Canad. Math. Bull. Vol. **29** (1), 1986

ON THE CRITERIA OF D.D. ANDERSON FOR INVERTIBLE AND FLAT IDEALS

BY
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ABSTRACT. Let R be an integral domain. It is proved that if a nonzero ideal I of R can be generated by $n < \infty$ elements, then I is invertible (i.e., flat) if and only if $I(\cap Ra_i) = \cap Ia_i$ for all $\{a_1, \ldots, a_n\} \subset I$. The article's main focus is on torsion-free R-modules E which are LCM-stable in the sense that $E(Ra \cap Rb) = Ea \cap Eb$ for all $a, b \in R$. By means of linear relations, LCM-stableness is shown to be equivalent to a weak aspect of flatness. Consequently, if each finitely generated ideal of R may be 2-generated, then each LCM-stable R-module is flat. Finally, LCM-stableness of maximal ideals serves to characterize Prüfer domains, Dedekind domains, principal ideal domains, and Bézout domains amongst suitably larger classes of integral domains.

1. **Introduction**. Our starting point is the following recent result of D. D. Anderson ([1], Theorem 1): a nonzero ideal I of an integral domain R is invertible if and only if $I(\cap J_i) = \cap IJ_i$ for each collection $\{J_i\}$ of fractional ideals of R. A careful study of Anderson's proof reveals that attention may be restricted to fractional ideals J_i which are principal. In this spirit, we show in Theorem 2.2 that if I is finitely generated, then the focus of Anderson's criterion may be restricted still further, namely to finite index sets $\{i\}$ and principal (integral) ideals J_i.

In a related result, Anderson ([1], Theorem 2) showed that an ideal I of an integral domain R is R-flat if and only if $I(J_1 \cap J_2) = IJ_1 \cap IJ_2$ for all pairs J_1, J_2 of ideals of R. A more general result, with I replaced by an arbitrary torsion-free R-module E, was established several years earlier by Jensen ([11], Theorem 1). In view of the above sharpening of ([1], Theorem 1), it seems natural to attempt to relate the possible flatness of such an E to the property "$E(Ra \cap Rb) = Ea \cap Eb$ for all $a, b \in R$." This latter property has been studied under the name "LCM-stableness," in case E is an extension domain of R, by Gilmer [9] and, recently, Uda [16]. Uda has shown in fact that LCM-stableness is genuinely weaker than flatness; and has rephrased Richman's characterization of Prüfer domains [13] in terms of the LCM-stableness of overrings.

As Prüfer domains are characterized by the flatness of their ideals, the above result of Uda makes it natural to ask whether LCM-stableness of *ideals* also characterizes Prüfer domains. Indeed, this is so: see Proposition 3.7. In the presence of a mild

Received by the editors July 2, 1984.
AMS Subject Classification (1980): primary 13C11, 13F05; secondary 13G05, 13F15, 13E05, 18G15.

finiteness condition, LCM-stableness of just the maximal ideals serves to characterize Prüfer domains (Theorem 3.8). As a consequence (Corollary 3.9), one has new characterizations of Bézout domains, PID's, and Dedekind domains. However, analysis of the $D + M$ construction in Corollary 3.6 shows that such results fail in the absence of a suitable finiteness condition.

The key to the above results is Theorem 3.3(b), which explains the connection between LCM-stableness and flatness: an R-module E is LCM-stable if and only if each linear relation (*of length two*) of elements r_1, r_2 in R with coefficients in E is a linear consequence of linear relations of the r_i's with coefficients in R. As another consequence (Corollary 3.4), LCM-stableness is equivalent to flatness in case each finitely generated ideal of R is 2-generated.

Any unexplained material is standard, as in [8], [12].

2. **Invertible ideals**. We begin by recording a sharp version of what was established in a *proof* of D. D. Anderson ([1], Theorem 1).

PROPOSITION 2.1. *Let E be a nonzero ideal of an integral domain R. Then the following are equivalent:*

(1) $\cap EI_i = E(\cap I_i)$ *for each nonempty set* $\{I_i\}$ *of ideals of* R;

(2) $\cap Ea_i = E(\cap Ra_i)$ *for each subset* $\{a_i\}$ *of the quotient field of* R;

(3) E *is invertible;*

(4) E *is* R-*projective.*

It is known, by various module-theoretic results ([7] Theorem 1, [11], Corollary 1), that a nonzero ideal I of an integral domain R is invertible if (and only if) I is finitely generated and flat over R. Accordingly, it is of some interest to find conditions characterizing when a finitely generated nonzero ideal of an integral domain is flat (i.e., invertible). One such result appeared in ([4], Proposition 1). Another is given next, motivated by weakening the above condition (2), our variant of the criteria in [1].

THEOREM 2.2. *Let I be a nonzero n-generated ideal of an integral domain R, for some positive integer n. Then the following are equivalent:*

(1) $\cap Ia_i = I(\cap Ra_i)$ *for each finite subset* $\{a_i\}$ *of* R;

(2) $\cap Ia_i = I(\cap Ra_i)$ *for all* $\{a_1, \ldots, a_n\} \subset I$;

(3) I *is* R-*flat;*

(4) I *is invertible.*

PROOF. The above remarks established (3) $\Leftrightarrow$ (4). Moreover, (4) $\Rightarrow$ (1) by Proposition 2.1; and (1) $\Rightarrow$ (2) trivially. It thus remains only to prove that (2) $\Rightarrow$ (4).

The result is evident if I is a principal ideal. We may therefore assume that $n > 1$, and write $I = Ra_1 + Ra_2 + \ldots + Ra_n$. For each i, let $b_i = a_1 a_2 \ldots a_n a_i^{-1}$, the product of all the a_j's except a_i. Observe next that

$$I(Rb_1 \cap Rb_2 \cap \ldots \cap Rb_n) \supset Ib_1 \cap \ldots \cap Ib_n \supset Ra_1 a_2 \ldots a_n,$$

with the first inclusion following from (2) and the second inclusion following since

$a_1a_2 \ldots a_n = a_ib_i$. As one may similarly check that $I(\cap Rb_i) \subset Ra_1a_2 \ldots a_n$, it follows that $I(\cap Rb_i) = Ra_1a_2 \ldots a_n$, an invertible ideal. Hence, I is also invertible, completing the proof.

The proof that (2) $\Rightarrow$ (4) in Theorem 2.2 was motivated by an argument of Gilmer ([8], Theorem 25.2, $(c_1) \Rightarrow (d_1) \Rightarrow (a)$), possible revisiting an argument of Jensen ([10], Theorem 3). The results in question are characterizations of Prüfer domains. These are relevant since an integral domain R is a Prüfer domain if and only if each ideal of R is flat; that is, if and only if each nonzero 2-generated ideal of R is invertible (cf. [3], Theorem 4.2, [8], Theorem 22.1). We shall meet this theme again in Proposition 3.7. Next, we close the section by summarizing the import of the *proof* of Theorem 2.2 for 2-generated ideals.

COROLLARY 2.3. *Let a, b be elements, not both of which are zero, of an integral domain R. Then for $I = Ra + Rb$, the following are equivalent:*

(1) $Ia \cap Ib = I(Ra \cap Rb)$;

(2) *I is R-flat;*

(3) *I is invertible.*

3. **LCM-stableness and flatness**. For motivation, we begin by collecting some characterizations of flatness.

PROPOSITION 3.1. *Let R be an integral domain and E a torsion-free R-module. Then the following are equivalent:*

(1) $EI \cap EJ = E(I \cap J)$ *for all ideals I, J of R;*

(2) $\cap EI_i = E(\cap I_i)$ *for all finite sets $\{I_i\}$ of ideals of R;*

(3) $EI \cap Eb = E(I \cap Rb)$ *for all finitely generated ideals I of R and all elements $b \in R$;*

(4) *E is R-flat.*

PROOF. (1) $\Leftrightarrow$ (4): This is the content, when specialized to the commutative case, of a result of Jensen ([11], Theorem 1).

(3) $\Rightarrow$ (4): This follows from the above-cited *proof* of Jensen.

(1) $\Rightarrow$ (2): Induction.

(2) $\Rightarrow$ (3): Use $\{I_i\} = \{I, Rb\}$.

The proof is complete.

The equivalence of conditions (1), (2), and (4) in Proposition 3.1 was established for the special case in which E is an ideal of R by D. D. Anderson ([1], Theorem 2). We shall next introduce our main object of study, a weakening of condition (3) in Proposition 3.1.

Let R be an integral domain. By analogy with ([9], p. 50), we shall say that a torsion-free R-module E is LCM-*stable* (*over* R) in case $Ea \cap Eb = E(Ra \cap Rb)$ for all $a, b \in R$.

REMARK 3.2. It is evident from the result of Jensen (cf. Proposition 3.1) that if E is a flat module over an integral domain R, then E is LCM-stable over R. The converse, however, is false, as Uda ([16], Example 4.8) has shown via a suitable simple algebraic extension of integral domains.

For a deeper study of the relationship between "flat" and "LCM-stable," we introduce the following definition. Let R be an integral domain, E an R-module, and n a positive integer. We shall say that E is *n-flat* (*over* R) in case each relation $r_1e_1 + \ldots + r_ne_n = 0$ (with each $r_i \in R$, $e_i \in E$) is induced by suitable $f_j \in E (1 \leq j \leq m)$ and $r_{ij} \in R (1 \leq i \leq n; 1 \leq j \leq m)$ satisfying $e_i = \Sigma r_{ij}f_j$ for each i and $\Sigma r_i r_{ij} = 0$ for each j. The terminology is, of course, suggested by the result ([2], Corollary 1, p. 27) that, for R and E as above, E is R-flat if and only if E is n-flat over R for each $n \geq 1$.

THEOREM 3.3. *Let R be an integral domain and E an R-module. Then:*

(a) *E is 1-flat over R if and only if E is a torsion-free R-module.*

(b) *E is 2-flat over R if and only if E is LCM-stable over R.*

PROOF. (a) This may be proved by simple calculations. For instance, if E is 1-flat and $re = 0$ ($r \in R$, $e \in E$) with $r \neq 0$, then the equations $e = \Sigma r_{1j}f_j$ and $rr_{1j} = 0$ lead to $r_{1j} = 0$ for each j, whence $e = 0$. The details for the converse may be left to the reader.

(b) Suppose first that E is 2-flat over R. It is easy to see that E is then also 1-flat over R; hence, by (a), E is torsion-free over R. It remains to prove that $Ea \cap Eb \subset E(Ra \cap Rb)$ for all $a, b \in R$. Consider $g = ae = bf$ ($g, e, f \in E$). By 2-flatness, the relation $ae + b(-f) = 0$ induces equations $e = \Sigma r_{1j}h_j$, $-f = \Sigma r_{2j}h_j$, and $ar_{1j} + br_{2j} = 0$ for suitable $r_{1j}, r_{2j} \in R$ and $h_j \in E$. Since $ar_{1j} = b(-r_{2j}) \in Ra \cap Rb$, it follows that $g = \Sigma(ar_{1j})h_j \in (Ra \cap Rb)E$, as desired.

Conversely, let E be LCM-stable over R. To see that E is 2-flat, note that the proof of Jensen ([11], Theorem 1) adapts nearly *verbatim*, in view of (a). These details may be left to the reader, completing the proof.

COROLLARY 3.4. *Let R be an integral domain in which each finitely generated ideal is 2-generated. Then an R-module E is LCM-stable over R (if and) only if E is R-flat.*

PROOF. Remark 3.2 takes care of the parenthetical assertion. Conversly, let E be LCM-stable. To see that E is R-flat, it is enough (cf. [2], Proposition 1, p. 12) to prove that the canonical homomorphism $g : I \otimes_R E \rightarrow E$ is a monomorphism for each finitely generated ideal I of R. By hypothesis, $I = Ra + Rb$. Thus any element $t \in I \otimes_R E$ has the form $t = a \otimes e + b \otimes f$. If $g(t) = 0$, the construction of g yields $ae + bf = 0$, and so Theorem 3.3(b) produces equations $e = \Sigma r_{1j}h_j$, $f = \Sigma r_{2j}h_j$, and $ar_{1j} + br_{2j} = 0$; hence $t = \Sigma(ar_{1j} + br_{2j}) \otimes h_j = 0$, completing the proof.

In view of Theorem 3.3, Corollary 3.4 may be viewed as a companion for the result ([2], Proposition 3, p. 15) that if R is a Bézout domain, then an R-module E is torsion-free if and only if E is R-flat. Of course, Bézout domains are the most natural examples of integral domains satisfying the hypothesis of Corollary 3.4. A partial converse is available (via [6], Theorem 4): if an integrally closed integral domain R of finite Krull dimension satisfies the hypothesis of Corollary 3.4, then R is a Prüfer domain. We shall pursue such rings via the LCM-stable property later in this section.

First, though, we shall collect some useful ways in which LCM-stableness reflects the behavior of flatness.

PROPOSITION 3.5. *Let R be an integral domain and n a positive integer. Then:*

(a) *If E is n-flat over R and S is a multiplicatively closed subset of R, then E_S is n-flat over R_S.*

(b) *If $E \cong \varinjlim E_i$ where each E_i is n-flat over R, then E is n-flat over R.*

(c) *If (R, M) is quasilocal, I an n-flat ideal of R and $n \geq 2$, then either $I = MI$ or I is principal.*

PROOF. (a) Consider an n-term relation $\Sigma (r_i s^{-1})(e_i s^{-1}) = 0 \in E_S$, with $r_i \in R$, $s \in S$ and $e_i \in E$. As E is also 1-flat over R, Theorem 3.3(a) yields $\Sigma\, r_i e_i = 0 \in E$. Using n-flatness of E over R, we infer certain equations in R and E which, via the canonical maps $R \to R_S$ and $E \to E_S$, induce the required equations in R_S and E_S.

(b) This follows readily from the construction of direct limit. The main point is that any n-term relation $r_1 e_1 + \ldots + r_n e_n = 0$ ($r_k \in R$, $e_k \in E$) is induced by some relation $\Sigma\, r_k e_{ik} = 0$ where each e_{ik} is sent to e_k by the structure map $E_i \to E$.

(c) Since I is also 2-flat over R, we may apply *verbatim* an argument of Sally-Vasconcelos ([14], Lemma 2.1), completing the proof.

In view of Theorem 3.3(b), the case $n = 2$ of Proposition 3.5(a) asserts that LCM-stableness is preserved by localization: cf. ([16], Corollary 1.5(2)). Note also that the proof of Proposition 3.5(b) recovers stability of flatness under direct limit ([2], Proposition 9, p. 20). We turn next to an application of Proposition 3.5(c).

COROLLARY 3.6. *Let (V, M) be a valuation domain of the form $V = K + M$, where K is a field. Let k be a proper subfield of K. Set $R = k + M$. Then a proper ideal I of V is LCM-stable over R (if and) only if I is R-flat; that is, if and only if $I = MI$.*

PROOF. Remark 3.2 takes care of the parenthetical assertion, while ([5], Theorem 7) dispatches the final assertion. Therefore, it remains only to show that if I is nonzero and LCM-stable over R, then $I = MI$. By Proposition 3.5(c), we may assume instead that I is principal over R and seek a contradiction. Take $I = Ri$ for some $i \in R\backslash\{0\}$. Note that $Ki \subset VI = I = Ri$, whence cancellation of i yields $K \subset R$. Thus $K = k$, the desired contradiction, completing the proof.

An interesting direct calculation shows, in the context of Corollary 3.6 and without appeal to Proposition 3.5(c), that I is LCM-stable over R if and only if $I = MI$. We leave the details to the reader. Note also, via Proposition 3.5(c) and Nakayama's lemma, that any finitely generated LCM-stable ideal of $k + M$ must be principal. Of course, the condition $k \neq K$ assures that $k + M$ has some nonprincipal finitely generated ideals, for $k + M$ is not a Bézout domain (cf. [8], Exercise 12(3), p. 287). These observations help to motivate Proposition 3.7 below.

Besides its ideals, the most natural examples of torsion-free modules over an integral domain are afforded by its overrings. In this regard, Uda ([16], Proposition 1.7) has recently extended some work of Richman ([13], Lemma 1 and Theorem 1) by showing that if T is an overring of an integral domain R, then T is LCM-stable over R (if and) only if T is R-flat. We may find motivation for our study of LCM-stableness for ideals by pursuing additional analogies with Uda's studies of LCM-stableness for ring extensions. For instance, the ideal-theoretic analogue of ([16], Proposition 1.9) would assert, "If each 2-generated submodule of an R-module E is LCM-stable over R, then so is E." This assertion is easily established, as is the evident generalization for n-flatness. More substantially, the characterization of Prüfer domains in ([16], Corollary 1.8), based on the above-cited result on LCM-stableness for overrings, motivates the following.

PROPOSITION 3.7. *For an integral domain R, the following are equivalent:*
(1) *Each 2-generated ideal of R is LCM-stable over R;*
(2) *Each ideal of R is LCM-stable over R;*
(3) $(Ra + Rb)a \cap (Ra + Rb)b = (Ra + Rb)(Ra \cap Rb)$ *for all* $a, b \in R$;
(4) *R is a Prüfer domain.*

PROOF. (4) $\Rightarrow$ (2): As noted in §2, each ideal of a Prüfer domain is (2-) flat. Apply Theorem 3.3(b).

(2) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (3): Immediate from the definition of LCM-stableness.

(3) $\Rightarrow$ (4): By Corollary 2.3, (3) implies that each nonzero 2-generated ideal of R is invertible. As noted in §2, this condition in turn implies (4), completing the proof.

It is convenient to recall here that an integral domain R is said to be *finite-conductor* in case $Ra \cap Rb$ is a finitely generated ideal of R for each pair of elements a, b of R. The natural examples of finite-conductor domains are arbitrary GCD-domains and arbitrary coherent integral domains (cf. [3], Theorem 2.2). In particular, all UFD's, Noetherian integral domains, and Prüfer domains are finite-conductor domains. We shall next give a relevant characterization of Prüfer domains, by reworking an argument of Vasconcelos ([18], Lemma 3.9).

THEOREM 3.8. *R is a Prüfer domain if and only if R is a finite-conductor domain each of whose maximal ideals is LCM-stable.*

PROOF. The "only if" assertion is immediate from Proposition 3.7 and the above remarks. For the converse, we may suppose that (R, M) is quasilocal, since localization preserves the finite-conductor property and LCM-stableness (cf. Proposition 3.5(a)). By ([12], Theorems 63 and 64), we need only show that R is a Bézout domain, i.e. that $I = Ra + Rb$ is a principal ideal for each nonzero $a, b \in R$. To this end, consider the short exact sequence

$$0 \to K \to R \oplus R \xrightarrow{f} I \to 0$$

where $f(r, s) = ra - sb$ for each $r, s \in R$. Of course, $K \cong Ra \cap Rb \neq 0$. By the finite-conductor hypothesis, K is finitely generated over R and so by Nakayama's lemma, $\dim_{R/M}(K/MK) \geq 1$. By similar reasoning, it is enough to prove that $\dim_{R/M}(I/MI) \leq 1$, which would in turn follow by dimension-counting from a short exact sequence of the form

https://doi.org/10.4153/CMB-1986-004-4 Published online by Cambridge University Press

$$0 \to K/MK \to R/M \oplus R/M \to I/MI \to 0.$$

Accordingly, by applying $\cdot \otimes_R R/M$ to the first-displayed sequence, we have only to prove that $T = Tor_1^R(I, R/M)$ is 0.

From here on, our methods must differ from those in [18]. Let $g : I \otimes_R M \to R$ denote the multiplication map. As $T \cong \ker(g)$, it will suffice to prove that any element $e \in \ker(g)$ is trivial. Express e as $a \otimes m_1 + b \otimes m_2$ for suitable $m_1, m_2 \in M$. By construction of g, we have $am_1 + bm_2 = 0$. Since the hypothesis, as interpreted via Theorem 3.3(b), assures that M is 2-flat, we obtain $m_1 = \Sigma\, r_{1j} n_j$, $m_2 = \Sigma\, r_{2j} n_j$, and $ar_{1j} + br_{2j} = 0$ for suitable $n_j \in M$ and r_{1j}, $r_{2j} \in R$. Therefore, $e = \Sigma(ar_{1j} + br_{2j}) \otimes n_j = 0$, completing the proof.

By means of Corollary 3.6, it is easy to see that one cannot delete the "finite-conductor" hypothesis in Theorem 3.8: cf. ([5], Theorem 3).

Corollary 3.9(a) generalizes a result of Vasconcelos ([17], Proposition A) characterizing valuation domains. Corollary 3.9(c) sharpens the characterization of Dedekind domains as the Noetherian integral domains whose nonzero maximal ideals are invertible (cf. [12], Exercise 12, p. 73).

COROLLARY 3.9. (a) *R is a Bézout domain if and only if R is a GCD-domain each of whose maximal ideals is LCM-stable.*

(b) *R is a PID if and only if R is a UFD each of whose maximal ideals is LCM-stable.*

(c) *R is a Dedekind domain if and only if R is a Noetherian integral domain each of whose maximal ideals is LCM-stable.*

PROOF. By Proposition 3.7 and the above remarks concerning finite-conductor domains, the assertions are direct consequences of Theorem 3.8 and the following well-known material.

(a) Each invertible ideal of a GCD-domain is principal (cf. [12], Exercise 15, p. 42).

(b) If R is a UFD and a Prüfer domain, then R is a PID (cf. [8], Proposition 38.6).

(c) Each Noetherian Prüfer domain is a Dedekind domain.

The proof is complete.

The following direct proof of Corollary 3.9(a) is of some interest. As above, we may take (R, M) a quasilocal GCD-domain for which M is LCM-stable. If R is not a valuation domain, there exist $a, b \in M$ such that $a \nmid b$ and $b \nmid a$. Let $d = gcd(a, b)$. Thus $a = a_1 d$ and $b = b_1 d$, for suitable relatively prime $a_1, b_1 \in M$. Since R is an LCM-domain (i.e., a GCD-domain), $Ra \cap Rb = R(abd^{-1}) = Ra_1 b_1 d$. Observe that $a_1 b_1 d \in Ma \cap Mb$. As M is LCM-stable, $a_1 b_1 d \in M(Ra \cap Rb) = Ma_1 b_1 d$, whence $1 \in M$, the desired contradiction.

The reader may also wish to compare Corollary 3.9(a) with a result of Sheldon ([15], Theorem 3.7), giving a rather different characterization of Bézout domains within the class of GCD-domains.

In closing, we record an application of the above ideas. Let X and Y be algebraically independent indeterminates over a field k, set $R = k[X, Y]$, and consider $I = RX + RY$.

It is known (cf. [2], Exercise 3(a), p. 41) that I is not R-flat. We can actually show that I is not LCM-stable over R. Indeed, if I were LCM-stable, the Proposition 3.5(a) and Corollary 3.9(b) would imply that $S = k[X, Y]_I$ is a PID and hence of Krull dimension at most 1. However $\dim(S) = 2$, the desired contradiction.

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https://doi.org/10.4153/CMB-1986-004-4 Published online by Cambridge University Press