

FUNCTIONAL EQUATIONS OF GENERALIZED EPSTEIN ZETA FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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Let $S^{(n)}$ be the matrix of a positive definite quadratic form and $(\rho_1, \dots, \rho_{r-1}) \in \mathcal{C}^{r-1}$. Define

$$(1) \quad \zeta_{n_1, \dots, n_r}(S, \rho_1, \dots, \rho_{r-1}) = \sum \prod_{i=1}^{r-1} |S[U_i]|^{-\rho_i}.$$

Here the sum is over unimodular matrices $U^{(n)} = (U_i^{(n, N_i \rho_i)})$ which lie in a complete set of representatives for the equivalence relation $U \sim V$ if $U = VP$, with P unimodular and having block form

$$P = \begin{pmatrix} P_1^{(n_1)} & & * \\ \cdot & \cdot & \\ 0 & P_r^{(n_r)} & \end{pmatrix}.$$

The following notation shall be used throughout:

$A^{(n, m)}$ for an n by m matrix A

$A^{(n)} = A^{(n, n)}$

$|A|$ = determinant of A

$S[A] = {}^tASA$, tA being the transpose of A

$N_i = \sum_{j=1}^i n_j$, $i = 1, 2, \dots, r$, $N_r = n = \sum_{j=1}^r n_j$.

A unimodular matrix $U^{(n)}$ is one with integral entries and determinant ± 1 .

The function (1) is clearly a generalization of Epstein's zeta function, as well as a generalization of functions defined by Koecher [1], Maass [3], and Selberg [4], [5], [6]. It can also be viewed as an Eisenstein series; for which, see Langlands [2].

If $n_i = 1$ for all $i = 1, 2, \dots, r$ and $r = n$, denote the function defined by (1) as $\zeta_{(n)} = \zeta_{1, \dots, 1}$.

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We know from [7] that (1) converges for $\text{Re } \rho_i > \frac{1}{2}n$, for $i = 1, 2, \dots, r - 1$. It was also shown in [7] that

$$(2) \quad \zeta_{n_1, \dots, n_r}(S, \rho_{N_1}, \dots, \rho_{N_{r-1}}) = \zeta_{(n)}(S, \rho) \Big|_{\rho_i = 0, \text{ for } i \notin \{N_1, \dots, N_r\}}.$$

Also from [7], every permutation σ of $\{1, 2, \dots, n\}$ induces a functional equation of $\zeta_{(n)}(S, \rho)$. More precisely, one introduces the new variables z_i , $i = 1, 2, \dots, n$ with $\rho_i = z_{i+1} - z_i + \frac{1}{2}$. That means

$$\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_n \end{pmatrix} = \begin{pmatrix} -1+1 & & & \\ & -1+1 & 0 & \\ & & \ddots & \\ & & & \ddots & \\ & & & & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \vdots \\ \vdots \\ \vdots \\ \frac{1}{2} \end{pmatrix} = Az + \frac{1}{2}.$$

Define $\phi(x) = \Gamma(x)\zeta(2x)$. Here ζ denotes the ordinary Riemann zeta function. Now define

$$\Phi'(z) = |S|^{z_n - \frac{1}{2}} \pi^{-2 \sum_{j=1}^n jz_j} \prod_{1 \leq i < j \leq n} \phi\left(z_j - z_i + \frac{1}{2}\right).$$

Then the functional equations of $\zeta_{(n)}(S, \rho) = \zeta'_{(n)}(S, z)$, for $\rho = Az + \frac{1}{2}$, are

$$\Phi'(z^\sigma) \zeta'_{(n)}(S, z^\sigma) = \Phi'(z) \zeta'_{(n)}(S, z),$$

where $(z^\sigma)_i = z_{\sigma(i)}$, that is,

$$z^\sigma = P^\sigma z, \text{ where } P^\sigma = (\delta_{\sigma(i)j}), \delta_{ij} \text{ being}$$

the usual Kronecker delta.

One can rewrite the functional equations in terms of the variables ρ by defining

$$\Phi(\rho) = \Phi'(z), \text{ for } \rho = Az + \frac{1}{2};$$

$$\rho^\sigma = AP^\sigma A^{-1} \left(\rho - \frac{1}{2} \right) + \frac{1}{2};$$

$$F(\rho, \sigma) = \frac{\Phi(\rho)}{\Phi(\rho^\sigma)}.$$

Then the functional equations become

$$(3) \quad \zeta_{(n)}(S, \rho^\sigma) = F(\rho, \sigma) \zeta_{(n)}(S, \rho).$$

Note that we have defined the map $\rho \rightarrow \rho^\sigma$ to be the map induced on the

ρ coordinates by the map $z \rightarrow z^\sigma$. It is clear that $(\rho^\sigma)_i \in \mathbb{Q}[\rho_1, \dots, \rho_{n-1}]$, for $i = 1, \dots, n - 1$.

Now equation (2) suggests that some of the equations (3) induce functional equations or relations on the functions ζ_{n_1, \dots, n_r} . The purpose now is to discover which r -tuples $(n_1, \dots, n_r), (m_1, \dots, m_r)$ such that $\sum_{i=1}^r n_i = n = \sum_{i=1}^r m_i$ have the property that the corresponding zeta functions ζ_{n_1, \dots, n_r} and ζ_{m_1, \dots, m_r} are related by (3) for some σ . A result in this direction is proved in [7] for $r = 2$. Relations were obtained between $\zeta_{i, n-i}$ and $\zeta_{n-i, i}$ in that case for $i = 1, 2, \dots, n - 1$. This means that if $i = n - i$, a non-trivial functional equation is induced on $\zeta_{i, i}$ by some permutation of $\{1, 2, \dots, 2i\}$, in (3).

In order to state the problem more precisely, define $F_{n_1, \dots, n_r}(\rho, \sigma) = F(\rho, \sigma) \Big|_{\rho_j = 0, \text{ for } j \in \{N_1, \dots, N_r\}}$.

We say that for (n_1, \dots, n_r) and (m_1, \dots, m_r) such that $\sum_{i=1}^r n_i = n = \sum_{i=1}^r m_i$,

$$\zeta_{n_1, \dots, n_r} \sim \zeta_{m_1, \dots, m_r}$$

if and only if there exists a permutation σ of $\{1, 2, \dots, n\}$ such that setting $\rho_j = 0$, for $j \in \{N_1, \dots, N_r\}$ in (3), one obtains:

$$\begin{aligned} F_{n_1, \dots, n_r}(\rho, \sigma) \zeta_{n_1, \dots, n_r}(S, \rho_{N_1}, \dots, \rho_{N_{r-1}}) \\ = \zeta_{m_1, \dots, m_r}(S, \rho'_{M_1}, \dots, \rho'_{M_{r-1}}). \end{aligned}$$

Here $N_i = \sum_{j=1}^i n_j$ and $M_i = \sum_{j=1}^i m_j$. And $\rho'_{M_i} = (\rho^\sigma)_{M_i} \Big|_{\rho_j = 0, \text{ } j \in \{N_1, \dots, N_r\}}$;

So $\rho'_{M_i} \in \mathbb{Q}[\rho_{N_1}, \dots, \rho_{N_{r-1}}]$.

We shall soon see that “ \sim ” is indeed an equivalence relation.

Clearly $\zeta_{n_1, \dots, n_r} \sim \zeta_{m_1, \dots, m_r}$ under σ is equivalent to the condition that the ρ 's which are set equal to zero to produce ζ_{n_1, \dots, n_r} from $\zeta^{(n)}$ must be sent by the map $\rho \rightarrow \rho^\sigma$ to the ρ 's which are set equal to zero to produce ζ_{m_1, \dots, m_r} from $\zeta^{(n)}$.

Define $\mathcal{H}_{n_1, \dots, n_r}$ to be the set of $n - r$ hyperplanes defined by $z_{j+1} - z_j + \frac{1}{2} = 0$, for $j \in \{N_1, \dots, N_r\}$. Recall that $N_i = \sum_{j=1}^i n_j$, $i = 1, 2, \dots, r$. It is obvious that $\zeta_{n_1, \dots, n_r} \sim \zeta_{m_1, \dots, m_r}$ under σ is equivalent to requiring that the map $z \rightarrow z^\sigma$ take the set $\mathcal{H}_{n_1, \dots, n_r}$ one-to-one, onto the set $\mathcal{H}_{m_1, \dots, m_r}$.

THEOREM 1. For every (n_1, \dots, n_r) and (m_1, \dots, m_r) such that $\sum_{i=1}^r n_i = n = \sum_{i=1}^r m_i$.

$$\zeta_{n_1, \dots, n_r} \sim \zeta_{m_1, \dots, m_r}$$

if and only if there exists a permutation μ of $\{1, 2, \dots, r\}$ such that $n_i = m_{\mu(i)}$.

Proof. Suppose for some permutation μ of $\{1, 2, \dots, r\}$, $n_i = m_{\mu(i)}$. Define a permutation σ_μ of $\{1, 2, \dots, n\}$ as follows. Set $N_i = \sum_{j=1}^i n_j$ and $M_k = \sum_{j=1}^k m_j$. Then define

$$\sigma_\mu = \left(\begin{array}{c} \dots N_i - (n_i - 1) \dots N_i - j \dots N_i \dots \\ \dots M_{\mu(i)} - (n_i - 1) \dots M_{\mu(i)} - j \dots M_{\mu(i)} \dots \end{array} \right).$$

Then for $1 \leq j < n_i = m_{\mu(i)}$,

$$\begin{aligned} z_{\sigma_\mu(N_i - j + 1)} - z_{\sigma_\mu(N_i - j)} + \frac{1}{2} \\ = z_{M_{\mu(i)} - j + 1} - z_{M_{\mu(i)} - j} + \frac{1}{2}. \end{aligned}$$

This means that $z \rightarrow z^{\sigma_\mu}$ takes $\mathcal{H}_{n_1, \dots, n_r}$ one-to-one, onto $\mathcal{H}_{m_1, \dots, m_r}$.

Suppose next that $\zeta_{n_1, \dots, n_r} \sim \zeta_{m_1, \dots, m_r}$ under σ . Then $z \rightarrow z^\sigma$ takes $\mathcal{H}_{n_1, \dots, n_r}$ one-to-one, onto $\mathcal{H}_{m_1, \dots, m_r}$. The first claim is that there exists a permutation μ of $\{1, 2, \dots, r\}$ such that if

$$N_i = \sum_{j=1}^i n_j, \quad M_i = \sum_{j=1}^i m_j,$$

then $\sigma(N_i) = M_{\mu(i)}$.

To see this, note that

$$\left(z_{N_i} - z_{N_{i-1}} + \frac{1}{2} = 0 \right) \in \mathcal{H}_{n_1, \dots, n_r}, \quad \text{but} \quad \left(z_{N_{i+1}} - z_{N_i} + \frac{1}{2} = 0 \right) \notin \mathcal{H}_{n_1, \dots, n_r}.$$

And for all $k \in \{1, 2, \dots, n\} - \{N_1, N_2, \dots, N_r\}$,

$$\left(z_k - z_{k-1} + \frac{1}{2} = 0 \right) \in \mathcal{H}_{n_1, \dots, n_r}, \quad \text{and} \quad \left(z_{k+1} - z_k + \frac{1}{2} = 0 \right) \in \mathcal{H}_{n_1, \dots, n_r}.$$

Thus for σ to send $\mathcal{H}_{n_1, \dots, n_r}$ one-to-one, onto $\mathcal{H}_{m_1, \dots, m_r}$, it is necessary that $\sigma(N_i) = M_{\mu(i)}$, for some permutation μ of $\{1, 2, \dots, r\}$.

The next claim is that if $n_i > 1$, then $m_{\mu(i)} > 1$, and $\sigma(N_i - 1) = M_{\mu(i)} - 1$. If this is not the case, then $\left(0 = z_{N_i} - z_{N_{i-1}} + \frac{1}{2} \right) \in \mathcal{H}_{n_1, \dots, n_r}$ is mapped by

σ onto $(0 = z_{M_{\mu(i)}} - z_{\sigma(N_i-1)} + \frac{1}{2}) \notin \mathcal{H}_{m_1, \dots, m_r}$. This is a contradiction.

Thus, by induction, we obtain, for $1 \leq j < n_i$, that $\sigma(N_i - j) = M_{\mu(i)} - j$, $m_{\mu(i)} \geq n_i$. It must be shown that $m_{\mu(i)} = n_i$. If this were not so, then $n_i < m_{\mu(i)}$ and $0 = z_{M_{\mu(i)} - (n_i - 1)} - z_{M_{\mu(i)} - n_i} + \frac{1}{2} \in \mathcal{H}_{m_1, \dots, m_r}$. Let $M_{\mu(i)} - n_i = \sigma(x)$. We know that $M_{\mu(i)} - (n_i - 1) = \sigma(N_i - (n_i - 1))$. Now where can x lie in the interval between 1 and n ? It cannot lie between N_{k-1} and N_k for any k . If it did, then $\sigma(x)$ would be between $M_{\mu(k)}$ and $M_{\mu(k)} - m_{\mu(k)}$. This means x cannot exist, a contradiction.

Note that Theorem 1 implies that “ \sim ” is an equivalence relation.

Now define $[n_1, \dots, n_r]$ to be the equivalence class of all ζ_{m_1, \dots, m_r} which are related to ζ_{n_1, \dots, n_r} by \sim .

Define $C(n_1, \dots, n_r)$ to be the number of elements of $[n_1, \dots, n_r]$. And define $N(n, r)$ to be the number of distinct classes $[n_1, \dots, n_r]$ for fixed n and r .

COROLLARY. $N(n, r)$ is the number of partitions of n into r parts (disregarding order).

Thus $N(n, r)$ is also equal to the number of partitions of n having largest summand r .

Now define $E(n_1, \dots, n_r)$ as the number of functional equations of ζ_{n_1, \dots, n_r} induced by functional equations $\zeta_{(n)}(\rho) \sim \zeta_{(n)}(\rho^\sigma)$, for permutations σ of $\{1, 2, \dots, n\}$.

And lastly define $G(n, r)$ to be the number of functions ζ_{n_1, \dots, n_r} for fixed n and r . Clearly $G(n, r) = \binom{n-1}{r-1}$.

Given (n_1, \dots, n_r) , let τ permute the indices $\{1, 2, \dots, r\}$ so that

$$\begin{aligned} n_{\tau(1)} = n_{\tau(2)} = \dots = n_{\tau(r_1)} < n_{\tau(r_1+1)} = n_{\tau(r_1+2)} = \dots = n_{\tau(r_1+r_2)} < \dots \\ \dots = n_{\tau(r_1+r_2+\dots+r_{s-1})} < n_{\tau(r_1+\dots+r_{s-1}+1)} = \dots \\ \dots = n_{\tau(r_1+\dots+r_s)}. \end{aligned}$$

That is, using our usual notation for (r_1, r_2, \dots, r_s) , namely $R_j = \sum_1^j r_i$, we have

$$n_{\tau(R_{j+1})} = n_{\tau(R_{j+2})} = \dots = n_{\tau(R_{j+1})} < n_{\tau(R_{j+1}+1)},$$

for $j = 0, 1, \dots, s$. Here $R_0 = 0$.

THEOREM 2. $E(n_1, \dots, n_r) = (r_1)!(r_2)! \dots (r_s)!$.

Proof. The problem is to count the number of permutations σ of $\{1, 2, \dots, n\}$ such that $z \rightarrow z^\sigma$ fixes $\mathcal{H}_{n_1, \dots, n_r}$. By the proof of Theorem 1, such a permutation σ has the following properties: $\sigma(N_i) = N_{\mu(i)}$, for some permutation μ of $\{1, 2, \dots, r\}$. And $\sigma(N_i - j) = N_{\mu(i)} - j$, for $1 \leq j < n_i$. Finally $n_{\mu(i)} = n_i$.

Therefore, if $i = 1, \dots, s$,

$$\mu(\tau(R_i + j)) \in \{\tau(R_i + j) \mid j = 1, 2, \dots, r_{i+1}\}, \quad j = 1, 2, \dots, r_{i+1}.$$

There are exactly $(r_1)!(r_2)! \cdots (r_s)!$ such μ , and therefore the same number of σ .

COROLLARY. $C(n_1, \dots, n_r) = \frac{r!}{E(n_1, \dots, n_r)}.$

Proof. Clear.

We shall consider some examples:

1. $r = 2, n > 1$.

$$N(n, 2) = \left[\frac{n}{2} \right]$$

$$E(n_1, n_2) = 1 + \delta_{n_1 n_2}$$

$$G(n, 2) = n - 1$$

2. $r = n - 1, n > 1$.

$$N(n, n - 1) = 1$$

$$E(1, \dots, 1, 2) = (n - 1)!$$

$$G(n, n - 1) = n - 1$$

3. $n = 7$

- a. $r = 5$

$$N(7, 5) = 2$$

$$E(2, 2, 1, 1, 1) = 3! 2! \qquad C(2, 2, 1, 1, 1) = 10$$

$$E(3, 1, 1, 1, 1) = 4! \qquad C(3, 1, 1, 1, 1) = 5$$

$$G(7, 5) = 15$$

- b. $r = 4$

$$N(7, 4) = 3$$

$$E(4, 1, 1, 1) = 3! \qquad C(4, 1, 1, 1) = 4$$

$$E(3, 2, 1, 1) = 2! \qquad C(3, 2, 1, 1) = 12$$

$$E(2, 2, 2, 1) = 3! \qquad C(2, 2, 2, 1) = 4$$

$$G(7, 4) = 20$$

c. $\gamma = 3$

$$N(7, 3) = 4$$

$$E(5, 1, 1) = 2$$

$$E(4, 2, 1) = 1$$

$$E(3, 3, 1) = 2$$

$$E(3, 2, 2) = 2$$

$$G(7, 3) = 15$$

$$C(5, 1, 1) = 3$$

$$C(4, 2, 1) = 3!$$

$$C(3, 3, 1) = 3$$

$$C(3, 2, 2) = 3$$

Added in proof (October 18, 1971):

For another way of obtaining the results in this paper, see

Maass, H., *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Mathematics, Vol. 216, New York, Springer Verlag, 1971.

BIBLIOGRAPHY

- [1] Koecher, Max, "Über Dirichlet-Reihen mit Funktionalgleichung", J. Reine Angew. Math. **192** (1953), 1-23.
- [2] Langlands, R.P., "Eisenstein Series", *Algebraic Groups and Discontinuous Subgroups*, Proc. Symp. in Pure Math. IX, Providence, R.I., A.M.S., 1966.
- [3] Maass, H., "Some Remarks on Selberg's Zeta Functions", Proc. Internatl. Conf. on Several Complex Variables, U. of Maryland, College Park, Md., 1970.
- [4] Selberg, A., "Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces with Applications to Dirichlet Series", J. Indian Math. Soc. (1956), 47-87.
- [5] ———, "A New Type of Zeta Function Connected with Quadratic Forms", Report of the Institute in Theory of Numbers, U. of Colorado, Boulder, Colo., 1959, 207-210.
- [6] ———, "Discontinuous Groups and Harmonic Analysis", Proc. Internatl. Cong. of Math., Stockholm, 1962, 177-189
- [7] Terras, A., "A Generalization of Epstein's Zeta Function", Nagoya Math., J. **42** (1971),

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