1. $N \subseteq L$ implies $N^{\sigma} \subseteq L^{\sigma}$ for $N, L \in \mathscr{L}(M)$.
C2. $N \subseteq N^{\sigma}$.
C3. $(N^{\sigma})^{\sigma} = N^{\sigma}$.

The class of σ-torsion modules is closed under submodules, homomorphisms, extension and direct sums, while the class of σ-torsion free modules is closed under isomorphism, submodules, extension, injective hulls and direct products.

Notation. For subsets N and K of a module M, we set

$$(N:K) = \{r \in R:rK \subseteq N\}.$$

A nonempty set $\mathfrak{F}$ of left ideals of R is called a *Gabriel topology* on R if $0 \notin \mathfrak{F}$ and it satisfies:

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T1. If $I \in \mathfrak{J}$ and J is a left ideal of R such that $I \subseteq J$, then $J \in \mathfrak{J}$.
T2. If $I, J \in \mathfrak{J}$, then $I \cap J \in \mathfrak{J}$.
T3. If $I \in \mathfrak{J}$ and $r \in R$, then $(I:r) \in \mathfrak{J}$.
T4. If I is a left ideal of R and there exists $J \in \mathfrak{J}$ such that $(I:r) \in \mathfrak{J}$ for every $r \in J$, then $I \in \mathfrak{J}$.

For a given idempotent kernel functor σ on ${}_R\mathfrak{M}$, let $\mathfrak{J}_\sigma = \{I:I$ is a left ideal of R such that $I \subset {}^\sigma R\}$. Then it is easy to prove that $\mathfrak{J}_\sigma$ is a Gabriel topology on R.

Conversely, if $\mathfrak{J}$ is a Gabriel topology on R, then the functor $\sigma_{\mathfrak{J}}:{}_R\mathfrak{M} \to {}_R\mathfrak{M}$ given by

$$\sigma_{\mathfrak{J}}(M) = \{m \in M:Im = 0 \text{ for some } I \in \mathfrak{J}\},$$

for each left R-module M, is an idempotent kernel functor on ${}_R\mathfrak{M}$. Moreover, the map $F:\sigma \mapsto \mathfrak{J}_\sigma$ is a one-to-one correspondence between the class of all idempotent kernel functors on ${}_R\mathfrak{M}$ and the class of all Gabriel topologies on R with $G:\mathfrak{J} \mapsto \sigma_{\mathfrak{J}}$ as its inverse map.

If σ is an idempotent kernel functor and $\mathfrak{J}_\sigma$ is its corresponding Gabriel topology, then the closure, N^σ, of a submodule N of $M \in {}_R\mathfrak{M}$ is given by

$$N^\sigma = \{m \in M:(N:m) \in \mathfrak{J}_\sigma\}.$$

Given a Gabriel topology $\mathfrak{J}$, a module M is $\mathfrak{J}$-*injective* if every $f \in \mathrm{Hom}_R(I, M)$ with $I \in \mathfrak{J}$ can be extended to an element of $\mathrm{Hom}_R(R, M)$; equivalently, $\sigma_{\mathfrak{J}}(\tilde{M}/M) = 0$, where $\tilde{M}$ denotes the R-injective hull of M. Every module has a $\mathfrak{J}$-*injective hull*, $\mathfrak{J}(M)$, obtained as

$$\mathfrak{J}(M)/M = \sigma_{\mathfrak{J}}(\tilde{M}/M);$$

note that M is $\sigma_{\mathfrak{J}}$-dense in $\mathfrak{J}(M)$, and $\mathfrak{J}(M)$ is $\sigma_{\mathfrak{J}}$-closed in $\tilde{M}$. The *module of quotients*, $M_{\mathfrak{J}}$, of M with respect to $\mathfrak{J}$ is defined as $\mathfrak{J}(M/\sigma_{\mathfrak{J}}(M))$; $M_{\mathfrak{J}}$ is $\sigma_{\mathfrak{J}}$-torsion free and $\mathfrak{J}$-injective. $R_{\mathfrak{J}} = \mathfrak{J}(R/\sigma_{\mathfrak{J}}(R))$ forms a ring called the *ring of quotients* of R with respect to $\mathfrak{J}$.

For more details on Gabriel topologies and torsion theories the reader is referred to [**10**].

A quadruple (R, M, N, S), where R and S are rings and ${}_RM_S$ and ${}_SN_R$ are bimodules, is called a *Morita context* if there exists an R-R module homomorphism $(,):M \otimes_S N \to R$ and an S-S module homomorphism $[,]:N \otimes_R M \to S$ such that

$$m_1[n_1, m_2] = (m_1, n_1)m_2$$

and

$$n_1(m_1, n_2) = [n_1, m_1]n_2$$

for all $m_i \in M$, $n_i \in N$, $i = 1, 2$.

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If M is a left R-module, $M^* = \mathrm{Hom}_R(M, R)$ and $B = \mathrm{Hom}_R(M, M)$, then (R, M, M^*, B) is a Morita context with $(m, f) = mf$ for all $m \in M$ and $f \in M^*$, and $[f, m]$ is defined by $m_1[f, m] = (m_1, f)m$ for all $m, m_1 \in M, f \in M^*$; it is called the *standard Morita context for* ${}_RM$.

3. The lattice isomorphism. It is shown in [**8**] that, if M is a left R-module such that the standard context (R, M, M^*, B) satisfies the nondegeneracy condition:

ND. $[M^*, m] \neq 0$ whenever $0 \neq m \in M$;

and if σ is an idempotent kernel functor on ${}_R\mathfrak{M}$ such that $\sigma(M) = 0$ and $(M, M^*) \in \mathfrak{J}_\sigma$, then there is an idempotent kernel functor $\bar{\sigma}$ on ${}_B\mathfrak{M}$ which is naturally derived from σ, namely the kernel functor corresponding to the topology $\bar{\mathfrak{J}}$ defined as follows:

$$\bar{\mathfrak{J}} = \{I{:}I \text{ is a left ideal of } B \text{ and } I \supseteq [M^*, K], \text{ where } K \subset {}^\sigma M\}.$$

For easy reference, we denote the above hypotheses on M and its standard Morita context by (H). We assume henceforth, unless otherwise indicated, that (R, M, M^*, B) and σ satisfy hypothesis (H), with $\bar{\sigma}$ and $\bar{\mathfrak{J}}$ defined as above.

3.1 *Examples.* Hypothesis (H) is satisfied in case ${}_BM$ is a generator, for, in that case, the trace ideal, (M, M^*), equals R and hence $(M, M^*) \in \mathfrak{J}_\sigma$ for every kernel functor on ${}_R\mathfrak{M}$, and also: $[M^*, m] = 0$ for $m \in M$ implies

$$Rm = (M, M^*)m = M[M^*, m] = 0,$$

so that $m = 0$, which shows that ND holds.

However, ${}_RM$ need not be a generator for hypothesis (H) to hold. For example, let ${}_RM$ be a torsionless, faithful R-module, where R is a semiprime ring, and consider the standard context (R, M, M^*, B). Then the nondegeneracy condition holds (cf. e.g. [**8**], Proposition 6). Let σ be the Goldie kernel functor, so that $\sigma(M) = 0$ means M is nonsingular. Then it is not difficult to see that (M, M^*) is essential in ${}_RR$, so that $(M, M^*) \in \mathfrak{J}_\sigma$ since σ is the Goldie kernel functor; for, let $0 \neq a \in R$, we will show that

$$Ra \cap (M, M^*) \neq 0.$$

Since ${}_RM$ is faithful, there exists a nonzero $m_0 \in M$ such that $am_0 \neq 0$. Let $0 \neq f \in M^*$ be such that $[f, am_0] \neq 0$; f exists by ND. There is $m_1 \in M$ with

$$m_1[f, am_0] \neq 0,$$

so we have $0 \neq (m_1, f)am_0$ and, in particular,

$$0 \neq (m_1, f)a = (m_1, fa) \in (M, M^*) \cap Ra.$$

Here, M need not be a generator unless R is left pre-PF (= every faithful left ideal generates mod-R), as can be seen from the following result ([**5**], Proposition 1F p. 165):

Every torsionless faithful left R-module generates mod-R if and only if R is a left pre-PF ring.

A particularly simple example of a module which is not a generator but satisfies (H) is obtained by taking M to be any essential ideal, K, in a commutative semi prime ring R, such that K is not finitely generated and projective. For, in that case, ND holds since K is torsionless over a semiprime ring, the trace T is essential in R since $T \supseteq K$, and K is not a generator since an ideal of a commutative ring is a generator only if it is finitely generated and projective.

The following lemma groups together some known results which will be needed in the proof of Theorem 3.3.

LEMMA 3.2. (a) *If $N_1 \subset N_2 \subset N_3$ is a trio of submodules and N_i is σ-dense (respectively, σ-closed) in N_{i+1}, for $i = 1, 2$, then N_1 is σ-dense (respectively, σ-closed) in N_3.*

(b) *If $h: {}_RM \to {}_RL$ is a homomorphism, L is σ-torsion free and* ker $h \subset {}^{\sigma}M$, *then* ker $h = M$, *that is, h is the zero homomorphism.*

Proof. (a) This follows because the σ-torsion (respectively σ-torsion free) modules are closed under extension.

(b) Set $K = \ker h$. For any $x \in M$, we have $(K, x) \subset {}^{\sigma}R$: for, if r_1 is any element of R, then $r_1x \in M$ and $K \subset {}^{\sigma}M$ imply that there is a left ideal $J \in \mathfrak{J}_{\sigma}$ such that $J(r_1x) \subseteq K$, hence $(Jr_1)x \subseteq K$ and $Jr_1 \subseteq (K:x)$; showing that $(K:x) \subset {}^{\sigma}R$. Now $(K:x)(xh) \subseteq Kh = 0$ and $(K:x) \subset {}^{\sigma}R$ imply that xh is σ-torsion. But $xh \in L$ which is σ-torsion free; hence $xh = 0$ and $M = \ker h$.

THEOREM 3.3. *If (R, M, M^*, B) and σ satisfy* (H), *then the maps $W \to I_B(W)$ and $J \to (MJ)^{\sigma}$ determine a lattice isomorphism between the lattice, $\mathscr{L}^{\sigma}(M)$, of σ-closed submodules, W, of M and the lattice, $\mathscr{L}^{\bar{\sigma}}(B)$, of $\bar{\sigma}$-closed left ideals, J, of B.*

Proof. It is easy to see that the two maps are order-preserving. Since $(MJ)^{\sigma}$ is σ-closed by definition, we need to show that $I_B(W)$ is $\bar{\sigma}$-closed whenever W is σ-closed and that the two maps are bijective.

Assume that $W = W^{\sigma}$ is σ-closed so that M/W is σ-torsion free. To show that $I_B(W)$ is $\bar{\sigma}$-closed, let $b + I_B(W)$ be a $\bar{\sigma}$-torsion element of $B/I_B(W)$ and show that this element is the zero element in $B/I_B(W)$.

Since $b + I_B(W)$ is $\bar{\sigma}$-torsion, there is $J \in \tilde{\mathfrak{F}}$ such that $Jb \subseteq I_B(W)$. Hence there is a σ-submodule K of M such that $J \supseteq [M^*, K]$, and so $[M^*, K]b \subseteq I_B(W)$. Then

$$(M, M^*)Kb = M[M^*, K]b \subseteq W,$$

and hence, since $(M, M^*) \in \mathfrak{F}_\sigma$, every element of Kb is σ-torsion in M/W. But M/W is σ-torsion free, hence $Kb = 0$ in M/W, or $Kb \subseteq W$. The composition $h = b\pi$ of b with the canonical projection $\pi:M \rightarrow M/W$ is such that $K \subseteq \ker h$; hence, by Lemma 3.2 (b), since $K \subset {}^\sigma M$, we have $h = 0$, that is, $Mb \subseteq W$ and $b \in I_B(W)$, or $b + I_B(W) = 0$ as required.

It remains to prove that the two maps are inverses of each other, that is, that

$$[MI_B(W)]^\sigma = W \quad \text{and} \quad I_B[(MJ)^\sigma] = J,$$

for W σ-closed and J $\bar{\sigma}$-closed.

Clearly, $MI_B(W) \subseteq W$ implies $[MI_B(W)]^\sigma \subseteq W^\sigma = W$. To show the reverse inclusion, let $x \in W$. We have

$$M[M^*, x] = (M, M^*)x \subseteq W,$$

that is,

$$[M^*, x] \subseteq I_B(W) \quad \text{and} \quad (M, M^*)x = M[M^*, x] \subseteq MI_B(W).$$

Since $(M, M^*) \in \mathfrak{F}_\sigma$, this shows that

$$(MI_B(W):x) \in \mathfrak{F}_\sigma$$

hence $x \in [MI_B(W)]^\sigma$, so that

$$W \subseteq [MI_B(W)]^\sigma \quad \text{and} \quad W = [MI_B(W)]^\sigma.$$

Finally, assume that $J = J^{\bar{\sigma}}$ is a $\bar{\sigma}$-closed left ideal of B. It is easy to see that

$$J \subseteq I_B(MJ) \subseteq I_B[(MJ)^\sigma].$$

To show that $J \subset {}^{\bar{\sigma}}I_B(MJ)$, let $c \in I_B(MJ)$, so that $Mc \subseteq MJ$. Then, for any $y \in M$, we have

$$yc = \sum_{i=1}^{n} m_i b_i, \quad \text{with } m_i \in M,\ b_i \in J,\ i = 1, \ldots, n.$$

Hence, for any $m \in M$ and $g \in M^*$, we have

$$m[g, y]c = (m, g)yc = (m, g)\left(\sum_{i=1}^{n} m_i b_i\right) = \sum_{i=1}^{n} (m, g)m_i b_i$$

$$= \sum_{i=1}^{n} m[g, m_i]b_i = m \sum_{i=1}^{n} [g, m_i]b_i = mb,\ b \in J;$$

here, $[g, m_i]b_i \in J$, for $i = 1, \ldots, n$, since J is a left ideal of B. Hence $[g, y]c \in J$ for each $g \in M^*$ and $y \in M$, that is, $c \in I_B(MJ)$ implies $[M^*, M]c \subseteq J$. Since $[M^*, M] \in \bar{\mathfrak{F}}$, this shows that $J \subset {}^{\bar{\sigma}}I_B(MJ)$.

Next, let $b \in I_B[(MJ)^\sigma]$, so that $Mb \subseteq (MJ)^\sigma$. Then, for each $m \in M$, there is $I \in \mathfrak{F}_\sigma$ such that $I(mb) \subseteq MJ$. Let

$$K = \{m \in M{:}mb \in MJ\}.$$

K is clearly an R-submodule of M. Also, $K \subset {}^{\sigma}M$, for, if $y \in M$, then $yb \in (MJ)^\sigma$, hence there is $I \in \mathfrak{F}_\sigma$ such that $I(yb) \subseteq MJ$, hence $Iy \subseteq K$. We have

$$M[M^*, K]b = (M, M^*)Kb \subseteq (M, M^*)MJ \subseteq MJ,$$

which implies $[M^*, K]b \subseteq I_B(MJ)$. Since $[M^*, K] \in \bar{\mathfrak{F}}$, this shows that

$$I_B(MJ) \subset {}^{\bar{\sigma}}I_B[(MJ)^\sigma].$$

Now, $J \subset {}^{\bar{\sigma}}I_B(MJ)$ and $I_B(MJ) \subset {}^{\bar{\sigma}}I_B[(MJ)^\sigma]$ imply, by Lemma 3.2 (a), that

$$J \subset {}^{\bar{\sigma}}I_B[(MJ)^\sigma],$$

that is $I_B[(MJ)^\sigma]/J$ is $\bar{\sigma}$-torsion; but J is $\bar{\sigma}$-closed, that is $I_B[(MJ)^\sigma]/J$ is $\bar{\sigma}$-torsion free; hence $J = I_B[(MJ)^\sigma]$ and the proof is complete.

4. The Goldie topology. We consider now the case of the Goldie torsion functor or, equivalently, the Goldie topology, $\mathcal{G}$; here $\mathcal{G} = \{{}_RI{:}{}_RI \subseteq {}_RJ$ where J and $(I{:}r)$ are essential left ideals of R for each $r \in J\}$; that is, $\mathcal{G}$ is the smallest Gabriel topology which contains the set of essential left ideals of R (cf. **[10]**, p. 148).

The next lemma is a restatement of some known results which will be needed in the sequel.

LEMMA 4.1. (a) *For any idempotent kernel functor σ, a σ-dense submodule N of a σ-torsionfree module M is essential in M.*

(b) *If σ is the Goldie kernel functor, then the converse of* (a) *holds, that is, an essential submodule of a σ-torsionfree module is σ-dense.*

(c) *When σ is the Goldie kernel functor and $\sigma(M) = 0$, then for any two submodules W_1, W_2 of M, we have $W_1 \cap W_2 =$ if and only if $W_1^\sigma \cap W_2^\sigma = 0$.*

Proof. (a) For $N \subset {}^{\sigma}M$ and $0 \neq m \in M$, with $\sigma(M) = 0$, we have

$$(N{:}m) \in \mathfrak{F}_\sigma \quad \text{and} \quad 0 \neq (N{:}m)m \subseteq Rm \cap N.$$

(b) By **[11]** ((2.3), p. 104), essential submodules of σ-torsionfree modules are σ-dense if and only if σ-torsionfree $\mathfrak{F}_\sigma$-injective modules are injective. Since, when σ is the Goldie kernel functor, $\mathcal{G}$-injectives are injective (cf. e.g. [**10**], p. 204), the result follows.

(c) Since σ is the Goldie kernel functor, this is simply a well-known property of essential extensions.

We still keep our hypothesis, (H), on M and its standard context, noting that, with the Goldie topology on R, $\sigma(M) = 0$ means ${}_RM$ is *nonsingular*, that is, that 0 is the only element of M which is annihilated by an essential left ideal of R. It is natural to expect here that, when $\mathfrak{F} = \mathcal{G}$, then $\bar{\mathfrak{F}}$ should be the Goldie topology on B. The next lemma shows that this expectation is justified.

LEMMA 4.2. *If ${}_RM$ and its standard context satisfy* (H), *and $\mathcal{T} = \mathcal{G}$, the Goldie topolgy on R, then $\bar{\mathcal{T}}$ is the Goldie topology on B.*

Proof. Recall that

$$\bar{\mathfrak{F}} = \{{}_RI:I \supseteq [M^*, K], \text{ where } K \subset {}^{\sigma}M\}.$$

We will show that $I \in \bar{\mathfrak{F}}$ if and only if I is essential in B, and this will show that $\bar{\mathfrak{F}}$ is the Goldie topology on B. The reason that the Goldie topology on B coincides with the set of essential left ideals of B is that ${}_BB$ is $\bar{\sigma}$-torsionfree, by Theorem 8 of [**8**] since $\sigma(M) = 0$, hence, by Lemma 4.1 (a), every $\bar{\sigma}$-dense left ideal of B is essential in B.

Assume that $I \in \bar{\mathfrak{F}}$, that is $I \supseteq [M^*, K]$, where $K \subset {}^{\sigma}M$. By Lemma 4.1 (a), since $\sigma(M) = 0$, this implies that K is essential in M. To show that I is essential in B, let $0 \neq b \in B$. Since $Mb \neq 0$, there is $m_1 \in M$ such that

$$0 \neq m_1b \in K \cap Mb.$$

By ND, $[M^*, m_1b] \neq 0$, and since $[M^*, K] \subseteq I$, we have $[M^*, m_1b] \subseteq I$, hence

$$0 \neq [M^*, m_1b] = [M^*, m_1]b \subseteq I \cap Bb.$$

Since b was any nonzero element of B, this shows that I is essential in B.

Conversely, assume that I is essential in B. Let

$$K = \{m \in M:[M^*, m] \subseteq 1\}.$$

It is easy to see that K is an R-submodule of M. To see that K is essential in M, let $0 \neq m_0 \in M$. By ND, $[M^*, m_0] \neq 0$, hence there is $0 \neq b_0 \in [M^*, m_0] \cap I$, say $b_0 = [f_0, m_0]$, for some $f_0 \in M^*$. Let $0 \neq m_1 \in M$ be such that $m_1b_0 \neq 0$ and set $r_0 = (m_1, f_0)$. Then

$$0 \neq m_1b_0 = m_1[f_0, m_0] = (m_1, f_0)m_0 = r_0m_0.$$

Now, given any $f \in M^*$, if m is any element of M, we have:

$$m[f, r_0m_0] = (m, f)(r_0m_0) = (m, f)(m_1b_0) = ((m, f)m_1)b_0 \\ = (m[f, m_1])b_0 = m([f, m_1]b_0);$$

therefore, $[f, r_0m_0] = [f, m_1]b_0 \in I$ since $b_0 \in I$. Hence, $[M^*, r_0m_0] \subseteq I$, so that $0 \neq r_0m_0 \in K \cap Rm_0$, and we have shown that K is essential in M. Now, by Lemma 4.1 (b), $K \subset {}^{\sigma}M$; thus $I \supseteq [M^*, K]$ implies $I \in \widetilde{\mathfrak{F}}$.

Recall that a submodule N of a module M is said to be a *complement submodule* if N has no proper essential extension in M. ${}_RM$ is said to be a *CS-module* in case every complement submodule of M is a direct summand of M. R is a left *CS-ring* if ${}_RR$ is a CS-module ([**1**]). The notation $d({}_RM)$ indicates the Goldie dimension of M, and $Q^l_{\max}(B)$ will be used to denote the maximal left quotient ring of B.

In the next theorem, Theorem 3.3 is applied in the case of the Goldie kernel functor to show how properties of M transfer to B and conversely.

THEOREM 4.3. *If ${}_RM$ is a nonsingular left R-module such that the standard Morita context (R, M, M^*, B) satisfies: $[M^*, m] \neq 0$ whenever $0 \neq m \in M$, and $(M, M^*) \in \mathcal{G}$, where $\mathcal{G}$ is the Goldie topology on R, then:*

(i) *B is a left nonsingular ring with maximal left quotient ring*

$$Q^l_{\max}(B) \cong \widetilde{B} \cong \mathrm{Hom}_R(\widetilde{M}, \widetilde{M});$$

(ii) *B is a left CS-ring if and only if ${}_RM$ is a CS-module*;

(iii) *$d({}_RM) < \infty$ if and only if $d({}_BB) < \infty$, and in this case $d({}_RM) = d({}_BB)$ and B is semisimple artinian.*

Proof. (i) By [**8**], Theorem 8, $\sigma(M) = 0$ implies $\overline{\sigma}(B) = 0$, hence, since by Lemma 4.2 $\overline{\sigma}$ is the Goldie kernel functor of B, this means ${}_BB$ is nonsingular. It is well known that the maximal left quotient ring of a left nonsingular ring is isomorphic to its injective hull (cf. e.g. [**10**], p. 149). Moreover, since $\sigma(M) = 0$, we have $M_{\mathfrak{F}} = \mathfrak{F}(M)$, the $\mathfrak{F}$-injective hull of M; and since $\mathfrak{F}$ is the Goldie topology, $\mathcal{G}$, and every $\mathcal{G}$-injective module is injective, $M_{\mathfrak{F}}$ is injective; that is (see Section 2), M is σ-dense, hence essential, in the σ-torsionfree and injective module $M_{\mathfrak{F}}$, showing that $M_{\mathfrak{F}} = \widetilde{M}$, the injective hull of M. Now, by Corollary 10 of [**8**], we have the ring of quotients

$$B_{\mathfrak{F}} \cong \mathrm{Hom}_R(M_{\mathfrak{F}}, M_{\mathfrak{F}}) = \mathrm{Hom}_R(\widetilde{M}, \widetilde{M});$$

and, by Lemma 3.2 of [**11**], we have $B_{\overline{\mathcal{J}}} = Q^l_{\max}(B)$.

(ii) Assume that B is a left CS-ring and let W be a complement submodule of M. By Theorem 3.3, $I_B(W)$ is a complement left ideal of B;

hence, since B is left CS, $I_B(W)$ is a direct summand of B. Therefore, $I_B(W) = Be$, where $e = e^2 \in B$, and we have

$$W = [MI_B(W)]^\sigma = [MBe]^\sigma = [Me]^\sigma = Me;$$

that is, W is a direct summand of M, and M is a CS-module.

Conversely, assume that M is a CS-module and let J be a complement left ideal of B. Then $(MJ)^\sigma$ is a complement submodule of M, and hence a direct summand of M. Therefore, $(MJ)^\sigma = Me$, where $e = e^2 \in B$. Then

$$J = I_B[(MJ)^\sigma] = I_B[Me],$$

by Theorem 3.2, and it is easy to see that $I_B[Me] = Be$: clearly, $e \in I_B(Me)$, implying that $Be \subseteq I_B(Me)$ since $I_B(Me)$ is a left ideal of B; on the other hand, if $b \in I_B(Me)$, then, for any $m \in M$, we have $mb = m_1e$ for some $m_1 \in M$, and hence $mbe = m_1e^2 = m_1e = mb$, showing that $b = be \in Be$. Hence $J = Be$, that is J is a direct summand of B and B is left CS.

(iii) Assume that $d({}_BB) = m$. Suppose that $\sum_{i=1}^{m+1} \oplus W_i$ is a direct sum of submodules of M, and consider the sum $\sum_{i=1}^{m+1} I_B(W_i)$ of left ideals of B. If

$$b \in I_B(W_i) \cap \sum_{j \neq i} I_B(W_j),$$

then

$$Mb \subseteq W_i \cap \sum_{j \neq i} W_j = 0,$$

hence $b = 0$ and the sum $\sum_{i=1}^{m+1} I_B(W_i)$ is direct, contradicting $d({}_BB) = m$. Therefore

$$d({}_RM) \leqq m = d({}_BB).$$

Now assume that $d({}_RM) = n$, and suppose that $\sum_{i=1}^{n+1} \oplus J_i$ is a direct sum of $n + 1$ nonzero left ideals of B. By Lemma 4.1 (c) (and by induction), the sum $\sum_{i=1}^{n+1} J_i^{\bar{\sigma}}$ is also direct, hence we may assume, without loss of generality, that

$$J_i = J_i^{\bar{\sigma}}, \quad i = 1, \ldots, n + 1.$$

Let

$$W_i = (MJ_i)^\sigma \cap \left[\sum_{k \neq i} (MJ_k)^\sigma\right]^\sigma.$$

Note that, since the join $(MJ_1)^\sigma \vee (MJ_2)^\sigma$ of $(MJ_1)^\sigma$ and $(MJ_2)^\sigma$ in $\mathscr{L}^\sigma(M)$ is $[(MJ_1)^\sigma + (MJ_2)^\sigma]^\sigma$, and since

$$I_B:\mathscr{L}^\sigma({}_RM) \to \mathscr{L}^{\bar{\sigma}}({}_BB)$$

is a lattice isomorphism, we have

$$\begin{aligned} I_B\{[(MJ_1)^\sigma + (MJ_2)^\sigma]^\sigma\} &= I_B\{(MJ_1)^\sigma \vee (MJ_2)^\sigma\} \\ &= I_B[(MJ_1)^\sigma] \vee I_B[(MJ_2)^\sigma] \\ &= J_1 \vee J_2 \\ &= (J_1 + J_2)^{\bar{\sigma}}. \end{aligned}$$

Hence, by induction,

$$\begin{aligned} I_B(W_i) &= I_B[(MJ_i)^\sigma] \cap I_B\left\{\left[\sum_{k\neq i} (MJ_k)^\sigma\right]^\sigma\right\} = J_i \cap \left[\sum_{k\neq i} J_k\right]^{\bar{\sigma}} \\ &= 0. \end{aligned}$$

Now, by the nondegeneracy condition, if U is a nonzero submodule of M, then $[M^*, x] \neq 0$ for $0 \neq x \in U$, hence

$$0 \neq [M^*, x] \subseteq I_B(U).$$

Hence $I_B(W_i) = 0$ implies $W_i = 0$, showing that the sum $\sum_{i=1}^{n+1} (MJ_i)^\sigma$ is direct, contrary to $d({}_RM) = n$. Hence $d({}_BB) \leqq n = d({}_RM)$, and we can conclude that $d({}_RM) = d({}_BB)$.

Finally, to see that $\tilde{B}$ is semisimple artinian, we use Theorem 3.1 of [**11**], which states that $\tilde{B}$ is semisimple artinian if and only if there is no infinite independent family of $\bar{\mathfrak{T}}$-torsionfree left ideals of B. But here, since B is left nonsingular, that is Goldie torsionfree or $\bar{\mathfrak{T}}$-torsionfree, every left ideal of B is $\bar{\mathfrak{T}}$-torsionfree. Thus B is semisimple artinian if and only if $d({}_BB) < \infty$.

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