## Chapter 1

# Introduction

Fluid flows exhibit rich behavior—regular flow in viscous regime; patterns and chaos in weakly nonlinear regime; and turbulence in strongly nonlinear regime. Such complex behavior arises due to nonlinearity. A good understanding of some of these features, specially turbulence, is still lacking even after sustained efforts extending over two centuries.

Hydrodynamics broadly deals with the analysis of the Navier–Stokes equations and their generalization to magnetohydrodynamics, convection, passive scalars, rotating flows, etc. The large-scale structures in such flows contribute to various instabilities, patterns, and chaos, but turbulence is governed by structures at all scales. Researchers have studied these phenomena using the energy exchanges among various Fourier modes, a topic which has not been studied in great detail. In this book we attempt to fill this gap by focusing on energy transfers in fluid flows, and their role in turbulence. Note, however, that energy transfer formalism developed in the book are general, and they could be applied to study pattern formation and chaos as well.

The energy transfers in fluid flows are generic, and they arise in linear as well as in nonlinear systems. A pendulum is an example of a linear system that exhibits a periodic exchange of kinetic energy and potential energy during its oscillations. Similar transfers occur in surface gravity waves and in internal gravity waves. Also note that in the unstable configuration of the pendulum (vertically standing up), it is the potential energy that drives the kinetic energy and makes the pendulum unstable. The buoyancy-driven instabilities—thermal instability, Rayleigh–Taylor instability—have energy transfers similar to that in the unstable configuration of the pendulum.

In nonlinear hydrodynamics, the energy transfers among the interacting modes arise due to quadratic nonlinearities, for example, $\mathbf{u}\cdot\nabla\mathbf{u}$ in fluids, $\mathbf{u}\cdot\nabla T$ in thermal convection, $\nabla\times(\mathbf{u}\times\mathbf{b})$ and $\mathbf{b}\cdot\nabla\mathbf{b}$ in magnetohydrodynamics, where $\mathbf{u},\mathbf{b}$, and $T$ are respectively the velocity, magnetic, and temperature fields. Therefore, we develop a general formulation to compute the energy transfers that help us formulate interactions among the participating modes, energy flux, shell-to-shell and ring-to-ring energy transfers, etc. The energy transfers among the large-scale modes help us understand patterns and chaos. On the other hand, the energy flux, and the shell-to-shell and ring-to-ring energy transfers provide useful information about turbulence in hydrodynamics, magnetohydrodynamics, and buoyancy-driven flows. In this book, we present these topics thematically, duly emphasizing the common features among them. Also, the present book focusses on the nonlinear energy transfers.

In the next section we introduce a generic nonlinear equation that contains basic features of nonlinear energy transfers.

## 1.1 A Generic Nonlinear Equation

In a nonlinear system, nonlinearity induces interactions among the Fourier modes. These interactions are illustrated using the following equation:

$$\frac{\partial}{\partial t}f(x,t)=\frac{\partial^2}{\partial x^2}f(x,t)+a[f(x,t)]^2, \tag{1.1}$$

where $a$ is a constant. In this equation, $f^2$ is the nonlinear term. We assume the field $f$ to be contained in a periodic box of length $L$. We decompose the field $f(x)$ into Fourier modes $f(k)$:

$$f(x,t)=\sum_k f(k,t)\exp(ikx), \tag{1.2}$$

where $k=2n\pi/L$ with $n$ as an integer. The corresponding inverse transform is given by

$$f(k,t)=\frac{1}{L}\int_0^L dx f(x,t)\exp(-ikx). \tag{1.3}$$

Using this definition, we derive an equation for $f(k,t)$ in Fourier space. We start with

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$$\frac{\partial}{\partial t}\sum f(k)\exp(ikx) = \frac{\partial^2}{\partial x^2}\sum f(k)\exp(ikx) + a\sum f(p)\exp(ipx)\sum f(q)\exp(iqx). \quad (1.4)$$

Using the orthogonality relation,

$$\frac{1}{L}\int_0^L dx \exp(i(p+q-k)x) = \delta_{p+q,k}, \quad (1.5)$$

we obtain the requisite equation:

$$\frac{d}{dt}f(k) = -k^2 f(k) + a\sum_p f(p)f(k-p). \quad (1.6)$$

Note that the nonlinear term has become a convolution in Fourier space.

In the absence of the nonlinear term, Eq. (1.6) reduces to diffusion equation:

$$\frac{d}{dt}f(k) = -k^2 f(k) \quad (1.7)$$

that has a very simple solution—each Fourier mode decays exponentially as

$$f(k,t) = f(0)\exp(-k^2 t). \quad (1.8)$$

However, the nonlinear term, $a\sum_p f(p)f(k-p)$, of Eq. (1.6), couples the Fourier modes in an intricate manner. In Eq. (1.6), nonlinear interactions involve three Fourier modes, for example, $f(k)$, $f(p)$, and $f(k-p)$. In general, there is no analytical solution for Eq. (1.6) because of the nonlinearity. Note however that analytical solution may be possible for some specific cases.

The modal energy $|f(k)|^2/2$ is the energy of the Fourier mode $f(k)$. By multiplying Eq. (1.6) with $f^*(k)$ and summing the resulting equation with its complex conjugate, we obtain an equation for the time evolution of $|f(k)|^2$ as

$$\frac{d}{dt}\frac{1}{2}|f(k)|^2 = -2k^2\frac{1}{2}|f(k)|^2 + a\Re\left[\sum_p f(k-p)f(p)f^*(k)\right], \quad (1.9)$$

where $\Re(.)$ stands for the real part of the argument. The last term of Eq. (1.9) represents the nonlinear energy transfer. The modal energy decays exponentially in the absence of such transfer. Note, however, that the nonlinear term could stabilize the modal energy or make it time dependent. We remark that on many

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occasions, energy is more convenient to analyze than the Fourier modes. For example, $|f(k)|^2$ is real in contrast to complex $f(k)$.

In the aforementioned equation, the decaying term $-k^2|f(k)|^2$ provides energy transfer from *coherent energy* to *heat* or *dissipation*. The nonlinear term $\Re(f(k-p)f(p)f^*(k))$ represents the energy transfers from the modes $f(p)$ and $f(k-p)$ to $f(k)$. Note that these energy transfers occur among various length scales, and it differs from the energy transport in real space, for example, $\int f^2 d\mathbf{S}$ where $d\mathbf{S}$ is an elemental area.

The aforementioned behavior is a generic feature of nonlinear partial differential equations including the Navier–Stokes equation. In the present book, we will present energy transfer formalism for hydrodynamics, magnetohydrodynamics, scalar flows, buoyancy-driven flows, etc. Most of our discussion will be focused on incompressible flows.

Before we start our detailed discussion on energy transfers, we outline the contents of the book.

## 1.2 Outline of the Book

The book is divided into four parts. Part I contains discussion on various aspects of hydrodynamics, while Parts II–IV cover more complex applications. Though the book focuses on energy transfers, we also cover physics of the flows under consideration—hydrodynamics, scalar flows, magnetohydrodynamics, thermal convection, etc.

In Part I, Chapters 2 and 3 describe the governing equations of hydrodynamics in real and Fourier spaces respectively. The formalism of energy transfers in hydrodynamics is developed in Chapter 4. Chapters 5 and 7 describe phenomenologies of three-dimensional and two-dimensional hydrodynamic turbulence respectively. Here we describe energy fluxes and other related diagnostics. Chapter 6 contains a discussion on enstrophy transfers in turbulent flows.

In Chapter 8, we describe helical turbulence. Chapter 9 introduces Craya–Herring and helical basis that are very useful for describing flow properties. In Chapter 10 we briefly describe field-theoretic treatment of energy transfers. These computations provide many useful insights into turbulence dynamics. Chapter 11 contains formulation of energy transfer in anisotropic turbulence.

Energy transfers in real space—energy flow from one scale to another—can be described using structure function. This scheme was first derived by Kolmogorov (1941a,b,c). In Chapter 12, we describe this formalism, and relate it to the energy transfers in Fourier space. Note, however, that the present book focuses on energy transfers in Fourier space.

In nature and in engineering, we encounter flows that advect scalar, vector, or tensor fields along with it. There are many examples of such flows, as listed in Table 1.1. In subsequent chapters, we will cover properties, especially related to the energy transfers, of such flows.

**Table 1.1** Examples of scalars, vectors, and tensors in flows.

| Field | Examples |
|---|---|
| Scalar | Densities of dust particles and pollution, fluid density, temperature, binary fluid |
| Vector | Magnetic field, flock velocity, dipoles |
| Tensor | Polymer, elastic fluid |

In Part II (Chapters 13 to 17), we systematically cover flows with scalars. We describe properties of passive scalar flows, stably stratified flows, and thermal convection. We also cover binary fluid mixture in this part.

In Part III (Chapters 18 to 25), we describe flows with vectors that include passive vector flows, magnetohydrodynamics (MHD), and electron MHD. The topics covered are energy transfers in MHD, turbulence phenomenologies, dynamo, etc.

In Part IV (Chapters 26 and 31), we include miscellaneous topics that go beyond scalar and vector flows. These topics include tensor flows, rotating turbulence, shell model of turbulence, Burgers turbulence, and compressible turbulence. The last two topics in this part are on compressible hydrodynamics—an exception to the theme of the book. We conclude in Chapter 32.

There are a large number of excellent textbooks and reviews on various kinds of turbulent flows (hydrodynamic, magnetohydrodynamic, buoyancy-driven). In Table 1.2 we list some of the important references, especially those connected to our discussion. The present book attempts to highlight common features in the aforementioned flows using energy transfers, energy fluxes, etc., and show that energy transfer formalism provides important insights into turbulent flows.

**Table 1.2** Key references.

| | |
|---|---|
| Hydrodynamic turbulence | Leslie (1973); McComb (1990, 2014); Frisch (1995); Mathieu and Scott (2000); Pope (2000); Davidson (2004); Lesieur (2008); Kraichnan (1959); Alexakis and Biferale (2018) |
| Craya–Herring basis | Sagaut and Cambon (2008); Lesieur (2008) |
| Buoyancy-driven turbulence | Manneville (2014); Verma (2018); Lohse and Xia (2010); Verma et al. (2017) |
| MHD turbulence | Biskamp (2003); Davidson (2017); Verma (2004) |
| Shell model | Ditlevsen (2010); Plunian et al. (2012); Verma and Kumar (2016) |
| Anisotropy | Davidson (2013); Verma (2017, 2018) |

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