Glasgow Math. J. **59** (2017) 729–741.
doi:10.1017/S0017089516000525.

RELATIVE HILBERT CO-EFFICIENTS

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(Received 21 December 2015; accepted 12 July 2016; first published online 1 March 2017)

Abstract. Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let $I \subseteq J$ be two $\mathfrak{m}$-primary ideals with I a reduction of J. For $i = 0, \dots, d$, let $e_i^J(A)$ ($e_i^I(A)$) be the ith Hilbert coefficient of J (I), respectively. We call the number $c_i(I, J) = e_i^J(A) - e_i^I(A)$ the ith relative Hilbert coefficient of J with respect to I. If $G_I(A)$ is Cohen–Macaulay, then $c_i(I, J)$ satisfy various constraints. We also show that vanishing of some $c_i(I, J)$ has strong implications on depth $G_{J^n}(A)$ for $n \gg 0$.

1991 *Mathematics Subject Classification.* Primary 13A30, Secondary 13D40, 13D07.

1. Introduction. Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let J be an $\mathfrak{m}$-primary ideal. The Hilbert–Samuel function of A with respect to J is $H_J(n) = \lambda(A/J^{n+1})$, (here, $\lambda(-)$ denotes the length). It is well known that H_J is of polynomial type, *i.e*. there exists $P_J(X) \in \mathbb{Q}[X]$ such that $H_J(n) = P_J(n)$ for all $n \gg 0$. We write

$$P_J(X) = e_0^J(A)\binom{X+d}{d} - e_1^J(A)\binom{X+d-1}{d-1} + \cdots + (-1)^d e_d^J(A).$$

Then, the numbers $e_i^J(A)$ for $i = 0, 1, \dots, d$ are the Hilbert coefficients of A with respect to J. The number $e_0^J(A)$ is called the multiplicity of A with respect to J.

Now assume for convenience A has infinite residue field. Then, J has a minimal reduction $\mathfrak{q}$ generated by a system of parameters of A. Let $Gr_J(A) = \bigoplus_{n\geq 0} J^n/J^{n+1}$ be the associated graded ring of A with respect to J. There has been a lot of research regarding properties of J and $\mathfrak{q}$ and the depth properties of $Gr_J(A)$. For example, if $J^2 = \mathfrak{q}J$, then we say J has minimal multiplicity and in this case $Gr_J(A)$ is Cohen–Macaulay (see [**14**]).

In the context of this paper, we consider minimal reduction to be an absolute reduction of J. The main new idea of this paper is that it is convenient to consider reduction I of J not necessarily minimal but having the crucial property that $Gr_I(A)$ is Cohen–Macaulay. We note that if $\mathfrak{q}$ is a minimal reduction of J then it is generated by system of parameters of A and so necessarily $Gr_{\mathfrak{q}}(A)$ is Cohen–Macaulay.

As I is a reduction of J, then necessarily $e_0^I(A) = e_0^J(A)$. Let

$$c_i(I, J) = e_i^J(A) - e_i^I(A) \quad \text{for } i \geq 1.$$

Then, we say that $c_i(I, J)$ is the ith *relative Hilbert coefficient* of J with respect to I. We note that if $\mathfrak{q}$ is a minimal reduction of J, then $e_i^{\mathfrak{q}}(A) = 0$ for $i \geq 1$ and so $c_i(\mathfrak{q}, J) = e_i^J(A)$ for $i \geq 1$.

Let us recall the classical Northcott's inequality

$$e_1^J(A) \geq e_0^J(A) - \lambda(A/J)$$

(see [**10**]). But $e_0^J(A) = \lambda(A/\mathfrak{q})$ where $\mathfrak{q}$ is a minimal reduction of J. So Northcott's inequality can be rewritten as

$$e_1^J(A) \geq \lambda(J/\mathfrak{q}).$$

Furthermore, if equality hods, then by Huneke (see [**4**]) and Ooishi (see [**11**]), $Gr_J(A)$ is Cohen–Macaulay.

Our first result which easily follows from a deep result of Huckaba and Marley [**5**, Theorem 4.7] is the following.

THEOREM 1. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and let $I \subset J$ be $\mathfrak{m}$-primary ideals with I a reduction of J. Assume $Gr_I(A)$ is Cohen–Macaulay. Then,*

$$c_1(I, J) \geq \lambda(J/I).$$

If equality holds, then $Gr_J(A)$ is also Cohen–Macaulay.

We give a different proof of Theorem 1. Although it's longer than the proof using Huckaba and Marley result, it has the advantage that it's techniques can be generalized to prove other results.

In [**8**], Narita proved that $e_2^J(A) \geq 0$. Furthermore, if dim $= 2$, then $e_2^J(A) = 0$ if and only if the reduction number of J^n is 1 for $n \gg 0$. In particular, $Gr_{J^n}(A)$ is Cohen–Macaulay for $n \gg 0$.

Our generalization of Narita's result is:

THEOREM 2. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring and let $I \subset J$ be $\mathfrak{m}$-primary ideals with I a reduction of J. Assume $Gr_I(A)$ is Cohen–Macaulay. Then,*

$$c_2(I, J) \geq 0.$$

If $\dim A = 2$ *and $c_2(I, J) = 0$, then $Gr_{J^n}(A)$ is Cohen–Macaulay for $n \gg 0$.*

If we assume J is integrally closed, then we have the following result:

THEOREM 3. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension* 2. *Let $I \subset J$ be $\mathfrak{m}$-primary ideals with I a reduction of J. Assume J is integrally closed and $Gr_I(A)$ is*

https://doi.org/10.1017/S0017089516000525 Published online by Cambridge University Press

Cohen–Macaulay. If $c_1(I, J) = \lambda(J/I) + 1$*, then*

$$2\lambda(J/I) \leq \lambda(\widetilde{J^2}/I^2) \leq 2\lambda(J/I) + 1.$$

If $\lambda(\widetilde{J^2}/I^2) = 2\lambda(J/I) + 1$*, then* $Gr_{J^n}(A)$ *is Cohen–Macaulay for all* $n \gg 0$.

In Theorem 3, $\widetilde{J^2}$ denotes the Ratlif–Rush closure of J^2 (see [**13**]).

Narita gave an example which shows that $e_3^J(A)$ of an $\mathfrak{m}$-primary ideal J can be negative. Recall that an ideal J is said to be normal if J^n is integrally closed for all $n \geq 1$. In [**6**], Itoh proved that if $\dim A \geq 3$ and J is a normal ideal, then $e_3^J(A) \geq 0$. We prove:

THEOREM 4. *Let* $(A, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 3$. *Let* $I \subset J$ *be* $\mathfrak{m}$*-primary ideals with* I *a reduction of* J. *Assume* J *is normal and* $Gr_I(A)$ *is Cohen–Macaulay. Then,*

$$c_3(I, J) \geq 0.$$

If $d = 3$ *and* $c_3(I, J) = 0$*, then* $Gr_{J^n}(A)$ *is Cohen–Macaulay for all* $n \gg 0$.

In the main body of the paper, we consider a more general situation $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a finite map with $\dim A = \dim B$, I an $\mathfrak{m}$-primary A-ideal, J an $\mathfrak{n}$-primary ideal with IB a reduction of J. We now describe in brief the contents of this paper. In section 2, we discuss a few prelimary results that we need. In section 3, we prove Theorem 1. In section 4, we prove Theorems 2, 3. We prove Theorem 4 in section 5. Finally, in section 6, we give a few examples which illustrate our results.

2. Preliminaries.

Throughout this paper, we will use the following hypothesis unless otherwise stated.

HYPOTHESIS. Let $(A, \mathfrak{m}) \xrightarrow{\psi} (B, \mathfrak{n})$ be a local homomorphism of Cohen–Macaulay local rings such that

(1) B is finite as an A-module and $\dim A = \dim B$.
(2) $I \subset A$ and $J \subset B$ are ideals such that $\psi(I)B$ is a reduction of J.

REMARK 2.1. $\psi(I)B$ is not necessarily a minimal reduction of J.

REMARK 2.2. Note that $B/\mathfrak{n}$ is a finite extension of $A/\mathfrak{m}$. Set $\delta = \dim_{A/\mathfrak{m}} B/\mathfrak{n}$. Then, for any B-module N of finite length, we have $\lambda_A(N) = \delta\lambda_B(N)$.

The following result gives a necessary and sufficient condition for $\psi(I)B$ to be a reduction of J.

LEMMA 2.3. *Let* $\psi : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ *be a local homomorphism of Cohen–Macaulay local rings such that* B *is a finite* A*-module and* $\dim A = \dim B$. *Let* I *be an ideal in* A *and let* J *be an ideal in* B *with* $\psi(I) \subset J$. *Then,* $Gr_J(B)$ *is finitely generated as a* $Gr_I(A)$*-module if and only if* $\psi(I)B$ *is a reduction of* J.

Proof. Suppose $\psi(I)B$ is a reduction of J. Let $c \geq 1$ be such that $\psi(I)J^n = J^{n+1}$ for all $n \geq c$. Then, $J^{n+1}/J^{n+2} = \psi(I)(J^n/J^{n+1})$ for all $n \geq c$. Therefore, $Gr_J(B)$ is a finite $Gr_I(A)$-module.

Conversely, suppose that $Gr_J(B)$ is a finite $Gr_I(A)$-module. Then, there exists n_0 such that $Gr_I(A)_1 Gr_J(B)_n = Gr_J(B)_{n+1}$ for all $n \geq n_0$. Thus, for $n \geq n_0$

$$\begin{aligned}\frac{J^{n+1}}{J^{n+2}} &= \frac{I}{I^2}.\frac{J^n}{J^{n+1}} \\ &= \frac{\psi(I)J^n + J^{n+2}}{J^{n+2}}.\end{aligned}$$

So $J^{n+1} = \psi(I)J^n + J^{n+2}$. Therefore, by Nakayama's lemma, $J^{n+1} = \psi(I)J^n$ for all $n \geq n_0$. □

REMARK 2.4. Let $\mathcal{R}(I, A) = A[It] = \bigoplus_{n\geq 0} I^n t^n$ be the Rees ring of A with respect to I. If M is a finite A module, then set $\mathcal{R}(I, M) = \bigoplus_{n\geq 0} I^n M t^n$ the Rees module of M with respect to I. It can also be easily shown $\psi(I)B$ is a reduction of J if and only if $\mathcal{R}(J, B)$ is a finite $\mathcal{R}(I, A)$-module.

REMARK 2.5. Set $W = \bigoplus_{n\geq 0} J^{n+1}/I^{n+1}B$. Then, we have

$$0 \longrightarrow \mathcal{R}(I, B) \longrightarrow \mathcal{R}(J, B) \longrightarrow W(-1) \longrightarrow 0$$

an exact sequence of $\mathcal{R}(I, A)$ modules . If $\psi(I)B$ is a reduction of J, then $W(-1)$ and hence W are finite $\mathcal{R}(I, A)$-modules.

The following is our main object to study associated graded modules and Hilbert coefficients.

DEFINITION 2.6. Let M be an A-module. Set $L^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. Then, the $A-$module $L^I(M)$ can be given an $\mathcal{R}(I, A)$-module structure as follows. The Rees ring $\mathcal{R}(I, A)$ is a subring of $A[t]$ and so $A[t]$ is an $\mathcal{R}(I, A)$-module. Therefore, $M[t] = M \otimes_A A[t]$ is an $\mathcal{R}(I, A)$-module. The exact sequence

$$0 \longrightarrow \mathcal{R}(I, M) \longrightarrow M[t] \longrightarrow L^I(M)(-1) \longrightarrow 0$$

defines an $\mathcal{R}(I, A)$-module structure on $L^I(M)(-1)$ and so on $L^I(M)$. Notice $L^I(M)$ is not necessarily a finitely generated $\mathcal{R}(I, A)$-module.

REMARK 2.7. Let x be M superficial with respect to I and set $u = xt \in \mathcal{R}(I, A)_1$. Notice that $L^I(M)/uL^I(M) = L^I(M/xM)$.

By [**12**, Proposition 5.2], we have the following:

REMARK 2.8. Let $x \in I\backslash I^2$. Then, $x^* \in Gr_I(A)_1$ is $Gr_I(M)$-regular if and only if $u = xt \in \mathcal{R}(I, A)_1$ is $L^I(M)$-regular.

REMARK 2.9. Let $\psi : A \longrightarrow B$ be as before and $\dim A = d$. Assume that I is $\mathfrak{m}$-primary and J is $\mathfrak{n}$-primary. Define $L^I(B) = \bigoplus_{n\geq 0} B/I^{n+1}B$ and $L^J(B) = \bigoplus_{n\geq 0} B/J^{n+1}$. As $L^J(B)$ is a $\mathcal{R}(J, B)$-module and so as a $\mathcal{R}(I, A)$-module. For each $n \geq 0$, we have

$$0 \longrightarrow \frac{J^{n+1}}{I^{n+1}B} \longrightarrow \frac{B}{I^{n+1}B} \longrightarrow \frac{B}{J^{n+1}} \longrightarrow 0$$

an exact sequence of A-modules. So we get

$$0 \longrightarrow W \longrightarrow L^I(B) \longrightarrow L^J(B) \longrightarrow 0$$

https://doi.org/10.1017/S0017089516000525 Published online by Cambridge University Press

an exact sequence of $\mathcal{R}(I, A)$-modules. Therefore,

$$\sum \lambda_A \left(\frac{J^{n+1}}{I^{n+1}B} \right) z^n = \frac{h_B^I(z) - \delta h_B^J(z)}{(1-z)^{d+1}}.$$

Note that $h_B^I(1) - \delta h_B^J(1) = 0$. So we can write

$$\delta h_B^J(z) = h_B^I(z) + (z-1)r(z).$$

Therefore, we have

(1) $\delta e_0^J(B) = e_0^I(B)$;
(2) $\delta e_i^J(B) = e_i^I(B) + (r^{(i-1)}(1)/(i-1)!)$ for $i \geq 1$.

REMARK 2.10. If $\delta e_1^J(B) \neq e_1^I(B)$, then $\dim W = d$.

We need the following technical result.

LEMMA 2.11. *Let $\psi : A \longrightarrow B$ be as before. Assume the residue field of A is infinite. Then, there exists $x \in I$ such that*

(1) *x is A superficial with respect to I;*
(2) *$\psi(x)$ is B superficial with respect to J.*

Proof. Note that $Gr_J(B)$ is a finite $Gr_I(A)$-module. Also, ψ induces a natural map $\hat{\psi} : Gr_I(A) \longrightarrow Gr_J(B)$. Let $z \in Gr_I(A)_1$ be $Gr_I(A) \oplus Gr_J(B)$ filter regular. Then, note that $\hat{\psi}(z)$ is $Gr_J(B)$ filter regular. Let $x \in I$ be such that $x^* = z$. Then, clearly x is A superficial with respect to I. Also, note that $\psi(x)^* = \hat{\psi}(z)$. So $\psi(x)$ is B superficial with respect to J. □

The following result easily follows by induction on the dimension of the ring.

COROLLARY 2.12. *Let $\psi : A \longrightarrow B$ be as before. Assume that the residue field of A is infinite. Let* $\dim A = d \geq 1$. *Then, there exist $\underline{x} = x_1, \ldots, x_d \in I$ such that*

(1) *$\underline{x}$ is an A superficial sequence with respect to I;*
(2) *$\psi(\underline{x})$ is a B superficial sequence with respect to J.*

Proof. Follows easily by induction on d and using Lemma 2.11. □

If $\psi(I)B \subsetneq J$, then $W \neq 0$. Moreover, we have the following result:

LEMMA 2.13. *Let $\psi : A \longrightarrow B$ be as before. Assume that $Gr_I(B)$ is Cohen–Macaulay. Then, the following are equivalent:*

(1) $\delta e_1^J(B) = e_1^I(B)$.
(2) $Gr_I(B) = Gr_J(B)$.

Proof. Clearly (2) implies (1). So we assume (1) and prove (2). We first prove that $Gr_J(B)$ is Cohen–Macaulay. By Sally's machine and Lemma 2.11, we may assume that $\dim B = 1$. Now consider the exact sequence

$$0 \longrightarrow W \longrightarrow L^I(B) \longrightarrow L^J(B) \longrightarrow 0,$$

where $W = \bigoplus J^{n+1}/I^{n+1}B$. As $\delta e_1^J(B) = e_1^I(B)$, we get $\lambda(W) < \infty$. Let $\mathfrak{M}$ be the unique homogeneous maximal ideal of $\mathcal{R}(I, A)$ and $H^i(-) = H^i_{\mathfrak{M}}(-)$. As $Gr_I(B)$ is Cohen–Macaulay by Remark 2.8, we get $H^0(L^I(B)) = 0$. So we get $W = 0$. Therefore, $L^I(B) = L^J(B)$. So $H^0(L^J(B)) = 0$. Thus, $Gr_J(B)$ is Cohen–Macaulay.

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Now assume $\dim B \geq 2$. We prove $L^I(B) = L^J(B)$. This will prove the result. Note that depth $W \geq 1$. Set $u := xt$ where x is A-superficial with respect to I and $\psi(x)$ is B superficial with respect to J. Then, we have an exact sequence

$$0 \longrightarrow \frac{W}{uW} \longrightarrow \frac{L^I(B)}{uL^I(B)} \longrightarrow \frac{L^J(B)}{uL^J(B)} \longrightarrow 0.$$

By induction and Remark 2.7, we get $L^I(B)/uL^I(B) = L^J(B)/uL^J(B)$. So $W = uW$. By graded Nakayama's Lemma, $W = 0$. Hence, $L^I(B) = L^J(B)$. □

3. Extension of Northcott's inequality. The following result easily follows from a result due to Huckaba and Marley (see [**5**, Theorem 4.7]). However, our techniques to prove this extend to prove our other results.

Theorem 3.1. *Let $\psi : A \longrightarrow B$ be as before. Assume that I is $\mathfrak{m}$-primary and J is $\mathfrak{n}$-primary. If $Gr_I(B)$ is Cohen–Macaulay, then $\delta e_1^J(B) \geq e_1^I(B) + \lambda(J/IB)$. If equality holds then $Gr_J(B)$ is also Cohen–Macaulay.*

Proof. By Sally's machine, we may assume that $\dim A = 1$. Set $W_i = J^{i+1}/I^{i+1}B$ and $W = \bigoplus_{i \geq 0} W_i$. Note that $\lambda(W_i) = e_0(W)$ for $i \gg 0$. Let (x) be a minimal reduction of I. Let $u = xt \in \mathcal{R}(I, A)_1$. Then, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_i & \longrightarrow & L^I(B)_i & \longrightarrow & L^J(B)_i & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & W_{i+1} & \longrightarrow & L^I(B)_{i+1} & \longrightarrow & L^J(B)_{i+1} & \longrightarrow & 0
\end{array}$$

a commutative diagram with exact rows. As $Gr_I(B)$ is Cohen–Macaulay so x^* is $Gr_I(B)$ regular. So by Remark 2.8, u is $L^I(B)$ regular. Thus, $L^I(B)_i \xrightarrow{u} L^I(B)_{i+1}$ is injective. As $W \subset L^I(B)$, we also get $W_i \xrightarrow{u} W_{i+1}$ is injective. Therefore, $\lambda(W_0) \leq \lambda(W_1) \leq \cdots \leq \lambda(W_i) = e_0(W)$ for $i \gg 0$. So

$$\begin{aligned}
\delta e_1^J(B) &= e_1^I(B) + e_0(W) \\
&\geq e_1^I(B) + \lambda(J/IB).
\end{aligned}$$

Now suppose $\delta e_1^J(B) = e_1^I(B) + \lambda(J/IB)$. Then, $e_0(W) = \lambda(J/IB)$. Thus, $\lambda(W_n) = \lambda(W_0)$ for all $n \geq 0$. So we get $W_i \xrightarrow{u} W_{i+1}$ is an isomorphism. By Snake lemma, we get

$$L^J(B)_i \xrightarrow{u} L^J(B)_{i+1}$$

is injective. Therefore, u is $L^J(B)$ regular. By Remark 2.8, x^* is $Gr_J(B)$-regular. Hence, $Gr_J(B)$ is Cohen–Macaulay. □

Now we give an example where Theorem 3.1 holds.

Example 3.2. Let $A = \mathbb{Q}[|X, Y, Z, W|]/(XY - YZ, XZ + Y^3 - Z^2)$. Let x, y, z, w be the images of X, Y, Z, W in A, respectively. Set $\mathfrak{m} = (x, y, z, w)$. Then, $(A, \mathfrak{m})$ is a two-dimensional Cohen–Macaulay local ring. Let $I = (x, y, w)$. Note that I is $\mathfrak{m}$-primary and z is integral over I. So I is a reduction of $\mathfrak{m}$. It is proved in [**7**, Example 3.6], $Gr_I(A)$ is Cohen–Macaulay. Using CoCoA (see [**1**]), we have computed $e_1^I(A) = 6$,

$e_1^{\mathfrak{m}}(A) = 7$ and $\lambda(A/I) = 2$. So $e_1^{\mathfrak{m}}(A) = e_1^I(A) + \lambda(\mathfrak{m}/I)$. Hence, by Theorem 3.1, we get $Gr_{\mathfrak{m}}(A)$ is Cohen–Macaulay.

The following example shows that the condition $Gr_I(B)$ is Cohen–Macaulay and is essential.

EXAMPLE 3.3. Let $A = \mathbb{Q}[|X, Y, Z, U, V, W|]/(Z^2, ZU, ZV, UV, YZ - U^3, XZ - V^3)$, with X, Y, Z, U, V, W inderterminates. Let x, y, z, u, v, w be the images of X, Y, Z, U, V, W in A. Set $\mathfrak{m} = (x, y, z, u, v, w)$. Then, $(A, \mathfrak{m})$ is a three-dimensional Cohen–Macaulay local ring. Let $I = (x, y, u, w)$. Note that $v^4 = vxz = 0$ and $z^2 = 0$ in A. Thus, z, v are integral over I. So I is a reduction of $\mathfrak{m}$. Let $J = (x, y, w)$. Then, J is a minimal reduction of I. Using CoCoA (see **[1]**), we have checked that

$$P_I(t) = \frac{4 + t^2 + t^3}{(1-t)^3} \quad \text{and} \quad P_{\mathfrak{m}}(t) = \frac{1 + 3t + 3t^3 - t^4}{(1-t)^3}.$$

We also checked $\lambda(I/J) = 2$, $\lambda(I^2/I^2 \cap J) = 1$ and $\lambda(I^3/I^3 \cap J) = 0$. Therefore, by **[5**, Theorem 4.7], we get depth $Gr_I(A) < 3$. Hence, $Gr_I(A)$ is not Cohen–Macaulay. Also, note that the h-polynomial of $Gr_{\mathfrak{m}}(A)$ has negative coefficient. So $Gr_{\mathfrak{m}}(A)$ is also not Cohen–Macaulay. It is easy to see that $e_1^{\mathfrak{m}}(A) = e_1^I(A) + \lambda(\mathfrak{m}/I)$.

4. The case of dimension two.

Let $\mathfrak{a}$ be an ideal in a Notherian ring S and M a finite S module. Then, for $n \geq 1$, $\widetilde{\mathfrak{a}^n M} := \cup_{k\geq 1}(\mathfrak{a}^{n+k}M :_M \mathfrak{a}^k)$ is called the *Ratliff–Rush* closure of $\mathfrak{a}^n M$.

In general, if $\mathfrak{a} \subset \mathfrak{b}$ are two ideals in a ring S, then it does not imply that $\widetilde{\mathfrak{a}} \subset \widetilde{\mathfrak{b}}$. However, for reduction of ideals, we have the following:

PROPOSITION 4.1. *Let S be a Notherian ring and let $\mathfrak{a} \subset \mathfrak{b}$ be a reduction of $\mathfrak{b}$. Then, $\widetilde{\mathfrak{a}^n} \subset \widetilde{\mathfrak{b}^n}$ for all $n \geq 1$.*

Proof. Let $x \in \widetilde{\mathfrak{a}^n}$. So $x\mathfrak{a}^k \subset \mathfrak{a}^{n+k}$ for some k. Thus, $x\mathfrak{a}^k\mathfrak{b}^r \subset \mathfrak{a}^{n+k}\mathfrak{b}^r$ for all $r \geq 0$. As $\mathfrak{a} \subset \mathfrak{b}$ is a reduction so $\mathfrak{a}\mathfrak{b}^s = \mathfrak{b}^{s+1}$ for $s \gg 0$. Choose $r \geq s$. Then, $\mathfrak{a}^k\mathfrak{b}^r = \mathfrak{b}^{k+r}$. Therefore, $x\mathfrak{b}^{k+r} \subset \mathfrak{b}^{n+k+r}$. Thus, $x \in \widetilde{\mathfrak{b}^n}$. □

4.2. Let M be an A-module. Define $\widetilde{L}^I(M) = \bigoplus_{n\geq 0} M/\widetilde{I^{n+1}M}$. Then, $\widetilde{L}^I(M)$ is an $\mathcal{R}(\widetilde{I}, A)$-module so $\mathcal{R}(I, A)$-module. Set

$$\widetilde{L}^J(B) = \bigoplus_{n\geq 0} B/\widetilde{J^{n+1}} \quad \text{and} \quad \widetilde{W} = \bigoplus_{n\geq 0} \widetilde{J^{n+1}}/I^{n+1}B.$$

Then, we have

$$0 \longrightarrow \widetilde{W} \longrightarrow L^I(B) \longrightarrow \widetilde{L}^J(B) \longrightarrow 0$$

an exact sequence of $\mathcal{R}(I, A)$ modules. Note that $h_B^I(1) = \delta h_B^J(1) = \delta \widetilde{h}_B^J(1)$. Therefore, we can write

$$\delta \widetilde{h}_B^J(z) = h_B^I(z) + (z-1)\widetilde{r}(z) \quad \text{and} \quad H_{\widetilde{W}}(z) = \frac{\widetilde{r}(z)}{(1-z)^d}.$$

Therefore,

(1) $\delta\widetilde{e}_0^J(B) = e_0^I(B)$.

(2) $\delta\widetilde{e}_i^J(B) = e_i^I(B) + \widetilde{r}^{(i-1)}(1)/(i-1)!$.

Now we extend a famous result of Narita concerning second Hilbert coefficient (see [**8**]).

PROPOSITION 4.3. *Let $\psi : A \longrightarrow B$ be as before and* $\dim A \geq 2$. *Let I be* $\mathfrak{m}$*-primary and J be* $\mathfrak{n}$*-primary. Assume that $Gr_I(B)$ is Cohen–Macaulay. Then,*

$$\delta e_2^J(B) \geq e_2^I(B).$$

Proof. We may assume that $\dim A = 2$. Let $\mathfrak{M}$ be the unique homogeneous maximal ideal of $\mathcal{R}(I, A)$. Let $H^i(-) := H_{\mathfrak{M}}(-)$ be the ith local cohomology module. As $Gr_I(B)$ is Cohen–Macaulay, so $H^i(Gr_I(B)) = 0$ for $i = 0, 1$. Also note that $H^0(\widetilde{Gr}_J(B)) = 0$. By Remark 2.8, $H^0(\widetilde{L}^J(B)) = 0$ and $H^i(L^I(B)) = 0$ for $i = 0, 1$. As we have

$$0 \longrightarrow \widetilde{W} \longrightarrow L^I(B) \longrightarrow \widetilde{L}^J(B) \longrightarrow 0$$

an exact sequence of $\mathcal{R}(I, A)$ modules. By considering long exact sequence in local cohomology, we get $H^i(\widetilde{W}) = 0$ for $i = 0, 1$. Hence, $\widetilde{W}$ is Cohen–Macaulay. So $\widetilde{r}^{(1)}(1) \geq 0$. Also note that $e_2^I(B) \geq 0$.

Now,

$$\begin{aligned}\delta\widetilde{e}_2^J(B) &= e_2^I(B) + \widetilde{r}^{(1)}(1). \\ &\geq e_2^I(B).\end{aligned}$$

□

REMARK 4.4. From the proof of Proposition 4.3, one can see that $\widetilde{W}$ is Cohen–Macaulay if $\dim A = 2$.

By analysing the case of equality in the above Theorem, we prove:

THEOREM 4.5. *Let $\psi : A \longrightarrow B$ be as before and* $\dim A = 2$. *Let I be* $\mathfrak{m}$*-primary and J be* $\mathfrak{n}$*−primary. Assume that $Gr_I(B)$ is Cohen–Macaulay. Suppose $\delta e_2^J(B) = e_2^I(B)$. Then, $\widetilde{Gr}_J(B)$ is Cohen–Macaulay. Consequently, $Gr_{J^n}(B)$ is Cohen–Macaulay for* $n \gg 0$.

Proof. We have $\delta e_2^J(B) = e_2^I(B) + \widetilde{r}^{(1)}(1)$. By Remark 4.4, we get $\widetilde{W}$ is Cohen–Macaulay. So $\widetilde{r}^{(1)}(1) \geq 0$. By hypothesis, $\delta e_2^J(B) = e_2^I(B)$. So $\widetilde{r}^{(1)}(1) = 0$. Therefore, $\widetilde{r}(z) = c$(constant) and hence

$$H_{\widetilde{W}}(z) = \frac{c}{(1-z)^2}.$$

Let x, y be an I superficial sequence. Set $u = xt$, $v = yt$. Then, $u, v \in \mathcal{R}(I, A)_1$. Now, set $\overline{\widetilde{W}} = \widetilde{W}/u\widetilde{W}$, $\overline{L^I(B)} = L^I(B)/uL^I(B)$ and $\overline{\widetilde{L}^J(B)} = \widetilde{L}^J(B)/u\widetilde{L}^J(B)$. As u is $\widetilde{W} \oplus L^I(B) \oplus \widetilde{L}^J(B)$-regular, we get

$$0 \longrightarrow \overline{\widetilde{W}} \longrightarrow \overline{L^I(B)} \longrightarrow \overline{\widetilde{L}^J(B)} \longrightarrow 0$$

an exact sequence. So we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\widetilde{W}_i} & \longrightarrow & \overline{L^I(B)_i} & \longrightarrow & \overline{\widetilde{L}^J(B)_i} & \longrightarrow & 0 \\
 & & \downarrow v & & \downarrow v & & \downarrow v & & \\
0 & \longrightarrow & \overline{\widetilde{W}_{i+1}} & \longrightarrow & \overline{L^I(B)_{i+1}} & \longrightarrow & \overline{\widetilde{L}^J(B)_{i+1}} & \longrightarrow & 0
\end{array}$$

with exact rows. As $Gr_I(B)$ and $\widetilde{W}$ are Cohen–Macaulay, we get that v is $\overline{L^I(B)} \oplus \overline{\widetilde{W}}$-regular. So $\overline{L^I(B)_i} \xrightarrow{v} \overline{L^I(B)_{i+1}}$ and $\overline{\widetilde{W}_i} \xrightarrow{v} \overline{\widetilde{W}_{i+1}}$ are injective. As $\lambda(\overline{\widetilde{W}_i}) = \lambda(\overline{\widetilde{W}_{i+1}})$ for all i. We get $\overline{\widetilde{W}_i} \xrightarrow{v} \overline{\widetilde{W}_{i+1}}$ is an isomorphism. By Snake Lemma, it follows that $\overline{\widetilde{L}^J(B)_i} \xrightarrow{v} \overline{\widetilde{L}^J(B)_{i+1}}$ is injective. So depth $\widetilde{L}^J(B) \geq 2$. Hence, by Remark 2.8, depth $\widetilde{Gr_J}(B) \geq 2$. So $\widetilde{Gr_J}(B)$ is Cohen–Macaulay. In particular, $Gr_{J^n}(B)$ is Cohen–Macaulay for $n \gg 0$ (see [**12**]). □

Now we give an example where Theorem 4.5 holds.

EXAMPLE 4.6. Let $A = \mathbb{Q}[|X, Y, Z, U, V|]/(Z^2, ZU, ZV, UV, Y^2Z - U^3, X^2Z - V^3)$, with X, Y, Z, U, V inderterminates. Let x, y, z, u, v be the images of X, Y, Z, U, V in A. Set $\mathfrak{m} = (x, y, z, u, v)$. Then, $(A, \mathfrak{m})$ is a two-dimensional Cohen–Macaulay local ring. Let $I = (x, y, z, u)$ and $J = (x, y, z, u, v^2)$. Note that $v^4 - vx^2z = 0$ in A. Thus, v is integral over I. So I is a reduction of J. Let $\mathfrak{q} = (x, y)$. Then, $\mathfrak{q}$ is a minimal reduction of I. Using CoCoA (see [**1**]), we have computed $e_1^I(A) = 4$ and $e_2^I(A) = e_2^J(A) = 1$. We also checked that $\lambda(I/\mathfrak{q}) = 3$, $\lambda(I^2/I^2 \cap \mathfrak{q}) = 1$. By [**5**, Theorem 4.7], we get $Gr_I(A)$ is Cohen–Macaulay. Hence, by Theorem 4.5, we get $\widetilde{Gr_J}(A)$ Cohen–Macaulay. Hence, $Gr_{J^n}(A)$ is Cohen–Macaulay for $n \gg 0$.

For integrally closed ideals we prove:

THEOREM 4.7. *Let $\psi : A \longrightarrow B$ be as before and* $\dim A = 2$. *Let I be $\mathfrak{m}$-primary and J be $\mathfrak{n}$-primary. Assume that $Gr_I(B)$ is Cohen–Macaulay. Suppose J is integrally closed and*

$$\delta e_1^J(B) = e_1^I(B) + \lambda(J/IB) + 1.$$

Then,

(*a*) $2\lambda(J/IB) \leq \lambda(\widetilde{J^2}/I^2B) \leq 2\lambda(J/IB) + 1$;
(*b*) *if* $\lambda(\widetilde{J^2}/I^2B) = 2\lambda(J/IB) + 1$. *Then, $\widetilde{Gr_J}(B)$ is Cohen–Macaulay. Consequently, $Gr_{J^n}(B)$ is Cohen–Macaulay for $n \gg 0$.*

Proof. By Remark 4.4, $\widetilde{W}$ is Cohen–Macaulay. Let

$$H_{\widetilde{W}}(z) = \sum_{n \geq 0} \lambda_A(\widetilde{W}_n) z^n = \frac{\widetilde{r}(z)}{(1-z)^2}$$

be the Hilbert series of $\widetilde{W}$. Note that all the co-efficients of $\widetilde{r}(z)$ are positive. Write

$$\widetilde{r}(z) = r_0 + r_1 z + \cdots + r_s z^s.$$

Then, we have $r_0 = \lambda(\widetilde{J}/IB) = \lambda(J/IB)$ and $\lambda(\widetilde{J^2}/I^2B) = 2r_0 + r_1$. We have

$$\begin{aligned} \delta e_1^J(B) &= e_1^I(B) + \widetilde{r}(1). \\ &= e_1^I(B) + \lambda(J/IB) + 1. \end{aligned}$$

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Therefore, $\widetilde{r}(1) = \lambda(J/IB) + 1$. Hence, $r_1 + \cdots + r_s = 1$. So (a) follows.

Suppose (b) holds. *i.e.*

$$\lambda(\widetilde{J^2}/I^2B) = 2\lambda(J/IB) + 1.$$

Then, $r_1 = 1$ and $r_j = 0$ for $j \geq 2$. Let x, y be an I superficial sequence. Set $u = xt$. Then, $u \in \mathcal{R}(I, A)_1$. Also, set $U = \widetilde{W}/u\widetilde{W}$. Then, we have

$$0 \longrightarrow U \longrightarrow \frac{L^I(B)}{uL^I(B)} \longrightarrow \frac{\widetilde{L}^J(B)}{u\widetilde{L}^J(B)} \longrightarrow 0$$

an exact sequence. Note that $\lambda(U_n) = \lambda(U_2)$ for $n \geq 2$. Also note that $v = yt$ in $\mathcal{R}(I, A)$ is $L^I(B)/uL^I(B)$ regular. So v is U regular. So we have

$$0 \longrightarrow U(-1) \longrightarrow U \longrightarrow \frac{U}{vU} \longrightarrow 0$$

an exact sequence. By Hilbert series $(U/vU)_j = 0$ for $j \geq 2$. Now by setting $K = \ker\left((\widetilde{L}^J(B)/u\widetilde{L}^J(B))(-1) \xrightarrow{v} \widetilde{L}^J(B)/u\widetilde{L}^J(B)\right)$, we get by Snake Lemma $K_j = 0$ for $j \geq 2$. Also note that $K_0 = 0$.

Claim ($K_1 = 0$). To prove the claim set $\mathcal{F} = \{\widetilde{J^n}\}$. Then, $\mathcal{F}$ is a filtration on B. Then, $\overline{\mathcal{F}} = \{\widetilde{J^n} + (x)/(x)\}$ is the quotient filtration on B/xB. Put $\mathfrak{q} = J/(x) = \bar{\mathcal{F}}_1$. We may assume that $\mathfrak{q}$ is integrally closed. As $\widetilde{J^n} = J^n$ for $n \gg 0$. So we get $\bar{\mathcal{F}}_n = \mathfrak{q}^n$ for $n \gg 0$.

We prove that $\overline{\mathcal{F}_2} : y = \bar{\mathcal{F}}_1 = \mathfrak{q}$. Let $a \in \overline{\mathcal{F}_2} : y$. So $ya \in \overline{\mathcal{F}_2}$. So $y^{n+1}a \in y^n\overline{\mathcal{F}_2} \subset \overline{\mathcal{F}_{2+n}} = \mathfrak{q}^{n+2}$ for $n \gg 0$. This implies $a \in \widetilde{q} = \underline{\mathfrak{q}} = \mathfrak{q}$. It follows that $K_1 = 0$. Thus, K is zero. So $\widetilde{L}^J(B)$ has depth ≥ 2. This implies $\widetilde{Gr}_J(B)$ is Cohen–Macaulay. So $Gr_{J^n}(B)$ is Cohen–Macaulay for $n \gg 0$. $\square$

Here, we give an example where our Theorem 4.7 holds:

Example 4.8. Let $A = \mathbb{Q}[|X, Y, Z, W|]/(X^2 - Y^2Z, XY^4 - Z^2)$. Let x, y, z, w denotes the images of X, Y, Z, W in A, respectively. Let $I = (x, y, w)$ and $\mathfrak{m} = (x, y, z, w)$. Then, $(A, \mathfrak{m})$ is a two-dimensional Cohen–Macaulay local ring. Using CoCoA (see **[1]**), we have computed

$$P_I(t) = \frac{2 + 2t}{(1-t)^2} \quad \text{and} \quad P_{\mathfrak{m}}(t) = \frac{1 + 2t + t^2}{(1-t)^2}.$$

We have $e_1^{\mathfrak{m}}(A) = e_1^I(A) + \lambda(\mathfrak{m}/I) + 1$. We have also checked that $\lambda(\widetilde{\mathfrak{m}^2}/I^2) = 3 = 2\lambda(\mathfrak{m}/I) + 1$. Since I has minimal multiplicity, $Gr_I(A)$ is Cohen–Macaulay. Hence, by Theorem 4.7 (b), we get $\widetilde{Gr}_{\mathfrak{m}}(A)$ is Cohen–Macaulay.

5. The case of third Hilbert coefficient. In this section, we deal with the third Hilbert coefficients and generalize a remarkable result of Itoh for normal ideals (see **[6]**).

Theorem 5.1. *Let $A = B$ and $\psi = id_A$. Let* $\dim A \geq 3$. *Let I, J be* $\mathfrak{m}$*-primary ideals and I is a reduction of J. Suppose $Gr_I(A)$ is Cohen–Macaulay and J is asymptotically*

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normal. Then,

$$e_3^J(A) \geq e_3^I(A).$$

Proof. By standard argument, it suffices to consider $\dim A = 3$. As J is asymptotically normal by [**3**, Theorem 3.1], there exists n_0 such that depth $Gr_{J^n}(A) \geq 2$ for all $n \geq n_0$.

Now set $n \geq n_0$, $T = I^n$, $K = J^n$, and $W = \bigoplus_{i\geq 0} K^{i+1}/T^{i+1}$. Then, we get an exact sequence:

$$0 \longrightarrow W \longrightarrow L^T(A) \longrightarrow L^K(A) \longrightarrow 0$$

of $\mathcal{R}(T, A)$-modules. Note that depth $L^K(A) \geq 2$. So we get W is Cohen–Macaulay of dimension 3. Thus, $e_i(W) \geq 0$ for $0 \leq i \leq 3$. Hence,

$$\begin{aligned} e_3^K(A) &= e_3^T(A) + e_2(W). \\ &\geq e_3^T(A). \end{aligned}$$

By [**9**, Remark 2], we have $e_3^J(A) = e_3^{J^n}(A)$ and $e_3^I(A) = e_3^{I^n}(A)$ for all $n \geq 1$. Therefore, $e_3^J(A) \geq e_3^I(A)$. □

By analysing the case of equality, we prove the following:

THEOREM 5.2. *Let $A = B$ and $\psi = id_A$. Assume* $\dim A = 3$*. Let I, J be two $\mathfrak{m}$-primary ideals and I is a reduction of J. Suppose $Gr_I(A)$ is Cohen–Macaulay and J is asymptotically normal. If $e_3^J(A) = e_3^I(A)$, then $Gr_{J^n}(A)$ is Cohen–Macaulay for all $n \gg 0$.*

Proof. As J is asymptotically normal by [**3**, Theorem 3.1], there exists n_0 such that for all $n \geq n_0$, depth $Gr_{J^n}(A) \geq 2$. Now set $n \geq n_0$, $T = I^n$, $K = J^n$, and $W = \bigoplus_{i\geq 0} K^{i+1}/T^{i+1}$. From the proof of Theorem 5.1, we see that W is Cohen–Macaulay of dimension 3 and $e_3^J(A) = e_3^I(A) + e_2(W)$.

Suppose $e_3^I(A) = e_3^J(A)$. Then, $e_2(W) = 0$. So the Hilbert series of W is given by

$$H_W(s) = \frac{r_0 + r_1 s}{(1-s)^3}.$$

Let x, y, z be a $K \oplus T$ superficial sequence. Set $u = xt$, $v = yt$, and $w = zt$. Note that $u, v, w \in \mathcal{R}(T, A)_1$. Also, set $U = W/(u, v)W$, $\overline{L^K(A)} = L^K(A)/(u, v)L^K(A)$ and $\overline{L^T(A)} = L^T(A)/(u, v)L^T(A)$. Then, we get an exact sequence

$$0 \longrightarrow U \longrightarrow \overline{L^T(A)} \longrightarrow \overline{L^K(A)} \longrightarrow 0.$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(-1) & \longrightarrow & \overline{L^T(A)}(-1) & \longrightarrow & \overline{L^K(A)}(-1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & U & \longrightarrow & \overline{L^T(A)} & \longrightarrow & \overline{L^K(A)} & \longrightarrow & 0 \end{array}$$

Also note that Hilbert series of U/wU is given by

$$H_{U/wU}(s) = r_0 + r_1 s.$$

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Therefore, $(U/wU)_j = 0$ for $j \geq 2$. Now, set $E = ker(\overline{L^K(A)}(-1) \xrightarrow{w} \overline{L^K(A)})$. Note that $E_j = 0$ for all $j \geq 2$ by Snake Lemma. Also, $E_0 = 0$.

CLAIM ($E_1 = 0$). To prove this set, $\mathcal{F} = \{K^m\}$. Then, $\mathcal{F}$ is a filtration on A. Also, $\overline{\mathcal{F}} = \{K^m + (x, y)/(x, y)\}$ is the quotient filtration on $A/(x, y)A$. Put $\mathfrak{q} = K/(x, y) = \bar{\mathcal{F}}_1$. We may assume that $\mathfrak{q}$ is integrally closed. Note $\bar{\mathcal{F}}_m = \mathfrak{q}^m$ for $m \geq 1$.

We prove that $\overline{\mathcal{F}_2} : z = \bar{\mathcal{F}}_1 = \mathfrak{q}$. Let $a \in \overline{\mathcal{F}_2} : z$. So $za \in \overline{\mathcal{F}_2} = \mathfrak{q}^2$. This implies $a \in \widetilde{q} \subset \bar{\mathfrak{q}} = \mathfrak{q}$. It follows that $E_1 = 0$. Thus, E is zero. So depth $\overline{L^K(A)} \geq 1$. Thus, depth $L^K(A) \geq 3$. Therefore, depth $Gr_K(A) \geq 3$. Hence, $Gr_K(A)$ is Cohen–Macaulay. □

6. Examples. In this section, we show that there are plenty of examples where Theorem 3.1 holds.

EXAMPLE 6.1. Let $(R, \mathfrak{m})$ be a regular local ring. Let $(B, \mathfrak{n}) = (R/\mathfrak{a}, \mathfrak{m}/\mathfrak{a})$ be a Cohen–Macaulay local ring. Suppose dim $R = t$ and dim $B = d$. Then, ht$(\mathfrak{a}) = t - d$. Set $g = t - d$. Then, there exists a regular sequence $\underline{u} = u_1, \ldots, u_g$ of length g. Set $A = R/(\underline{u})$. Then, we get a surjective ring homomorphism

$$A \overset{\psi}{\twoheadrightarrow} B.$$

Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}_A$. Set $I = (\mathfrak{q} :_A \mathfrak{m}_A)$. Clearly, $\mathfrak{q} \subset I \subset \mathfrak{m}_A$. By [**2**, Theorem 2.1], we get

$$I^2 = \mathfrak{q}I.$$

So $\psi(I^2) = \psi(\mathfrak{q})\psi(I)$. Thus, $Gr_I(B)$ has minimal multiplicity. As $\psi(\mathfrak{q})B$ is a minimal reduction $\psi(\mathfrak{m}_A) = \mathfrak{n}$, so we get $\psi(I)B$ is a reduction of $\psi(\mathfrak{m}_A) = \mathfrak{n}$. Hence by Theorem 3.1, we get

$$e_1^{\mathfrak{n}}(B) \geq e_1^I(B) + \lambda(\mathfrak{n}/I).$$

EXAMPLE 6.2. Suppose $(A, \mathfrak{m})$ is a Gorenstein local ring which not regular. Let J be any $\mathfrak{m}$-primary ideal. Set $I := (\mathfrak{q} :_A \mathfrak{m})$ where $\mathfrak{q}$ is a minimal reduction of J. Then,

$$e_1^J(A) \geq e_1^I(A) + \lambda(J/I).$$

Proof. It is enough to prove that I has minimal multiplicity. By [**2**, Theorem 2.1], we have $\mathfrak{q} \subset I \subset J$ and $I^2 = \mathfrak{q}I$. Thus, I has minimal multiplicity. □

EXAMPLE 6.3. Let $(A, \mathfrak{m}) \overset{\psi}{\to} (B, \mathfrak{n})$ be a local homomorphism of Cohen–Macaulay local rings with dim $A =$ dim B. Let I be an $\mathfrak{m}$-primary ideal in A. If A is regular and $Gr_I(A)$ is Cohen–Macaulay, then $Gr_I(B)$ is Cohen–Macaulay. If $\psi(I)B$ is a reduction of J in B, then

$$e_1^J(B) \geq e_1^I(B) + \lambda(J/I).$$

Proof. By Auslander–Buchsbaum formula, we get B is free as an A-module. As $Gr_I(A)$ is Cohen–Macaulay, so $Gr_I(B)$ is Cohen–Macaulay. Hence, by Theorem 3.1, we get the inequality. □

EXAMPLE 6.4. Let $(A, \mathfrak{m})$ be Cohen–Macaulay local ring and I be an $\mathfrak{m}$-primary ideal. Let $I \subset J \subset \bar{I}$ (integral closure of I). If $Gr_I(A)$ is Cohen–Macalay, then by Theorem 3.1, we get

$$e_1^J(A) \geq e_1^I(A) + \lambda(J/I).$$

If equality holds above, then $Gr_J(A)$ is Cohen–Macaulay.

ACKNOWLEDGEMENTS. The authors are deeply grateful to the referee for the careful reading of the paper and the helpful suggestions. Also, the first and last authors would like to thank IIT Bombay especially Department of Mathematics for it's hospitality during the preparation of this work.

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