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INDECOMPOSABLE 1-FACTORIZATIONS OF THE COMPLETE MULTIGRAPH

CHARLES J. COLBOURN, MARLENE J. COLBOURN

and ALEXANDER ROSA

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Abstract

The existence of 1-factorizations of the complete multigraph λK_n which cannot be decomposed into 1-factorizations with smaller λ is studied.

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1. Introduction

Any 1-factorization of the complete graph K_{2n} provides a schedule for the $2n - 1$ rounds of a simple round robin tournament for $2n$ teams, with each team meeting each other team exactly once. If each team is to meet each other team exactly λ times, one schedule for such a multiple round robin tournament is obtained by combining any λ schedules (whether identical or not) for a single round robin tournament. In graph-theoretic terms, combining any λ 1-factorizations of K_{2n} yields a 1-factorization of λK_{2n}.

One might ask the converse question: given a 1-factorization of λK_{2n}, $\lambda > 1$, is it the union of λ 1-factorizations of K_{2n}? It is readily seen that the answer can be "no"; it suffices to take the 15 distinct 1-factors of K_6, remove the 5 1-factors of the unique 1-factorization, and observe that the remaining 10 1-factors cannot be partitioned into two 1-factorizations of K_6. A more general question would be as

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follows: given a 1-factorization of λK_{2n}, $\lambda > 1$, can it be written as a union of 1-factorizations of $\lambda' K_{2n}$ and $\lambda'' K_{2n}$ for some λ', $\lambda'' < \lambda$, for which $\lambda' + \lambda'' = \lambda$? If a 1-factorization cannot be written in this way, we call it *indecomposable*. We examine the existence of indecomposable 1-factorizations of K_{2n} in this paper, and show that there exist indecomposable 1-factorizations of λK_{2n} for arbitrarily high values of λ. We also settle existence of indecomposable 1-factorizations for $2 \leqslant \lambda \leqslant 6$, leaving a few small open cases.

2. Main results

A *1-factorization* of the complete multigraph λK_{2n} is a pair (V, F) where V is the vertex set of K_{2n}, and F is a collection of $\lambda(2n-1)$ 1-factors. A comprehensive survey of research on 1-factorizations of complete graphs is given in [3]. If no two members of F are identical as 1-factors (i.e., no 1-factors are "repeated"), the 1-factorization is said to be *simple*. We denote a 1-factorization of λK_{2n} by $OF(2n, \lambda)$; when it is indecomposable, we denote it by $IOF(2n, \lambda)$.

In what follows we need an auxiliary result on the existence of 1-factorizations of certain graphs. For $x \in Z_n$, define $|x|$ as x if $0 \leqslant x \leqslant \lfloor n/2 \rfloor$, and $-x$ if $\lfloor n/2 \rfloor < x < n$. For $n \geqslant 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, let $G(n, L)$ be the regular graph with vertex set Z_n and edge set E given by $\{x, y\} \in E$ if and only if $|x - y| \in L$.

LEMMA 1. *Let n be an even positive integer, and let $\emptyset \neq L \subseteq \{1, 2, \ldots, n/2\}$. Then $G(n, L)$ has a 1-factorization if and only if $n/\gcd(j, n)$ is even for at least one $j \in L$.*

PROOF. See [2, 4].

Our first result shows that there are indecomposable 1-factorizations with arbitrarily high index λ.

THEOREM 2. *There exists a simple $IOF(2n, n-1)$ whenever $2n-1$ is a prime.*

PROOF. Let $V = Z_{2n-1} \cup \{\infty\}$, and let θ be a generator of Z_{2n-1}. Let F be the 1-factor $\{\{2i-1, 2i\} | 1 \leqslant i < n\} \cup \{\{0, \infty\}\}$. Let $F_i = \theta^i F$, $0 \leqslant i \leqslant n-2$, and let $FF = \{F_i | 0 \leqslant i \leqslant n-2\} \bmod 2n-1$. Then (V, FF) is an $OF(2n, n-1)$, which by construction is simple. Let us show that it is also indecomposable. Assume that there exists an $OF(2n, \lambda)(V, F')$ with $F' \subseteq FF$, $\lambda < n-1$. Consider all of the pairs $\{x, y\}$, $x, y \in Z_{2n-1}$ having $|x - y| = 1$. There are $2n-1$

such pairs, each of which is contained in exactly λ factors of F'. On the other hand, F' contains m 1-factors for some $m < 2n - 1$ whose edges $\{x, y\}$ are such that $|x - y| = 1$, and each of these 1-factors contains $2n - 2$ such edges. Thus we have $\lambda(2n - 1) = m(2n - 2)$, which is a contradiction.

Any nonempty set of edges of a 1-factor F is a *subfactor* of F. An $OF(2n, \lambda)$ (V, F) is said to be a *sub-OF* of an $OF(2s, \lambda)$ (W, G) if $V \subseteq W$, and for each $f \in F$ there is a $g \in G$ such that f is a subfactor of g. We also say that (V, F) is *embedded* in (W, G).

THEOREM 3. *Any $OF(2n, \lambda)$ can be embedded in a simple $OF(2s, \lambda)$ for $s \geqslant 2n$ provided $\lambda \leqslant 2n - 1$.*

PROOF. Let (V, F) be an $OF(2n, \lambda)$ with $V = \{v_1, v_2, \ldots, v_{2n}\}$ and $F = \{F_{i,j} | 1 \leqslant i \leqslant 2n - 1, 1 \leqslant j \leqslant \lambda\}$. Note that (V, F) is not required to be simple. However, we may assume without loss of generality that if (V, F) contains repeated 1-factors whenever $F_{i,j}$ and $F_{k,l}$ are identical as 1-factors, then $i \neq k$.

Let $w = s - n$, and consider the complete graph K_{2w} with vertex set Z_{2w} (we assume here that $V \cap Z_{2w} = \varnothing$). The graph $G(2w, \{w - n + 1, w - n + 2, \ldots, w\})$ is regular of degree $2n - 1$, and by Lemma 1 has a 1-factorization. Let H_i, $1 \leqslant i \leqslant 2n - 1$ be the 1-factors of such a 1-factorization. We construct a set K of 1-factors on the $2s$ vertices $V \cup Z_{2w}$, taking $K = \{K_{i,j} = F_{i,j} \cup H_i | 1 \leqslant i \leqslant 2n - 1, 1 \leqslant j \leqslant \lambda\}$. K is a set of $\lambda(2n - 1)$ distinct 1-factors.

The remaining 1-factors involve edges between V and Z_{2w}, and are constructed as follows. Let $A = \{A_r | 1 \leqslant r \leqslant w - n\}$ be a set of $w - n$ disjoint pairs: $A_r = \{a_r, b_r\}$, $a_r, b_r \in Z_{2w}$, $|a_r - b_r| = r$, $A_r \cap A_q = \varnothing$ for $r \neq q$. Such a set A always exists and is easy to construct by taking a Skolem or hooked Skolem $(w - 1)$-sequence and omitting from it the $n - 1$ pairs with largest differences $w - n + 1, \ldots, w - 1$.

Let $Y = \{y_1, \ldots, y_{2n}\} = Z_{2w} - \bigcup_{r=1}^{w-n} A_r$. Define, for $i \in Z_{2w}$, $M_i = \{\{v_t, y_t + i\} | 1 \leqslant t \leqslant 2n\} \cup \{\{a_r + i, b_r + i\} | 1 \leqslant r \leqslant w - n\}$. Clearly M_i is a 1-factor of K_{2s} on $V \cup Z_{2w}$. Now let C be any $\lambda \times 2n$ Latin rectangle, and let b_j be the permutation given by the jth row of C. Let $M_{i,j} = \{\{v_t, y_{tb_j} + i\} | 1 \leqslant t \leqslant 2n\} \cup \{\{a_r + i, b_r + i\} | 1 \leqslant r \leqslant w - n\}$. It is straightforward to verify that $M = \{M_{i,j} | i \in Z_{2w}, 1 \leqslant j \leqslant \lambda\}$ is a set of $2w\lambda$ distinct 1-factors, and further that $(V \cup Z_{2w}, K \cup M)$ is a simple $OF(2s, \lambda)$ containing the (not necessarily simple) $OF(2n, \lambda)(V, F)$.

COROLLARY 4. *If there exists an $IOF(2n, \lambda)$ with $\lambda \leqslant 2n - 1$, there exists a simple $IOF(2s, \lambda)$ for $s \geqslant 2n$.*

Before proceeding further, we observe that any $OF(4, \lambda)$, $\lambda > 1$, is trivially decomposable. Thus if an $IOF(2n, \lambda)$ exists for $\lambda > 1$, then $n \geqslant 3$.

THEOREM 5. *A simple* $IOF(2n, 2)$ *exists if and only if* $n \geqslant 3$.

PROOF. An $IOF(6, 2)$ exists by Theorem 2 (and also by the remarks in the introduction). Theorem 3 then gives a simple $IOF(2n, 2)$ for all $n \geqslant 6$. It remains only to exhibit solutions for $n = 4$ and 5. One simple $IOF(8, 2)$ has $V = Z_7 \cup \{\infty\}$, and $F = F' \cup F''$ where $F' = \{\{0, \infty\}, \{1, 6\}, \{2, 3\}, \{4, 5\} \bmod 7\}$ and $F'' = \{\{0, \infty\}, \{1, 5\}, \{2, 4\}, \{3, 6\} \bmod 7\}$.

An $IOF(10, 2)$ is developed similarly with $F' = \{\{0, \infty\}, \{1, 4\}, \{2, 6\}, \{3, 7\}, \{5, 8\} \bmod 9\}$ and $F'' = \{\{0, \infty\}, \{1, 3\}, \{2, 4\}, \{5, 6\}, \{7, 8\} \bmod 9\}$.

THEOREM 6. *A simple* $IOF(2n, 3)$ *exists if and only if* $n \geqslant 4$.

PROOF. An exhaustive search easily verifies that there is no $IOF(6, 3)$, whether simple or not. Theorem 2 yields a simple $IOF(8, 3)$, and then Theorem 3 gives a simple $IOF(2n, 3)$ for every $n \geqslant 8$. It remains only to give simple $IOF(2n, 3)$ for $n = 5, 6$, and 7; these are given in the appendix.

THEOREM 7. *A simple* $IOF(2n, 4)$ *exists if and only if* $n \geqslant 4$.

PROOF. Necessity is obvious. For sufficiency, Theorem 3 together with a simple $IOF(2n, 4)$ for $n = 4, 5, 6$, and 7 is enough; these IOFs are given in the appendix.

THEOREM 8. *A simple* $IOF(2n, 5)$ *exists for* $n = 5, 6, 7$ *and all* $n \geqslant 10$.

PROOF. A simple $IOF(12, 5)$ exists by Theorem 2, and a simple $IOF(10, 5)$ and $IOF(14, 5)$ are given in the appendix; Theorem 3 then gives simple $IOF(2n, 5)$ for all $n \geqslant 10$.

THEOREM 9. *A simple* $IOF(2n, 6)$ *exists for all* $n \geqslant 6$.

PROOF. A simple $IOF(14, 6)$ exists by Theorem 2, and a simple $IOF(12, 6)$ is given in the appendix. Theorem 3, together with a *non*simple $IOF(8, 6)$ given in the appendix, give simple $IOF(2n, 6)$ for all $n \geqslant 8$.

Of course, the application of the techniques developed does not merely apply to small values of λ; for example, we have

THEOREM 10. (i) *A simple* $IOF(2n, \lambda)$ *for* $\lambda = 8$ *or* 9 *exists for* $n = 6, 7$ *and all* $n \geqslant 12$.

(ii) *A simple* $IOF(2n, 10)$ *exists for* $n = 7$ *and all* $n \geqslant 14$.

(iii) *A simple* $IOF(2n, 12)$ *exists for all* $n \geqslant 16$.

PROOF. Simple $IOF(12,8)$, $IOF(14,8)$, $IOF(12,9)$, $IOF(14,9)$ and $IOF(14,10)$ and a nonsimple $IOF(16,12)$ are given in the appendix. The rest follows from Theorem 3.

3. Conclusions and open problems

There are exactly three nonisomorphic $OF(6,2)$'s of which exactly one is indecomposable [5]. There exists no indecomposable $IOF(6,3)$, whether simple or not. This can be determined by exhaustive search. Virtually nothing else is known about the enumeration of $OF(2n, \lambda)$'s for $\lambda > 1$.

One might ask what is the maximum $\lambda = \lambda(2n)$ such that there exists a simple $IOF(2n, \lambda)$. Taking all distinct 1-factors of K_{2n} obviously produces a simple $OF(2n, (2n-3)!!)$, where $n!!$ is the product of all odd numbers up to n. Thus $\lambda(2n) \leqslant (2n-3)!! - 1$. One has $\lambda(6) = 2$, but nothing else seems to be known about $\lambda(2n)$.

Let us mention one other (undoubtedly difficult) problem concerning 1-factorizations of λK_{2n}. Suppose $P = (p_1, p_2, \ldots, p_k)$ is a partition of the number $(2n-3)!!$. Is it possible to partition the 1-factors of K_{2n} on V into subsets $F_1, \ldots, F_k$ such that each (V, F_i) is an $IOF(2n, p_i)$? Let us call P *admissible* if the answer is yes. It is easily seen that $(1,2)$ is the only admissible partition for $n = 3$. Cameron [1] has shown that for $n = 4$, the partition (1*15) is admissible but it follows from Theorems 5–7 that many other partitions are admissible for $n = 4$.

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Appendix

We list here the "base" 1-factors for the $IOF(2n, \lambda)$'s referred to in Section 2. These were produced using a straightforward backtracking algorithm by computer. The vertex set is always taken to be $Z_{2n-1} \cup \{\infty\}$.

Simple $IOF(10,3)$

$\{0,\infty\}$	$\{1,8\}$	$\{2,3\}$	$\{4,5\}$	$\{6,7\}$
$\{0,\infty\}$	$\{1,3\}$	$\{2,7\}$	$\{4,6\}$	$\{5,8\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,5\}$	$\{3,8\}$	$\{4,7\}$

Simple $IOF(12,3)$

$\{0,\infty\}$	$\{1,7\}$	$\{2,3\}$	$\{4,5\}$	$\{6,10\}$	$\{8,9\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,9\}$	$\{3,5\}$	$\{4,6\}$	$\{8,10\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,7\}$	$\{3,10\}$	$\{5,8\}$	$\{6,9\}$

Simple $IOF(14,3)$

$\{0,\infty\}$	$\{1,12\}$	$\{2,3\}$	$\{4,5\}$	$\{6,7\}$	$\{8,10\}$	$\{9,11\}$
$\{0,\infty\}$	$\{1,10\}$	$\{2,11\}$	$\{3,12\}$	$\{4,7\}$	$\{5,8\}$	$\{6,9\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,10\}$	$\{3,8\}$	$\{4,9\}$	$\{5,11\}$	$\{6,12\}$

Simple $IOF(8,4)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,6\}$	$\{4,5\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,3\}$	$\{5,6\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,4\}$	$\{3,5\}$
$\{0,\infty\}$	$\{1,5\}$	$\{2,4\}$	$\{3,6\}$

Simple $IOF(10,4)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$	$\{7,8\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,7\}$	$\{3,5\}$	$\{6,8\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,6\}$	$\{3,5\}$	$\{4,8\}$
$\{0,\infty\}$	$\{1,3\}$	$\{2,6\}$	$\{4,7\}$	$\{5,8\}$

Simple $IOF(12,4)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,10\}$	$\{6,7\}$	$\{8,9\}$
$\{0,\infty\}$	$\{1,10\}$	$\{2,8\}$	$\{3,5\}$	$\{4,6\}$	$\{7,9\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,5\}$	$\{3,8\}$	$\{6,9\}$	$\{7,10\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,6\}$	$\{3,10\}$	$\{4,8\}$	$\{5,9\}$

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Simple $IOF(14,4)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,11\}$	$\{6,12\}$	$\{7,8\}$	$\{9,10\}$
$\{0,\infty\}$	$\{1,3\}$	$\{2,10\}$	$\{4,12\}$	$\{5,7\}$	$\{6,8\}$	$\{9,11\}$
$\{0,\infty\}$	$\{1,8\}$	$\{2,5\}$	$\{3,6\}$	$\{4,11\}$	$\{7,10\}$	$\{9,12\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,10\}$	$\{3,12\}$	$\{4,8\}$	$\{5,9\}$	$\{7,11\}$

Simple $IOF(10,5)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$	$\{7,8\}$
$\{0,\infty\}$	$\{1,5\}$	$\{2,6\}$	$\{3,7\}$	$\{4,8\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,3\}$	$\{5,7\}$	$\{6,8\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,5\}$	$\{3,6\}$	$\{4,8\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,4\}$	$\{3,5\}$	$\{6,8\}$

Simple $IOF(14,5)$

$\{0,\infty\}$	$\{1,8\}$	$\{2,3\}$	$\{4,5\}$	$\{6,7\}$	$\{9,10\}$	$\{11,12\}$
$\{0,\infty\}$	$\{1,3\}$	$\{2,4\}$	$\{5,7\}$	$\{6,12\}$	$\{8,10\}$	$\{9,11\}$
$\{0,\infty\}$	$\{1,11\}$	$\{2,12\}$	$\{3,10\}$	$\{4,7\}$	$\{5,8\}$	$\{6,9\}$
$\{0,\infty\}$	$\{1,10\}$	$\{2,11\}$	$\{3,7\}$	$\{4,8\}$	$\{5,9\}$	$\{6,12\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,8\}$	$\{3,11\}$	$\{4,9\}$	$\{5,10\}$	$\{7,12\}$

Nonsimple $IOF(8,6)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$	twice
$\{0,\infty\}$	$\{1,4\}$	$\{2,5\}$	$\{3,6\}$	
$\{0,\infty\}$	$\{1,3\}$	$\{2,5\}$	$\{4,6\}$	three times

Simple $IOF(12,6)$

$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$	$\{7,9\}$	$\{8,10\}$
$\{0,\infty\}$	$\{1,3\}$	$\{2,4\}$	$\{5,6\}$	$\{7,8\}$	$\{9,10\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,5\}$	$\{3,6\}$	$\{7,9\}$	$\{8,10\}$
$\{0,\infty\}$	$\{1,4\}$	$\{2,8\}$	$\{3,6\}$	$\{5,9\}$	$\{7,10\}$
$\{0,\infty\}$	$\{1,5\}$	$\{2,9\}$	$\{3,7\}$	$\{4,8\}$	$\{6,10\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,7\}$	$\{3,8\}$	$\{4,9\}$	$\{5,10\}$

Simple $IOF(12, 8)$

$\{0, \infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,10\}$	$\{6,7\}$	$\{8,9\}$
$\{0, \infty\}$	$\{1,2\}$	$\{3,8\}$	$\{4,5\}$	$\{6,7\}$	$\{9,10\}$
$\{0, \infty\}$	$\{1,5\}$	$\{2,7\}$	$\{3,9\}$	$\{4,8\}$	$\{6,10\}$
$\{0, \infty\}$	$\{1,6\}$	$\{2,7\}$	$\{3,10\}$	$\{4,8\}$	$\{5,9\}$
$\{0, \infty\}$	$\{1,9\}$	$\{2,7\}$	$\{3,6\}$	$\{4,10\}$	$\{5,8\}$
$\{0, \infty\}$	$\{1,10\}$	$\{2,4\}$	$\{3,5\}$	$\{6,8\}$	$\{7,9\}$
$\{0, \infty\}$	$\{1,9\}$	$\{2,10\}$	$\{3,6\}$	$\{4,7\}$	$\{5,8\}$
$\{0, \infty\}$	$\{1,10\}$	$\{2,6\}$	$\{3,5\}$	$\{4,8\}$	$\{7,9\}$

Simple $IOF(14, 8)$

$\{0, \infty\}$	$\{1,11\}$	$\{2,3\}$	$\{4,5\}$	$\{6,7\}$	$\{8,9\}$	$\{10,12\}$
$\{0, \infty\}$	$\{1,3\}$	$\{2,12\}$	$\{4,5\}$	$\{6,7\}$	$\{8,9\}$	$\{10,11\}$
$\{0, \infty\}$	$\{1,4\}$	$\{2,6\}$	$\{3,7\}$	$\{5,8\}$	$\{9,11\}$	$\{10,12\}$
$\{0, \infty\}$	$\{1,3\}$	$\{2,4\}$	$\{5,8\}$	$\{6,10\}$	$\{7,11\}$	$\{9,12\}$
$\{0, \infty\}$	$\{1,11\}$	$\{2,12\}$	$\{3,5\}$	$\{4,8\}$	$\{6,10\}$	$\{7,9\}$
$\{0, \infty\}$	$\{1,5\}$	$\{2,9\}$	$\{3,8\}$	$\{4,11\}$	$\{6,10\}$	$\{7,12\}$
$\{0, \infty\}$	$\{1,9\}$	$\{2,7\}$	$\{3,8\}$	$\{4,12\}$	$\{5,10\}$	$\{6,11\}$
$\{0, \infty\}$	$\{1,7\}$	$\{2,8\}$	$\{3,9\}$	$\{4,10\}$	$\{5,11\}$	$\{6,12\}$

Simple $IOF(12, 9)$

$\{0, \infty\}$	$\{1,4\}$	$\{2,10\}$	$\{3,5\}$	$\{6,8\}$	$\{7,9\}$
$\{0, \infty\}$	$\{1,9\}$	$\{2,4\}$	$\{3,5\}$	$\{6,8\}$	$\{7,10\}$
$\{0, \infty\}$	$\{1,2\}$	$\{3,6\}$	$\{4,7\}$	$\{5,8\}$	$\{9,10\}$
$\{0, \infty\}$	$\{1,10\}$	$\{2,4\}$	$\{3,6\}$	$\{5,8\}$	$\{7,9\}$
$\{0, \infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$	$\{7,8\}$	$\{9,10\}$
$\{0, \infty\}$	$\{1,6\}$	$\{2,3\}$	$\{4,9\}$	$\{5,10\}$	$\{7,8\}$
$\{0, \infty\}$	$\{1,6\}$	$\{2,7\}$	$\{3,8\}$	$\{4,9\}$	$\{5,10\}$
$\{0, \infty\}$	$\{1,5\}$	$\{2,9\}$	$\{3,7\}$	$\{4,8\}$	$\{6,10\}$
$\{0, \infty\}$	$\{1,7\}$	$\{2,6\}$	$\{3,10\}$	$\{4,8\}$	$\{6,10\}$

Simple $IOF(14,9)$

$\{0,\infty\}$	$\{1,4\}$	$\{2,3\}$	$\{5,7\}$	$\{6,8\}$	$\{9,10\}$	$\{11,12\}$
$\{0,\infty\}$	$\{1,2\}$	$\{3,6\}$	$\{4,5\}$	$\{7,9\}$	$\{8,10\}$	$\{11,12\}$
$\{0,\infty\}$	$\{1,2\}$	$\{3,4\}$	$\{5,8\}$	$\{6,7\}$	$\{9,11\}$	$\{10,12\}$
$\{0,\infty\}$	$\{1,11\}$	$\{2,4\}$	$\{3,6\}$	$\{5,8\}$	$\{7,9\}$	$\{10,12\}$
$\{0,\infty\}$	$\{1,9\}$	$\{2,10\}$	$\{3,6\}$	$\{4,8\}$	$\{5,12\}$	$\{7,11\}$
$\{0,\infty\}$	$\{1,9\}$	$\{2,11\}$	$\{3,7\}$	$\{4,10\}$	$\{5,8\}$	$\{6,12\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,6\}$	$\{3,8\}$	$\{4,11\}$	$\{5,10\}$	$\{9,12\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,10\}$	$\{3,9\}$	$\{4,8\}$	$\{5,12\}$	$\{7,11\}$
$\{0,\infty\}$	$\{1,8\}$	$\{2,7\}$	$\{3,11\}$	$\{4,10\}$	$\{5,9\}$	$\{7,12\}$

Simple $IOF(14,10)$

$\{0,\infty\}$	$\{1,12\}$	$\{2,3\}$	$\{4,8\}$	$\{5,7\}$	$\{6,9\}$	$\{10,11\}$
$\{0,\infty\}$	$\{1,12\}$	$\{2,6\}$	$\{3,5\}$	$\{4,7\}$	$\{8,9\}$	$\{10,11\}$
$\{0,\infty\}$	$\{1,6\}$	$\{2,5\}$	$\{3,4\}$	$\{7,9\}$	$\{8,10\}$	$\{11,12\}$
$\{0,\infty\}$	$\{1,2\}$	$\{3,8\}$	$\{4,7\}$	$\{5,6\}$	$\{9,11\}$	$\{10,12\}$
$\{0,\infty\}$	$\{1,11\}$	$\{2,4\}$	$\{3,5\}$	$\{6,12\}$	$\{7,8\}$	$\{9,10\}$
$\{0,\infty\}$	$\{1,11\}$	$\{2,12\}$	$\{3,10\}$	$\{4,7\}$	$\{5,8\}$	$\{6,9\}$
$\{0,\infty\}$	$\{1,10\}$	$\{2,6\}$	$\{3,12\}$	$\{4,8\}$	$\{5,9\}$	$\{7,11\}$
$\{0,\infty\}$	$\{1,9\}$	$\{2,7\}$	$\{3,8\}$	$\{4,12\}$	$\{5,10\}$	$\{6,11\}$
$\{0,\infty\}$	$\{1,7\}$	$\{2,8\}$	$\{3,9\}$	$\{4,10\}$	$\{5,11\}$	$\{6,12\}$
$\{0,\infty\}$	$\{1,8\}$	$\{2,11\}$	$\{3,7\}$	$\{4,9\}$	$\{5,10\}$	$\{6,12\}$

Nonsimple $IOF(16,12)$

$\{0,\infty\}$	$\{1,14\}$	$\{2,3\}$	$\{4,5\}$	$\{6,7\}$	$\{8,9\}$	$\{10,11\}$	$\{12,13\}$	twice
$\{0,\infty\}$	$\{1,4\}$	$\{2,7\}$	$\{3,14\}$	$\{5,13\}$	$\{6,12\}$	$\{8,10\}$	$\{9,11\}$	five times
$\{0,\infty\}$	$\{1,6\}$	$\{2,14\}$	$\{3,7\}$	$\{4,10\}$	$\{5,13\}$	$\{8,11\}$	$\{9,12\}$	
$\{0,\infty\}$	$\{1,10\}$	$\{2,5\}$	$\{3,14\}$	$\{4,9\}$	$\{6,13\}$	$\{7,11\}$	$\{8,12\}$	
$\{0,\infty\}$	$\{1,7\}$	$\{2,12\}$	$\{3,14\}$	$\{4,9\}$	$\{5,10\}$	$\{6,13\}$	$\{8,11\}$	
$\{0,\infty\}$	$\{1,11\}$	$\{2,8\}$	$\{3,14\}$	$\{4,10\}$	$\{5,12\}$	$\{6,9\}$	$\{7,13\}$	
$\{0,\infty\}$	$\{1,9\}$	$\{2,10\}$	$\{3,7\}$	$\{4,14\}$	$\{5,13\}$	$\{6,12\}$	$\{8,11\}$	

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Department of Computer Science
University of Waterloo
Waterloo, Ontario, N2L 3G1
Canada

Department of Mathematical Sciences
McMaster University
Hamilton, Ontario, L8S 4K1
Canada