ZERO-CYCLES ON NORMAL PROJECTIVE VARIETIES

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Abstract We prove an extension of the Kato–Saito unramified class field theory for smooth projective schemes over a finite field to a class of normal projective schemes. As an application, we obtain Bloch's formula for the Chow groups of 0-cycles on such schemes. We identify the Chow group of 0-cycles on a normal projective scheme over an algebraically closed field to the Suslin homology of its regular locus. Our final result is a Roitman torsion theorem for smooth quasiprojective schemes over algebraically closed fields. This completes the missing p-part in the torsion theorem of Spieß and Szamuely.

Contents

1. Introduction

1.1. Motivation

It is well known that the Chow group of 0-cycles on a smooth projective scheme over an appropriate field describes many other invariants of the scheme, such as Suslin homology, cohomology of K-theory sheaves, and abelianized étale fundamental groups. However, this is not the case when the underlying scheme is not projective. The latter case is a very challenging problem in the theory of algebraic cycles. The principal motivation of this paper is to explore whether the Levine–Weibel Chow group [50] of normal projective schemes can be used to solve this problem for those smooth quasiprojective schemes which are the regular loci of normal projective schemes. The results that we obtain in this paper suggest that this strategy is indeed a promising one. We describe our main results in some detail.

1.2. Levine–Weibel Chow group and class field theory

The aim of the class field theory in the geometric case is to describe the abelian étale coverings (which are extrinsic to the scheme) of a scheme over a finite field in terms of some arithmetic or geometric invariants (such as the Chow groups of 0-cycles) which are intrinsic to the scheme. Let k be a finite field and X an integral smooth projective scheme over k. Let $\mathrm{CH}_0^F(X)$ denote the classical [17] Chow group of 0-cycles and let $\mathrm{CH}_0^F(X)^0$ denote the kernel of the degree map $\deg_X \colon \mathrm{CH}_0^F(X) \to \mathbb{Z}$. Let $\pi_1^{\mathrm{ab}}(X)$ denote the abelianized étale fundamental group of X (see, for example, [74, §5.8]) and let $\pi_1^{\mathrm{ab}}(X)^0$ denote the kernel of the canonical map $\pi_1^{\mathrm{ab}}(X) \to \mathrm{Gal}\left(\overline{k}/k\right)$ induced by the structure map of X. The following is the main theorem of the geometric class field theory for smooth projective schemes. The case of curves was earlier proven by Lang [45], based on Artin's reciprocity theorem for local and global fields [2].

Theorem 1.1 (37, Theorem 1). *Let X be an integral smooth projective scheme over a finite field. Then the map $\phi_X^0 \colon CH_0^F(X)^0 \to \pi_1^{ab}(X)^0$, induced by sending a closed point to its associated Frobenius element, is an isomorphism of finite groups.*

If U is a smooth quasiprojective scheme over a finite field k which is not projective, then one does not know in general how to describe $\pi_1^{\mathrm{ab}}(U)$ in terms of 0-cycles. It was shown by Schmidt and Spieß [67] and Schmidt [66] that the tame quotient of $\pi_1^{\mathrm{ab}}(U)$ is described by the Suslin homology of U. But we do not yet know if the abelian covers of U with wild ramifications could be described in terms of the Chow group of 0-cycles on a compactification of U. Our main result in this direction provides a partial answer to this problem.

Let $\mathrm{CH}_0^{LW}(X)$ denote the Levine–Weibel Chow group of 0-cycles of a scheme X [50] (see §2.1 for a reminder of its definition).

Theorem 1.2. *Let X be an integral projective scheme over a finite field which is regular in codimension 1. Then the Frobenius substitution associated to the regular closed points gives rise to a reciprocity homomorphism*

$$\phi_X \colon \mathrm{CH}_0^{LW}(X) \to \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}}),$$

which restricts to a surjective homomorphism $\phi_X^0 \colon \mathrm{CH}_0^{LW}(X)^0 \twoheadrightarrow \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})^0$. The map ϕ_X^0 is an isomorphism of finite groups in one of the following cases:

(1) *X has only isolated singularities.*

(2) *X is regular in codimension 3 and its local rings satisfy Serre's S_4 condition.*

It readily follows that in these cases, the map ϕ_X is injective with uniquely divisible cokernel $\widehat{\mathbb{Z}}/\mathbb{Z}$ (see diagram (3.3)). Note also that the finiteness of the source and target of ϕ_X^0 is part of our assertion, and was not known before.

Without assumptions (1) or (2) in Theorem 1.2, we prove the following:

Theorem 1.3. *Let X be an integral projective scheme over a finite field which is regular in codimension 1. Then the reciprocity homomorphism of Theorem 1.2 induces an isomorphism of finite groups*

$$\phi_X \colon \mathrm{CH}_0^{LW}(X)/m \to \pi_1^{\mathrm{ab}}(X_{\mathrm{reg}})/m$$

for every integer $m \in k^\times$.

1.3. Bloch's formula for the Levine–Weibel Chow group

In the theory of algebraic cycles, Bloch's formula describes the Chow group of algebraic cycles of codimension d on a smooth scheme (of any dimension) over a field as the dth Zariski or Nisnevich cohomology of an appropriate Milnor or Quillen K-theory sheaf. A statement of this kind plays a central role in the study of algebraic cycles on smooth schemes. Bloch's formula for smooth schemes in the case $d = 1$ is classical, the case $d = 2$ is due to Bloch [8], and the general case is due to Quillen [60]. This formula for the Chow group of 0-cycles on smooth schemes in terms of the Milnor K-theory is due to Kato [33].

Bloch's formula for the Levine–Weibel Chow group is well known for singular curves [50, Proposition 1.4]. However, it is a very challenging problem in higher dimensions. This formula for singular surfaces over algebraically closed fields is due to Levine [48]. For projective surfaces over infinite fields, the formula was recently proven by Binda, Krishna, and Saito [5, Theorem 8.1]. Bloch's formula for the Levine–Weibel Chow group of singular projective schemes over nonalgebraically closed fields is yet unknown in any other case.

Suppose that X is a quasiprojective scheme of pure dimension d over a perfect[1] field k and $x \in X_{\mathrm{reg}}$ is a regular closed point. One then knows by [33, Theorem 2] that there is a canonical isomorphism $\mathbb{Z} \xrightarrow{\cong} K_0^M(k(x)) \xrightarrow{\cong} H_x^d\left(X, \mathcal{K}_{d,X}^M\right)$, where the latter is the Nisnevich cohomology with support, $K_n^M(R)$ is the Milnor K-theory on a ring R,

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[1]Perfectness is not required by [27, Lemma 3.7].

and $\mathcal{K}^M_{i,X}$ is the Nisnevich sheaf of Milnor K-theory on X (as defined, for instance, in [33, §0])[2]. Hence, using the 'forget support' map for x and extending it linearly to the free abelian group on all regular closed points of X, we get the cycle class homomorphism

$$\mathrm{cyc}_X \colon \mathcal{Z}_0\left(X_{\mathrm{reg}}\right) \to H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right). \tag{1.1}$$

As an application of Theorem 1.2 and the class field theory of Kato and Saito [35], we prove the following:

Theorem 1.4. *Let X be an integral projective scheme of dimension d over a finite field which satisfies one of the following:*

(1) *X has only isolated singularities.*

(2) *X is regular in codimension 3 and its local rings satisfy Serre's S_4 condition.*

Then the cycle class map induces an isomorphism

$$\mathrm{cyc}_X \colon \mathrm{CH}^{LW}_0(X) \xrightarrow{\cong} H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right).$$

In case of isolated singularities, one can also include the Zariski topology and Quillen K-theory sheaf $\mathcal{K}_{d,X}$ in Theorem 1.4 (see §2.4). We remark that the existence of cyc_X on $\mathrm{CH}^{LW}_0(X)$ is part of our assertion, and was not previously known.

1.4. Levine–Weibel Chow group and 0-cycles with modulus

Let X be an integral projective scheme of dimension $d \geq 2$ over a field which is regular in codimension 1. Let us assume that a resolution of singularities $f \colon \widetilde{X} \to X$ exists in the sense of Hironaka. Let $E \hookrightarrow \widetilde{X}$ denote the reduced exceptional divisor. Let $\mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ denote the Chow group of 0-cycles with modulus (see §3.3). It is not hard to see that the identity map of $\mathcal{Z}_0\left(X_{\mathrm{reg}}\right)$ induces a surjection $\mathrm{CH}^{LW}_0(X) \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ for all integers $m \geq 1$. The following application of Theorem 1.2 is an extension of the Bloch–Srinivas conjecture (which was proven for normal surfaces in [44]) to higher dimensions over finite fields:

Theorem 1.5. *Let X be an integral projective scheme of dimension $d \geq 2$ over a finite field which satisfies one of the following:*

(1) *X has only isolated singularities.*

(2) *X is regular in codimension 3 and its local rings satisfy Serre's S_4 condition.*

Let $f \colon \widetilde{X} \to X$ be a resolution of singularities with the reduced exceptional divisor E. Then the pullback map $f^* \colon \mathcal{Z}_0\left(X_{\mathrm{reg}}\right) \to \mathcal{Z}_0\left(\widetilde{X} \setminus E\right)$ induces an isomorphism

$$f^* \colon \mathrm{CH}^{LW}_0(X) \xrightarrow{\cong} \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$$

for all $m \gg 0$.

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[2]We could use the improved Milnor K-theory of Kerz instead, but it will make no difference in the top cohomology.

If X is defined over an algebraically closed field, Theorem 1.5 was proven by Gupta and Krishna [26, Theorem 1.8] (see also [39] and [40] for earlier results).

1.5. Levine–Weibel Chow group and Suslin homology

The Suslin homology was introduced by Suslin and Voevodsky [73] with the objective of constructing an analogue of the singular homology of topological spaces for algebraic schemes. It is now a part of the motivic cohomology with compact support of schemes, in the sense of $\mathbb{A}^1$-homotopy theory. Because of this, Suslin homology is now a well-studied theory, and there are many known results which can be used for its computation. On the contrary, the Levine–Weibel Chow group is only conjecturally a part of a motivic cohomology theory of singular schemes, and is much less accessible.

Our next main result, however, provides an identification between the two groups and provides a strong evidence that over algebraically closed fields of positive characteristic, the conjectural motivic cohomology of a normal projective scheme should coincide with the already-known motivic cohomology with compact support of its regular locus.

Theorem 1.6. *Let X be an integral projective scheme of dimension $d \geq 1$ over a field k which is regular in codimension 1. Then there is a canonical surjective homomorphism*

$$\theta_X \colon \mathrm{CH}_0^{LW}(X) \twoheadrightarrow H_0^S(X_{\mathrm{reg}}).$$

Furthermore, we have the following if k is algebraically closed:

(1) *θ_X is an isomorphism if $\mathrm{char}(k) > 0$.*
(2) *$\theta_X/n \colon \mathrm{CH}_0^{LW}(X)/n \to H_0^S(X_{\mathrm{reg}})/n$ is an isomorphism for all integers $n \neq 0$ if $\mathrm{char}(k) = 0$.*

We shall show (see §8.7) that the condition that k is algebraically closed in part (1) is essential. When $\mathrm{char}(k) = 0$, we expect that $\mathrm{Ker}(\theta_X)$ is a (large) divisible group.

1.6. Roitman torsion theorem for Suslin homology

Let k be an algebraically closed field. Let U be a smooth quasiprojective scheme over k which admits an open embedding $U \hookrightarrow X$, where X is smooth and projective over k. Then Spieß and Szamuely [72] showed that the Albanese map into the generalized Albanese variety (à la Serre) of U induces a homomorphism $\vartheta_U \colon H_0^S(U) \to \mathrm{Alb}_S(U)(k)$, which is an isomorphism on the prime-to-p torsion subgroups, where p is the exponential characteristic of k. This was a crucial breakthrough in eliminating the projectivity hypothesis from the famous Roitman torsion theorem for the Chow group of 0-cycles [63].

Geisser [18] subsequently showed that the prime-to-p condition in the torsion theorem of Spieß and Szamuely could be eliminated if one assumed resolution of singularities. The final result of this paper eliminates the prime-to-p condition from the torsion theorem of Spieß and Szamuely without assuming resolution of singularities.

Theorem 1.7. *Let U be a smooth quasiprojective scheme over an algebraically closed field k. Suppose that there exists an open immersion $U \subset X$ such that X is smooth and projective over k. Then the Albanese homomorphism*

$$\vartheta_U \colon H_0^S(U)_{\mathrm{tor}} \to \mathrm{Alb}_S(U)(k)_{\mathrm{tor}}$$

is an isomorphism.

1.7. Overview of proofs

The proofs of our main results broadly have two main steps, namely, the construction of the underlying maps and the proof that these maps are isomorphisms. The first part is achieved by means of a moving lemma for the Levine–Weibel Chow group. The heart of the problem is the more challenging second part.

Following an induction technique, we first prove our results for surfaces. To take care of higher dimensions, we establish new Lefschetz hypersurface[3] section theorems for several invariants of smooth quasiprojective (but not necessarily projective) and singular projective schemes. We may emphasize that these Lefschetz theorems are of independent interest, and we expect them to have several applications elsewhere. They allow us to reduce the proofs of the main results to the case of surfaces.

We prove the moving lemma (Lemma 2.4) and its consequences in §2. Using this, we construct the reciprocity map (Corollary 3.3) and prove the reciprocity isomorphism for surfaces (Theorem 3.6) in §3. We prove the Lefschetz theorems for the étale cohomology (Proposition 5.3) and the abelianized étale fundamental group (Theorem 5.4) in §5. They allow us to prove Theorem 1.2 in §6. We then combine this result with the class field theory of [35] to prove Theorems 1.4 and 1.5 in §6.

The heart of the proofs of Theorems 1.6 and 1.7 are the following two results: a Lefschetz hypersurface section theorem for the generalized Albanese variety of smooth quasiprojective schemes and an identification of Suslin homology with the Chow group of 0-cycles of a certain modulus pair. This first result is shown in §7, and the second in §8.3. The latter section also contains the proof of Theorem 1.6. We combine the Lefschetz theorem with a result of Geisser [18] to prove Theorem 1.7 in §9, following a delicate blowup trick.

1.8. Notations

We shall, in general, work with an arbitrary base field k even if our main results are over either finite or algebraically closed fields. We let $\mathbf{Sch}_k$ denote the category of quasiprojective k-schemes and $\mathbf{Sm}_k$ the category of smooth quasiprojective k-schemes. A product $X \times_k Y$ in $\mathbf{Sch}_k$ will be simply written as $X \times Y$. For a reduced scheme X, we let X_n denote the normalization of X.

For any excellent scheme X, we let X^o denote the regular locus of X. One knows that X^o is an open subscheme of X that is dense if X is generically reduced. We shall let

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[3]We call them hypersurface section theorems because we use hypersurfaces of large degree instead of only the hyperplanes used in some of the classical Lefschetz theorems.

X_{sing} denote the singular locus of X. If X is reduced, we shall consider X_{sing} as a closed subscheme of X with the reduced induced structure. For any Noetherian scheme X, we shall denote its étale fundamental group with a base point $x \in X$ by $\pi_1(X, x)$. We shall let $\pi_1^{\mathrm{ab}}(X)$ denote the abelianization of $\pi_1(X, x)$. One knows that $\pi_1^{\mathrm{ab}}(X)$ does not depend on the choice of the base point $x \in X$. We shall consider $\pi_1^{\mathrm{ab}}(X)$ a topological abelian group with its profinite topology.

For $X \in \mathbf{Sch}_k$ equidimensional, we let $X^{(i)}$ denote the set of codimension i points and $X_{(i)}$ the set of dimension i points on X for $i \geq 0$. We shall let $\mathcal{Z}_i(X)$ denote the free abelian group on $X_{(i)}$ and $\mathrm{CH}_i^F(X)$ the Chow group of cycles of dimension i as defined in [17, Chapter 1]. For an abelian group A, we shall denote the torsion and the p-primary torsion subgroups of A by A_{tor} and $A\{p\}$, respectively, for any prime p. For a commutative ring Λ, we shall write A_Λ for $A \otimes_{\mathbb{Z}} \Lambda$.

2. Zero-cycles on R_1-schemes

In this section, we shall recall the definition of the Levine–Weibel Chow group of an R_1-scheme and prove some preliminary results about this group. Recall from [52, p. 183] that a Noetherian scheme X is called an R_a-scheme if it is regular in codimension a, where $a \geq 0$. One says that X is an S_b-scheme if for all points $x \in X$, one has $\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min\{b, \dim(\mathcal{O}_{X,x})\}$. We shall say that a Noetherian scheme X is an $(R_a + S_b)$-scheme for $a \geq 0$ and $b \geq 0$ if it is an R_a- as well as an S_b-scheme. A Noetherian commutative ring will be called an R_a-ring (resp., S_b-ring) if its Zariski spectrum is so.

2.1. The Levine–Weibel Chow group

Let k be any field and X a reduced quasiprojective scheme of dimension $d \geq 2$ over k. Recall from [3, Definition 3.5] that a Cartier curve on X is a purely 1-dimensional reduced closed subscheme $C \hookrightarrow X$, none of whose irreducible components is contained in X_{sing}, and whose defining sheaf of ideals is a local complete intersection in $\mathcal{O}_X$ at every point of $C \cap X_{\mathrm{sing}}$. Let $k(C)$ denote the ring of total quotients for a Cartier curve C on X, and let $\{C_1, \ldots, C_r\}$ be the set of irreducible components of C. For $f \in \mathcal{O}_{C, C \cap X_{\mathrm{sing}}}^\times \subset k(C)^\times$, let $f_i \in k(C_i)^\times$ be the image of f under the projection $\mathcal{O}_{C, C \cap X_{\mathrm{sing}}}^\times \hookrightarrow k(C)^\times \twoheadrightarrow k(C_i)^\times$. We let $\mathrm{div}(f) = \sum_{i=1}^r \mathrm{div}(f_i) \in \mathcal{Z}_0(X^o)$ be the cycle associated to f. We let $\mathcal{R}_0^{LW}(X)$ denote the subgroup of $\mathcal{Z}_0(X^o)$ generated by $\mathrm{div}(f)$, where $C \subset X$ is a Cartier curve and $f \in \mathcal{O}_{C, C \cap X_{\mathrm{sing}}}^\times$. The Levine–Weibel Chow group of X is the quotient $\mathcal{Z}_0(X^o)/\mathcal{R}_0^{LW}(X)$ and is denoted by $\mathrm{CH}_0^{LW}(X)$.

It is immediate from the definition that $\mathrm{CH}_0^{LW}(X)$ coincides with $\mathrm{CH}_0^F(X)$ if X is regular. We also remark that the foregoing definition of $\mathrm{CH}_0^{LW}(X)$ is slightly different from that of [50]. However, this difference disappears if k is infinite, as a consequence of [49, Lemma 1.4]. If X is integral and projective over k, there is a degree map $\deg \colon \mathrm{CH}_0^{LW}(X) \to \mathbb{Z}$ and we let $\mathrm{CH}_0^{LW}(X)^0$ be the kernel of this map.

Let $\mathcal{R}_0^{\mathrm{lci}}(X)$ be the subgroup of $\mathcal{Z}_0(X^o)$ generated by $\nu_*(\mathrm{div}(f))$ for $f \in \mathcal{O}_{C, \nu^{-1}(X_{\mathrm{sing}})}^\times$, where $\nu \colon C \to X$ is a finite morphism from a reduced curve of pure dimension 1 over k

such that the image of none of the irreducible components of C is contained in X_{sing} and ν is a local complete intersection (lci) morphism at every point $x \in C$ such that $\nu(x) \in X_{\text{sing}}$. Such a curve C is called a good curve relative to X_{sing}. The lci Chow group of 0-cycles for X is the quotient $\text{CH}_0^{\text{lci}}(X) = \mathcal{Z}_0(X^o)/\mathcal{R}_0^{\text{lci}}(X)$. This modification of the Levine–Weibel Chow group was introduced in [3].

Clearly, the identity map of $\mathcal{Z}_0(X^o)$ induces a surjection $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0^{\text{lci}}(X)$. If k is infinite, we can say the following:

Lemma 2.1. *Assume that k is infinite and X is an R_1-scheme. Then the canonical map $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0^{\text{lci}}(X)$ is an isomorphism.*

Proof. Let $\nu \colon C \to X$ be a good curve relative to X_{sing}, and let $Z = \nu^{-1}(X_{\text{sing}})$. It suffices to show that $\nu_*(\text{div}(f)) \in \mathcal{R}_0^{LW}(X)$ for any $f \in \mathcal{O}_{C,Z}^\times$. Since ν is finite, we can find a factorization

$$
\begin{array}{ccc}
 & \mathbb{P}_X^n & \\
{\scriptstyle i}\nearrow & & \downarrow{\scriptstyle g} \\
C \xrightarrow[\nu]{} & X, &
\end{array}
\tag{2.1}
$$

where i is a closed immersion and g is the canonical projection. Setting $X' = \mathbb{P}_X^n$, we see that X' is an R_1-scheme and $X'_{\text{sing}} = g^{-1}(X_{\text{sing}})$. In particular, $i^{-1}(X'_{\text{sing}}) \subseteq Z$. Since ν is an lci morphism along X_{sing} and g is smooth, it follows that the closed immersion i is regular along X'_{sing}. In other words, C is embedded as a Cartier curve on X'. One deduces that $\text{div}(f) \in \mathcal{R}_0^{LW}(X')$.

Since X' is R_1 and k is infinite, it follows from [7, Lemma 2.1] (see also [49, Lemma 1.4]) that there are closed reduced curves $C_i' \subset X'$ and rational functions $f_i \in k(C_i')^\times$ such that $C_i' \subset X'^o$ and $\text{div}(f) = \sum_i \text{div}(f_i) \in \mathcal{Z}_0(X'^o)$. If $g(C_i')$ is a point, then clearly $g_*(\text{div}(f_i)) = 0$, as the map $C_i' \to X$ then factors through a regular closed point and $g_*(\text{div}(f_i))$ is already zero in the Chow group of the closed point. Otherwise, we let $C_i = g(C_i')$ and assume that the map $C_i' \to C_i$ is finite. Then each $C_i \subset X$ is clearly a Cartier curve, as it does not meet X_{sing}. Let $N_i \colon k(C_i')^\times \to k(C_i)^\times$ denote the norm map. We then have

$$
\nu_*(\text{div}(f)) = g_*(\text{div}(f)) = \sum_i g_*(\text{div}(f_i)) = \sum_i \text{div}(N_i(f_i)),
$$

and it is clear that $\text{div}(N_i(f_i)) \in \mathcal{R}_0^{LW}(X)$ for every i. $\qquad\square$

2.2. The moving lemma

One of the key ingredients for proving Theorem 1.2 is a moving lemma for the Levine–Weibel Chow group over finite fields, which we shall prove in this subsection.

Let k be any field and $X \in \mathbf{Sch}_k$ an integral R_1-scheme. Let $A \subset X$ be a closed subset of X of codimension ≥ 2 such that $X_{\text{sing}} \subseteq A$. We let $\mathcal{R}_0^{LW}(X, A)$ be the subgroup of $\mathcal{Z}_0(X \setminus A)$ generated by $\text{div}(f)$, where f is a nonzero rational function on an integral curve $C \subset X$ such that $C \cap A = \emptyset$. We define $\text{CH}_0^{LW}(X, A) := \mathcal{Z}_0(X \setminus A)/\mathcal{R}_0^{LW}(X, A)$.

We define $\mathrm{CH}_0^{\mathrm{lci}}(X,A)$ in an analogous way. The inclusion $\mathcal{Z}_0(X \setminus A) \hookrightarrow \mathcal{Z}_0(X^o)$ preserves the subgroups of rational equivalences. Hence, we get canonical maps

$$\mathrm{CH}_0^{LW}(X,A) \to \mathrm{CH}_0^{LW}(X), \qquad \mathrm{CH}_0^{\mathrm{lci}}(X,A) \to \mathrm{CH}_0^{\mathrm{lci}}(X).$$

Lemma 2.2. *The canonical surjection* $\mathrm{CH}_0^{LW}(X,A) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{lci}}(X,A)$ *is an isomorphism.*

Proof. Let $\nu \colon C \to X$ be a finite morphism from an integral curve such that $\nu(C) \cap A = \emptyset$. It suffices to show that $\nu_*(\mathrm{div}(f)) \in \mathcal{R}_0^{LW}(X,A)$ for any $f \in k(C)^\times$. But the proof of this is identical to that of [17, Theorem 1.4] (see the last part of the proof of Lemma 2.1). $\qquad\square$

We shall need the following application of the Bertini theorems of Altman and Kleiman [1] and Wutz [77] (which is a small modification of the Bertini theorem of Poonen [59]):

Lemma 2.3. *Assume that k is perfect, $\dim(X) \geq 2$, and $W \subset X_{(0)}^o$ is a finite set. Let $B \subset A$ be any closed subset containing X_{sing} such that $B \cap W = \emptyset$. We can then find a smooth integral curve $C \subset X$ containing W such that $C \cap B = \emptyset$ and $C \not\subset A$.*

Proof. We let $\nu \colon X_n \to X$ be the normalization morphism and let $\widetilde{B} = \nu^{-1}(B)$. We choose a dense open immersion $X_n \hookrightarrow Y$ such that Y is an integral projective normal k-scheme. We let B' be the Zariski closure of $\widetilde{B}$ in Y. Then $\dim(B') \leq \dim(X) - 2$. We let $S \subset B'$ be a finite closed subset whose intersection with every irreducible component of B' is nonempty. We fix a closed embedding $Y \hookrightarrow \mathbb{P}_k^n$. We choose a closed point $x \in \pi^{-1}(X^o \setminus A)$ and set $Z = W \cup \{x\}$.

By the Bertini theorems of Altman and Kleiman [1, Theorem 1] (when k is infinite) and Wutz [77, Theorem 3.1] (with $C = S$ and $T = H_{\mathrm{zar}}^0(S, \mathcal{O}_S^\times)$ when k is finite), we can find a hypersurface $H \subset \mathbb{P}_k^n$ containing Z and disjoint from S such that $Y_{\mathrm{reg}} \cap H$ is smooth. The condition $S \cap H = \emptyset$ implies that $\dim(B' \cap H) \leq \dim(B') - 1 \leq \dim(X) - 3$.

Since Y is normal, it follows from [23, Exposé XII, Corollaire 3.5] that $Y \cap H$ is connected. In particular, $Y_{\mathrm{reg}} \cap H$ is connected and smooth. Hence, it is integral. By iterating this process $\dim(X) - 1$ times, we get an integral curve $C' \subset Y$ containing Z such that $\dim(B' \cap C') \leq (\dim(X) - 2) - (\dim(X) - 1) < 0$. In particular, $B' \cap C' = \emptyset$ and $C' \not\subset \pi^{-1}(A)$. We let $C = \pi(C' \cap X_n)$. Then C satisfies the desired properties. $\qquad\square$

The moving lemma we want to prove is the following:

Lemma 2.4. *Let k be any field. Then the canonical map*

$$\mathrm{CH}_0^{LW}(X,A) \to \mathrm{CH}_0^{LW}(X) \tag{2.2}$$

is an isomorphism. The same holds also for the lci Chow group.

Proof. We can assume that $\dim(X) \geq 2$ because the lemma is trivial otherwise. Before we begin the proof, we note that one always has the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^{LW}(X,A) & \longrightarrow & \mathrm{CH}_0^{LW}(X) \\ \downarrow & & \downarrow \\ \mathrm{CH}_0^{\mathrm{lci}}(X,A) & \longrightarrow & \mathrm{CH}_0^{\mathrm{lci}}(X). \end{array} \qquad (2.3)$$

Suppose now that k is infinite. Since X is R_1, the top horizontal arrow is known to be an isomorphism (see [7, Lemma 2.1] and [49, Lemma 1.4]). The right (resp., left) vertical arrow is an isomorphism by Lemma 2.1 (resp., Lemma 2.2). It follows that the bottom horizontal arrow is an isomorphism.

We now assume that k is finite and prove the injectivity of formula (2.2). Let $\alpha \in \mathrm{CH}_0^{\mathrm{lci}}(X,A)$ be a cycle which dies in $\mathrm{CH}_0^{\mathrm{lci}}(X)$. Let $\ell_1 \neq \ell_2$ be two distinct prime numbers and let k_i be the pro-ℓ_i algebraic extension of k for $i=1,2$. For $i=1,2$, [3, Proposition 6.1] says that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^{\mathrm{lci}}(X,A) & \longrightarrow & \mathrm{CH}_0^{\mathrm{lci}}(X) \\ \downarrow & & \downarrow \\ \mathrm{CH}_0^{\mathrm{lci}}(X_{k_i},A_{k_i}) & \longrightarrow & \mathrm{CH}_0^{\mathrm{lci}}(X_{k_i}). \end{array} \qquad (2.4)$$

The bottom horizontal arrow is an isomorphism because k_i is infinite. It follows that α dies in $\mathrm{CH}_0^{\mathrm{lci}}(X_{k_i},A_{k_i})$. It follows from [3, Proposition 6.1] (by a straightforward modification, explained in [27, Proposition 8.5]) that there is a finite algebraic extension $k \hookrightarrow k_i'$ inside k_i such that α dies in $\mathrm{CH}_0^{\mathrm{lci}}(X_{k_i'},A_{k_i'})$. We apply [3, Proposition 6.1] once again to conclude that $\ell_i^n \alpha = 0$ for $i=1,2$ and some $n \gg 1$. Since ℓ_1^n and ℓ_2^n are relatively prime, it follows that $\alpha = 0$. We have thus shown that the map $\mathrm{CH}_0^{\mathrm{lci}}(X,A) \to \mathrm{CH}_0^{\mathrm{lci}}(X)$ is injective.

To show the same for the Levine–Weibel Chow group, we use diagram (2.3) again. The left vertical arrow is an isomorphism by Lemma 2.2. We just showed that the bottom horizontal arrow is injective. It follows that the top horizontal arrow is also injective.

It remains to show that formula (2.2) is surjective when k is finite. For this, we fix a closed point $x \in X^o$. We need to show that there is a 0-cycle $\alpha \in \mathcal{Z}_0(X^o)$ supported on $X^o \setminus A$ such that the cycle class $[x]$ coincides with α in $\mathrm{CH}_0^{LW}(X)$.

To that end, we let $C \subset X$ be a curve as in Lemma 2.3 with $W = \{x\}$ and $B = X_{\mathrm{sing}}$. We let $\mathrm{CH}_0^F(C,C \cap A)$ be the cokernel of the map $\mathcal{O}_{C,C \cap A}^\times \xrightarrow{\mathrm{div}} \mathcal{Z}_0(C \setminus A)$. Using the isomorphism $\mathrm{CH}_0^F(C) \cong H_{\mathrm{zar}}^1(C,\mathcal{O}_C^\times)$ and taking the colimit over the exact sequences for the cohomology of $\mathcal{O}_C^\times$ with support in finite closed subsets $S \subset C \setminus A$, one easily checks that the canonical map $\mathrm{CH}_0^F(C,C \cap A) \to \mathrm{CH}_0^F(C)$ is an isomorphism. In particular, the 0-cycle $[x]$ on C coincides with a 0-cycle $\alpha' \in \mathrm{CH}_0^F(C)$ supported on $C \setminus A$.

Since $C \cap X_{\mathrm{sing}} = \emptyset$, we have a push-forward map $\iota_* \colon \mathrm{CH}_0^F(C) \to \mathrm{CH}_0^{LW}(X)$, where $\iota \colon C \hookrightarrow X$ is the inclusion. Letting $\alpha = \iota_*(\alpha')$, we achieve our claim. This proves the surjectivity of formula (2.2) for the Levine–Weibel Chow group. Since the map

$\mathrm{CH}_0^{LW}(X) \to \mathrm{CH}_0^{\mathrm{lci}}(X)$ is surjective by definition, it follows that the map $\mathrm{CH}_0^{\mathrm{lci}}(X,A) \to \mathrm{CH}_0^{\mathrm{lci}}(X)$ is also surjective. This concludes the proof. $\qquad\square$

We now draw some consequences of the moving lemma. The first is an extension of Lemma 2.1 for integral R_1-schemes over finite fields.

Theorem 2.5. *Let k be any field and $X \in \mathbf{Sch}_k$ an integral R_1-scheme. Then the canonical map $\mathrm{CH}_0^{LW}(X) \to \mathrm{CH}_0^{\mathrm{lci}}(X)$ is an isomorphism.*

Proof. By Lemma 2.1, we assume that k is finite. We look at the commutative diagram (2.3). Its left vertical arrow is an isomorphism (without any condition on k), by Lemma 2.2. The two horizontal arrows are isomorphisms by Lemma 2.4. The theorem now follows. $\qquad\square$

The next two applications show that the Levine–Weibel Chow groups of R_1-schemes admit pullback and push-forward maps. Note that neither of these maps was previously known to exist.

Corollary 2.6. *Let k be any field and $X \in \mathbf{Sch}_k$ an integral R_1-scheme. Let $f \colon X' \to X$ be a morphism in $\mathbf{Sch}_k$ such that $X'_{\mathrm{sing}} \subseteq f^{-1}(X_{\mathrm{sing}})$ and the resulting map $f^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is finite and surjective. Then $f^* \colon \mathcal{Z}_0(X^o) \to \mathcal{Z}_0(X'^o)$ induces a pullback homomorphism*

$$f^* \colon \mathrm{CH}_0^{LW}(X) \to \mathrm{CH}_0^{LW}(X').$$

This is an isomorphism if f is the normalization morphism.

Proof. The proof of the existence of f^* is a routine construction following [17, Chapter 1], once we note that the map $f^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is finite and flat. The assertion that f^* is an isomorphism if f is the normalization follows directly from Lemma 2.4. $\qquad\square$

Corollary 2.7. *Let $f \colon X' \to X$ be a proper morphism of integral quasiprojective schemes over a field k. Assume that $f^{-1}(X_{\mathrm{sing}})$ has codimension ≥ 2 in X' and contains X'_{sing}. Then the push-forward between the cycle groups $f_* \colon \mathcal{Z}_0\left(X' \setminus f^{-1}(X_{\mathrm{sing}})\right) \to \mathcal{Z}_0(X^o)$ descends to a homomorphism*

$$f_* \colon \mathrm{CH}_0^{LW}(X') \to \mathrm{CH}_0^{LW}(X).$$

Proof. The hypothesis of the corollary implies that X' is R_1. We let $A' = f^{-1}(X_{\mathrm{sing}})$. Using Lemma 2.4 and the canonical map $\mathrm{CH}_0^{LW}(X,X_{\mathrm{sing}}) \to \mathrm{CH}_0^{LW}(X)$, it suffices to construct the homomorphism $f_* \colon \mathrm{CH}_0^{LW}(X',A') \to \mathrm{CH}_0^{LW}(X,X_{\mathrm{sing}})$. But this can be achieved by repeating the construction of the proper push-forward map for the classical Chow groups in [17, §1.4]. $\qquad\square$

2.3. The cycle class map

Let k be any field and $X \in \mathbf{Sch}_k$ of pure dimension d. Let $\mathcal{K}_{i,X}^M$ be the Zariski (or Nisnevich) sheaf of Milnor K-theory on X [33, §0]. Let $\mathcal{K}_{i,X}$ denote the Zariski (or Nisnevich) sheaf of Quillen K-theory on X. The product structures on the Milnor

and Quillen K-theory yield a natural map of sheaves $\mathcal{K}_{i,X}^M \to \mathcal{K}_{i,X}$. In formula (1.1), we defined the cycle class homomorphism (this could be trivial, for example, if $X^o = \emptyset$)

$$\mathrm{cyc}_X \colon \mathcal{Z}_0(X^o) \to H_{\mathrm{zar}}^d \left(X, \mathcal{K}_{d,X}^M \right). \tag{2.5}$$

Using the Thomason–Trobaugh spectral sequence for Quillen K-theory [75, Theorem 10.3], there are canonical maps $H_{\mathrm{zar}}^d(X, \mathcal{K}_{d,X}) \to H_{\mathrm{nis}}^d(X, \mathcal{K}_{d,X}) \to K_0(X)$. These maps fit into a commutative diagram [26, Lemma 3.2]

$$\tag{2.6}$$

$$
\begin{array}{ccc}
& H_{\mathrm{zar}}^d \left(X, \mathcal{K}_{d,X}^M \right) \longrightarrow H_{\mathrm{nis}}^d \left(X, \mathcal{K}_{d,X}^M \right) & \\
\mathrm{cyc}_X \nearrow & \downarrow \qquad\qquad \downarrow & \\
\mathcal{Z}_0(X^o) \longrightarrow H_{\mathrm{zar}}^d \left(X, \mathcal{K}_{d,X} \right) \longrightarrow H_{\mathrm{nis}}^d \left(X, \mathcal{K}_{d,X} \right) \longrightarrow K_0(X) &
\end{array}
$$

$$\widetilde{\mathrm{cyc}}_X$$

such that the composite map $\mathcal{Z}_0(X^o) \to K_0(X)$ is the cycle class map [50, Proposition 2.1] which takes a closed point $x \in X^o$ to the class $[k(x)] \in K_0(X)$. Note also that any closed point $x \in X^o$ has a class $[k(x)] \in K_0(X, D)$, where the latter is the 0th homotopy group of the S^1-spectrum $K(X, D)$ defined as the homotopy fiber of the pullback map of spectra $K(X) \to K(D)$ for any closed subscheme D supported on X_{sing}. Moreover, there is a factorization $\mathcal{Z}_0(X^o) \to K_0(X, D) \to K(X)$.

As an analogue of Bloch's formula, one asks whether the cycle class homomorphism $\mathcal{Z}_0(X^o) \to H_{\mathrm{nis}}^d \left(X, \mathcal{K}_{d,X}^M \right)$ factors through $\mathrm{CH}_0^{LW}(X)$ and whether the resulting map is an isomorphism when X is reduced. We shall prove some new results on this question in this paper. We consider a special case in what follows.

2.4. The case of isolated singularities

Let us now assume that X is reduced and equidimensional, and has only isolated singularities. Let S denote the finite set of singular points. We consider the sequence of Zariski sheaves of Milnor K-groups [33, §0]:

$$
\mathcal{K}_{m,X}^M \xrightarrow{\epsilon}
\left(
\begin{array}{c}
\coprod\limits_{x \in X^{(0)}} (i_x)_* K_m^M(k(x)) \\
\oplus \\
\coprod\limits_{P \in S} (i_P)_* K_m^M(\mathcal{O}_{X,P})
\end{array}
\right)
\xrightarrow{e_0}
\left(
\begin{array}{c}
\coprod\limits_{x \in X^{(1)}} (i_x)_* K_{m-1}^M(k(x)) \\
\oplus \\
\coprod\limits_{P \in S} \coprod\limits_{P \in \overline{\{x\}}, x \in X^{(0)}} (i_P)_* K_m^M(k(x))
\end{array}
\right)
\xrightarrow{e_1} \cdots
$$

$$\tag{2.7}$$

$$
\cdots \xrightarrow{e_{d-1}}
\left(
\begin{array}{c}
\coprod\limits_{x \in X^{(d)}} (i_x)_* K_{m-d}^M(k(x)) \\
\oplus \\
\coprod\limits_{P \in S} \coprod\limits_{P \in \overline{\{x\}}, x \in X^{(d-1)}} (i_P)_* K_{m-d+1}^M(k(x))
\end{array}
\right)
\xrightarrow{e_d}
\left(
\begin{array}{c}
0 \\
\oplus \\
\coprod\limits_{P \in S} (i_P)_* K_{m-d}^M(k(P))
\end{array}
\right)
\to 0.
$$

Here the map ϵ is induced by the inclusion into both terms and the other maps are given by the matrices

$$e_0 = \begin{pmatrix} \partial_1 & 0 \\ -\Delta & \epsilon \end{pmatrix}, e_1 = \begin{pmatrix} \partial_1 & 0 \\ \Delta & \partial_2 \end{pmatrix}, \cdots, e_d = \begin{pmatrix} 0 & 0 \\ \pm\Delta & \partial_2 \end{pmatrix},$$

with ∂_1 and ∂_2 being the differentials of the Gersten–Quillen complex for Milnor K-theory sheaves as described in [33] and Δs being the diagonal maps.

Lemma 2.8. *This sequence of maps forms a complex which gives a flasque resolution of the sheaf $\epsilon\left(\mathcal{K}_{m,X}^M\right)$ in Zariski and Nisnevich topologies.*

Proof. A similar complex for the Quillen K-theory sheaves is constructed in [58, §5], and it is shown there that this complex is a flasque resolution of $\epsilon(\mathcal{K}_{m,X})$. The same proof works here verbatim. On all stalks except at S, the exactness follows from [37, Proposition 10]. The exactness at the points of S is an immediate consequence of the way the differentials are defined in formula (2.7) (see [58] for details). $\square$

Proposition 2.9. *Let $X \in \mathbf{Sch}_k$ be integral of dimension $d \geq 1$ with only isolated singularities, and let τ denote Zariski or Nisnevich topology. Then there are canonical maps*

$$\mathrm{CH}_0^{LW}(X) \xrightarrow{\mathrm{cyc}_X} H_\tau^d\left(X, \mathcal{K}_{d,X}^M\right) \to H_\tau^d(X, \mathcal{K}_{d,X}) \to \mathrm{CH}_0^F(X),$$

in which the middle arrow is an isomorphism. Furthermore, all arrows in the middle square of diagram (2.6) are isomorphisms.

Proof. The case $d = 1$ is well known [50, Proposition 1.4]. We can thus assume that $d \geq 2$. Let S denote the singular locus of X and let $X_S^{(j)}$ denote the set of points $x \in X$ of codimension j such that $S \cap \overline{\{x\}} = \emptyset$. We first observe that the map of sheaves $\mathcal{K}_{d,X}^M \twoheadrightarrow \epsilon\left(\mathcal{K}_{d,X}^M\right)$ is generically an isomorphism, and the same holds for the Quillen K-theory sheaves. It follows [29, Exercise II.1.19, Lemma III.2.10] that the map $H_\tau^d\left(X, \mathcal{K}_{d,X}^M\right) \to H_\tau^d\left(X, \epsilon\left(\mathcal{K}_{d,X}^M\right)\right)$ is an isomorphism, and ditto for the Quillen K-theory sheaves. It follows from Lemma 2.8 that both $H_\tau^d\left(X, \mathcal{K}_{d,X}^M\right)$ and $H_\tau^d(X, \mathcal{K}_{d,X})$ are given by the middle homology of the complex $\mathcal{C}_X$:

$$\begin{pmatrix} \coprod_{x \in X^{(d-1)}} K_1(k(x)) \\ \oplus \\ \coprod_{P \in S} \coprod_{P \in \overline{\{x\}}} K_2(k(x)) \end{pmatrix} \xrightarrow{e_{d-1}} \begin{pmatrix} \coprod_{x \in X^{(d)}} K_0(k(x)) \\ \oplus \\ \coprod_{P \in S} \coprod_{P \in \overline{\{x\}}} K_1(k(x)) \end{pmatrix} \xrightarrow{e_d} \begin{pmatrix} 0 \\ \oplus \\ \coprod_{P \in S} K_0(k(P)) \end{pmatrix}.$$

On the other hand, letting $\mathcal{C}_X^0$ and $\mathcal{C}_X^{F,0}$ denote, respectively, the complexes

$$\coprod_{x \in X_S^{(d-1)}} K_1(k(x)) \xrightarrow{\mathrm{div}} \coprod_{x \in X_S^{(d)}} K_0(k(x)) \to 0,$$

$$\coprod_{x \in X^{(d-1)}} K_1(k(x)) \xrightarrow{\mathrm{div}} \coprod_{x \in X^{(d)}} K_0(k(x)) \to 0,$$

we see that there are canonical maps of chain complexes $\mathcal{C}_X^0 \hookrightarrow \mathcal{C}_X \twoheadrightarrow \mathcal{C}_X^{F,0}$. This yields canonical maps $H_1\left(\mathcal{C}_X^0\right) \to H_\tau^d\left(X, \mathcal{K}_{d,X}^M\right) \xrightarrow{\cong} H_\tau^d\left(X, \mathcal{K}_{d,X}\right) \to H_1\left(\mathcal{C}_X^{F,0}\right)$. It follows, however, from Lemma 2.4 that $H_1\left(\mathcal{C}_X^0\right) \cong \mathrm{CH}_0^{LW}(X)$. It is also clear that $H_1\left(\mathcal{C}_X^{F,0}\right) \cong \mathrm{CH}_0^F(X)$. The second assertion is clear. This concludes the proof. $\qquad\square$

3. Zero-cycles on surfaces

In this section, we shall define the reciprocity maps. We shall then restrict to the case of surfaces and prove several results which will essentially be enough to prove the main theorems of this paper in this special case.

3.1. The reciprocity maps

Let k be a finite field and $X \in \mathbf{Sch}_k$ an integral R_1-scheme of dimension $d \geq 1$. Given a closed point $x \in X^o$, we have the push-forward map $(\iota_x)_* \colon \pi_1(\mathrm{Spec}\ (k(x))) \to \pi_1(X^o)$, where $\iota \colon \mathrm{Spec}\ (k(x)) \hookrightarrow X^o$ is the inclusion. We let $\phi_X([x]) = (\iota_x)_*(F_x)$, where F_x is the Frobenius automorphism of $\overline{k(x)}$, which is the topological generator of $\mathrm{Gal}\left(\overline{k(x)}/k(x)\right) \cong \pi_1(\mathrm{Spec}\ (k(x))) \cong \widehat{\mathbb{Z}}$. Extending it linearly, we get the reciprocity map $\phi_X \colon \mathcal{Z}_0(X^o) \to \pi_1^{\mathrm{ab}}(X^o)$. By [35, Theorem 2.5], the cycle class map $\mathrm{cyc}_X \colon \mathcal{Z}_0(X^o) \to H_{\mathrm{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right)$ is surjective.

Lemma 3.1. *There exists a homomorphism* $\rho_X \colon H_{\mathrm{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right) \to \pi_1^{\mathrm{ab}}(X^o)$ *such that* $\phi_X = \rho_X \circ \mathrm{cyc}_X$.

Proof. We let $\mathcal{K}_{i,(X,Y)}^M = \mathrm{Ker}\left(\mathcal{K}_{i,X}^M \twoheadrightarrow \mathcal{K}_{i,Y}^M\right)$ for any closed subscheme $Y \subset X$. We now look at the diagram

$$\varprojlim_{Y} H_{\mathrm{nis}}^d\left(X, \mathcal{K}_{d,(X,Y)}^M\right) \xrightarrow{\cong} H_{\mathrm{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right) \qquad (3.1)$$

where Y ranges over all closed subschemes of X contained in X_{sing} and m ranges over all nonzero integers. The top horizontal arrow is the map induced on the cohomology by the inclusion of Nisnevich sheaves $\mathcal{K}_{d,(X,Y)}^M \hookrightarrow \mathcal{K}_{d,X}^M$. All such maps are isomorphisms, because $\dim(Y) \leq d-2$. This explains why the top horizontal arrow is an isomorphism. The cycle class map cyc_X clearly factors through $\mathcal{Z}_0(X^o) \to H_{\mathrm{nis}}^d\left(X, \mathcal{K}_{d,(X,Y)}^M\right)$ for any $Y \subset X_{\mathrm{sing}}$, because $H_{\{x\}}^d\left(X, \mathcal{K}_{d,(X,Y)}^M\right) \xrightarrow{\cong} H_{\{x\}}^d\left(X, \mathcal{K}_{d,X}^M\right)$ for any closed point $x \in X^o$. The limit of all these maps is cyc_X'. Hence, cyc_X is the composition of cyc_X' with the top horizontal arrow.

The middle vertical arrow is induced by the canonical surjections $H_{\text{nis}}^d \left(X, \mathcal{K}_{d,(X,Y)}^M \right) \twoheadrightarrow H_{\text{nis}}^d \left(X, \mathcal{K}_{d,(X,Y)}^M \right)/m$. The bottom horizontal arrow on the right is given by [35, Theorem 9.1(3)], and is an isomorphism. The arrows η_X and ψ_X are defined so that the triangles on the two sides of the middle vertical arrow commute. It follows from [35, Proposition 3.8(2)] that $\psi_X \circ \text{cyc}_X' = \phi_X$. We deduce from this that there is a unique homomorphism $\rho_X \colon H_{\text{nis}}^d \left(X, \mathcal{K}_{d,X}^M \right) \to \pi_1^{\text{ab}}(X^o)$, which factors ϕ_X via cyc_X. $\square$

Lemma 3.2. *For any closed subset $A \subset X$ of codimension ≥ 2 and containing X_{sing}, the reciprocity map ϕ_X descends to a homomorphism*

$$\phi_X \colon \text{CH}_0^{LW}(X, A) \to \pi_1^{\text{ab}}(X \setminus A).$$

Proof. If $d = 1$, then X is a connected smooth projective curve over k (note that k is perfect) with $A = \emptyset$, and one knows that $\text{CH}_0^{LW}(X) \cong \text{CH}_0^F(X)$. The lemma therefore follows from the classical class field theory in this case.

To prove that ϕ_X kills $\mathcal{R}_0^{LW}(X, A)$ in general, we need to show that $\phi_X(\text{div}(f)) = 0$ if f is a nonzero rational function on an integral projective curve $C \subset X \setminus A$. So we choose any such curve C and $f \in k(C)^\times$. Let $\nu \colon C_n \to C \hookrightarrow X$ be the composite map, where ν is the normalization morphism. We then have a commutative diagram [62, Lemma 5.1(1)]

$$
\begin{array}{ccc}
\mathcal{Z}_0(C_n) & \xrightarrow{\phi_{C_n}} & \pi_1^{\text{ab}}(C_n) \\
{\scriptstyle \nu_*} \downarrow & & \downarrow {\scriptstyle \nu_*} \\
\mathcal{Z}_0(X \setminus A) & \xrightarrow{\phi_X} & \pi_1^{\text{ab}}(X \setminus A).
\end{array}
\tag{3.2}
$$

Since $\text{div}((f)_C) = \nu_*(\text{div}((f)_{C_n}))$, this diagram reduces the problem to showing that $\phi_{C_n}(\text{div}((f)_{C_n})) = 0$. But this has already been shown. $\square$

In view of Lemma 2.4 (with $A = X_{\text{sing}}$), Lemma 3.2 implies the following:

Corollary 3.3. *The reciprocity map ϕ_X descends to a homomorphism*

$$\phi_X \colon \text{CH}_0^{LW}(X) \to \pi_1^{\text{ab}}(X^o).$$

It is clear that there is a commutative diagram (with exact rows)

$$
\begin{array}{ccccccc}
0 \to & \text{CH}_0^{LW}(X)^0 & \to & \text{CH}_0^{LW}(X) & \xrightarrow{\deg} & \mathbb{Z} \\
& {\scriptstyle \phi_X^0} \downarrow & & {\scriptstyle \phi_X} \downarrow & & \downarrow \\
0 \longrightarrow & \pi_1^{\text{ab}}(X^o)^0 & \longrightarrow & \pi_1^{\text{ab}}(X^o) & \longrightarrow & \widehat{\mathbb{Z}}.
\end{array}
\tag{3.3}
$$

3.2. Reciprocity map for surfaces

We assume now that X is an integral projective R_1-scheme of dimension 2 over a field k. By Proposition 2.9, there are canonical maps

$$\text{CH}_0^{LW}(X) \xrightarrow{\text{cyc}_X} H_\tau^2 \left(X, \mathcal{K}_{2,X}^M \right) \to K_0(X), \tag{3.4}$$

where $\tau = \text{zar/nis}$ and the composite arrow is the cycle class map to K-theory. We let $F^2 K_0(X)$ denote the image of this composite arrow. For any closed subscheme $D \subset X$ supported on X_{sing}, we let $F^2 K_0(X,D)$ denote the image of the cycle class map $\mathcal{Z}_0(X^o) \to K_0(X,D)$ (see §2.3). The main result about $\text{CH}_0^{LW}(X)$ is the following:

Proposition 3.4. *Under the foregoing assumptions, the following hold:*

(1) *There are cycle class maps* $\text{CH}_0^{LW}(X) \to F^2 K_0(X,D) \to F^2 K_0(X)$ *which are isomorphisms.*

(2) *The maps* $\text{CH}_0^{LW}(X) \xrightarrow{\text{cyc}_X} H_\tau^2\left(X, \mathcal{K}_{2,X}^M\right) \leftarrow H_\tau^2\left(X, \mathcal{K}_{2,(X,D)}^M\right)$ *are isomorphisms for* $\tau = \text{zar/nis}$.

Proof. By [44, Lemma 2.2], there is an exact sequence

$$SK_1(D) \to F^2 K_0(X,D) \to F^2 K_0(X) \to 0,$$

where $SK_1(D)$ is the kernel of the edge map $K_1(D) \to H_{\text{zar}}^0\left(D, \mathcal{O}_D^\times\right)$ in the Thomason–Trobaugh spectral sequence [75, Theorem 10.3]. But this edge map is an isomorphism, because D is a 0-dimensional k-scheme.

By Lemma 2.4 and [58, Theorem 2.2], there is an exact sequence

$$SK_1\left(X_{\text{sing}}\right) \to \text{CH}_0^{LW}(X) \to F^2 K_0(X) \to 0.$$

Item (1) now follows because $SK_1\left(X_{\text{sing}}\right) = 0$. To prove (2), it is enough to work with the Nisnevich topology by Proposition 2.9. We now have the maps $\text{CH}_0^{LW}(X) \xrightarrow{\text{cyc}_X} H_{\text{nis}}^2\left(X, \mathcal{K}_{2,X}^M\right) \twoheadrightarrow F^2 K_0(X)$. The map cyc_X is surjective by [35, Theorem 2.5] (see [27, §3.5] for an explanation as to why it suffices to know that X^o is regular instead of being nice). The map cyc_X in (2) is now an isomorphism by (1). The other map in (2) is an isomorphism for dimension reasons. $\qquad\square$

3.3. Zero-cycles on a surface and its desingularizations

Let k be any field. We recall the definition of the Chow group of 0-cycles with modulus from [6]. Let X be a quasiprojective scheme over k and $D \subset X$ an effective Cartier divisor. Given a finite map $\nu\colon C \to X$ from a regular integral curve whose image is not contained in D, we let $E = \nu^*(D)$. We say that a rational function f on C has modulus E if $f \in \text{Ker}\left(\mathcal{O}_{C,E}^\times \twoheadrightarrow \mathcal{O}_E^\times\right)$. We let $G(C,E)$ denote the multiplicative subgroup of such rational functions in $k(C)^\times$. Then $\text{CH}_0(X \mid D)$ is the quotient $\mathcal{Z}_0(X \setminus D)$ by the subgroup $\mathcal{R}_0(X \mid D)$ generated by $\nu_*(\text{div}(f))$ for all possible choices of $\nu\colon C \to X$ and $f \in G(C,E)$ as above.

Let $D \subset X$ be as before. We have seen in §2.3 that there is a cycle class homomorphism $\widetilde{\text{cyc}}_{X|D}\colon \mathcal{Z}_0(X^o \setminus D) \to K_0(X,D)$. If k is perfect and $X \in \mathbf{Sm}_k$, then it was shown in [3, Theorem 12.4] that $\text{cyc}_{X|D}$ descends to a group homomorphism

$$\text{cyc}_{X|D}\colon \text{CH}_0(X \mid D) \to K_0(X,D). \tag{3.5}$$

If $\dim(X) \leq 2$, this map exists without any condition on k by [41, Theorem 1.2]. We let $F^d K_0(X,D)$ be the image of this map if X is of pure dimension d.

Let us now assume that k is any field and X is an integral projective R_1-scheme of dimension 2 over k. Let $f\colon X_n \to X$ be the normalization morphism and $f'\colon \widetilde{X} \to X_n$ a resolution of singularities of X_n (assuming it exists). Let $E \subset \widetilde{X}$ be the reduced exceptional divisor. Then $\pi = f \circ f'\colon \widetilde{X} \to X$ is a resolution of singularities of X with reduced exceptional divisor E. We let $E' = \pi^{-1}(X_{\mathrm{sing}})_{\mathrm{red}}$ so that E' is a union of E and a finite set of closed points. Note that such a resolution of singularities exists (for example, by [51]). We write $S = X_{\mathrm{sing}}$ and $S' = (X_n)_{\mathrm{sing}}$ with reduced structures.

Proposition 3.5. *Let k be a field and X an integral and projective k-scheme of dimension 2 which is R_1. Let $\widetilde{X} \xrightarrow{f'} X_n \xrightarrow{f} X$ be the desingularization and normalization morphisms as before. Let $m \geq 1$ be any integer. With the foregoing notations, we have a commutative diagram*

$$(3.6)$$

$$
\begin{array}{ccccc}
\mathrm{CH}_0^{LW}(X) & \xrightarrow[\cong]{\mathrm{cyc}_{(X,mS)}} & F^2 K_0(X,mS) & \xrightarrow{\cong} & F^2 K_0(X) \\
{\scriptstyle\cong}\downarrow{\scriptstyle f^*} & & {\scriptstyle\cong}\downarrow{\scriptstyle f^*} & & {\scriptstyle\cong}\downarrow{\scriptstyle f^*} \\
\mathrm{CH}_0^{LW}(X_n) & \xrightarrow[\cong]{\mathrm{cyc}_{(X_n,mS')}} & F^2 K_0(X_n,mS') & \xrightarrow{\cong} & F^2 K_0(X_n) \\
{\scriptstyle f'^*}\downarrow & & {\scriptstyle f'^*}\downarrow & & {\scriptstyle f'^*} \\
\mathrm{CH}_0\left(\widetilde{X} \mid mE\right) & \xrightarrow[\mathrm{cyc}_{\widetilde{X}\mid mE}]{} & F^2 K_0\left(\widetilde{X},mE\right). &
\end{array}
$$

Moreover, all arrows are isomorphisms for $m \gg 1$.

Proof. It is clear that the two squares on the top are commutative. Furthermore, all arrows in these squares are isomorphisms, by Proposition 3.4 and Corollary 2.6. It is also easy to check using Lemma 2.4 that the pullback $f'^*\colon \mathcal{Z}_0(X_n^o) \xrightarrow{\cong} \mathcal{Z}_0\left(\widetilde{X} \setminus E\right)$ induces a pullback map $f'^*\colon \mathrm{CH}_0^{LW}(X_n) \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ which makes the bottom square commutative. The map $f'^*\colon F^2 K_0(X_n,mS') \to F^2 K_0\left(\widetilde{X},mE\right)$ is an isomorphism for all $m \gg 1$ by [44, Theorem 1.1]. It follows that all arrows in the diagram are isomorphisms for $m \gg 1$. $\qquad\square$

3.4. Reciprocity theorem for surfaces

Assume now that k is a finite field and X an integral projective R_1-scheme of dimension 2 over k. It follows from Proposition 2.9 and Corollary 3.3 that the cycle class and the reciprocity homomorphisms give rise to the degree-preserving maps

$$\mathrm{CH}_0^{LW}(X) \xrightarrow{\mathrm{cyc}_X} H_{\mathrm{nis}}^2\left(X,\mathcal{K}_{2,X}^M\right) \xrightarrow{\rho_X} \pi_1^{\mathrm{ab}}(X^o). \qquad (3.7)$$

Our main result on the class field theory of X is the following:

Theorem 3.6. *The cycle class and the reciprocity homomorphisms induce isomorphisms of finite groups*

$$\mathrm{CH}_0^{LW}(X)^0 \xrightarrow{\cong} H_{\mathrm{nis}}^2\left(X,\mathcal{K}_{2,X}^M\right)^0 \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X^o)^0. \qquad (3.8)$$

Proof. By Proposition 3.4, we only have to show that the composition of the two maps in formula (3.8) is an isomorphism of finite groups. We now choose a resolution of singularities $\pi\colon \widetilde{X} \to X$ (which exists by [51]) with the reduced exceptional divisor $E \subset \widetilde{X}$. For every integer $m \geq 1$, we then have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{CH}_0^{LW}\left(X, X_{\mathrm{sing}}\right)^0 & \xrightarrow{\ \rho_X\ } & \pi_1^{\mathrm{ab}}(X^o)^0 \\
{\scriptstyle \pi^*}\Big\downarrow & & \Big\downarrow \\
\mathrm{CH}_0\left(\widetilde{X}\mid mE\right)^0 & \xrightarrow{\ \rho_{\widetilde{X}\mid mE}\ } & \pi_1^{\mathrm{ab}}\left(\widetilde{X}, mE\right)^0,
\end{array}
\tag{3.9}
$$

where $\pi_1^{\mathrm{ab}}\left(\widetilde{X}, mE\right)$ is the abelianized étale fundamental group with modulus (see [5, §8.3] for the definition) and $\pi_1^{\mathrm{ab}}\left(\widetilde{X}, mE\right)^0$ is the kernel of the map $\pi_1^{\mathrm{ab}}\left(\widetilde{X}, mE\right) \to \mathrm{Gal}\left(\overline{k}/k\right)$.

The left vertical arrow in diagram (3.9) is an isomorphism for all $m \gg 0$ by Proposition 3.5. Combining this with [38, Theorem III] (if $\mathrm{char}(k) \neq 2$) and [5, Lemma 8.4, Theorem 8.5] (in general), it follows that all arrows in the diagram are isomorphisms for all $m \gg 0$. Moreover, these groups are finite by [5, Corollary 8.3]. $\qquad\square$

4. The Lefschetz condition

In this section, we shall prove one of Grothendieck's Lefschetz conditions as a prelude to our proof of the Lefschetz hypersurface section theorem for the étale fundamental groups of the regular loci of certain projective schemes over a field. All cohomology groups in this section will be with respect to the Zariski topology on schemes.

4.1. Reflexive sheaves

Let $(X, \mathcal{O}_X)$ be a locally ringed space. If $(X, \mathcal{O}_X)$ is an integral locally ringed space (that is, the stalks of $\mathcal{O}_X$ are integral domains), we let $\mathcal{K}_X$ denote the sheaf of field of fractions of $\mathcal{O}_X$. Recall that for a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X, the dual $\mathcal{E}^\vee$ is the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. There is a natural evaluation map $\mathrm{ev}_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E}^{\vee\vee}$ whose kernel is the subsheaf of torsion submodules of $\mathcal{E}$, where the latter is defined as the kernel of the canonical map $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. We denote either of these kernels by $\mathcal{E}_{\mathrm{tor}}$. If $\mathcal{E}$ is torsion-free, one calls $\mathcal{E}^{\vee\vee}$ the reflexive hull of $\mathcal{E}$. One says that $\mathcal{E}$ is reflexive if the evaluation map $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an isomorphism. The following lemma is elementary:

Lemma 4.1. *The dual of any sheaf of $\mathcal{O}_X$-modules is reflexive.*

Proof. We need to show that for any sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ with dual $\mathcal{N}$, the evaluation map $\mathrm{ev}_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}^\vee, \mathcal{O}_X)$ is an isomorphism. Since this is a local condition on X, we can assume that $X = \mathrm{Spec}(A)$ for a commutative ring A and represent $\mathcal{M}$ (resp., $\mathcal{N}$) by M (resp., N).

Now, $\mathrm{ev}_N(f) = 0$ implies that $f(x) = \mathrm{ev}_N(f)(\mathrm{ev}_M(x)) = 0$ for all $x \in M$. Equivalently, $f = 0$. This shows that ev_N is injective. To show the surjectivity, set $\alpha \in \mathrm{Hom}_A(N^\vee, A)$ and let $f_\alpha\colon M \to A$ be given by $f_\alpha(x) = \alpha(\mathrm{ev}_M(x))$. It is then clear that $\alpha = \mathrm{ev}_N(f_\alpha)$. $\qquad\square$

Let $(X, \mathcal{O}_X)$ be a locally ringed space with Noetherian stalks and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. Recall that $\mathcal{E}$ is said to satisfy the S_i property for some $i \geq 0$ if for every $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{E}_x$ satisfies Serre's S_i condition – that is, $\mathrm{depth}(\mathcal{E}_x) \geq \min\{i, \dim(\mathcal{E}_x)\}$ for every point $x \in X$. One says that X is a locally S_i-ringed space (or an S_i-scheme, if X is a Noetherian scheme) if it has Noetherian stalks and $\mathcal{O}_X$ satisfies the S_i property. The following is easy to verify using [11, Lemma 0AV6]:

Lemma 4.2. *Let $(X, \mathcal{O}_X)$ be a locally S_2-ringed space and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module. Then $\mathcal{E}^\vee$ satisfies the S_2 property. In particular, every reflexive coherent $\mathcal{O}_X$-module satisfies the S_2 property.*

Let k be a field. Suppose that X is an integral k-scheme and there is a locally closed immersion $X \hookrightarrow \mathbb{P}_k^N$. We let $\mathcal{O}_X^r(m)$ be the restriction of the sheaf $\left(\mathcal{O}_{\mathbb{P}_k^N}(m)\right)^{\oplus r}$ onto X. Let $\mathcal{E}$ be a reflexive coherent sheaf on X. We can then prove the following:

Lemma 4.3. *There are integers $q, q' \in \mathbb{Z}$, $r, r' \geq 1$, and a coherent sheaf $\mathcal{E}'$ together with exact sequences*

$$0 \to \mathcal{E} \to \mathcal{O}_X^r(q) \to \mathcal{E}' \to 0, \tag{4.1}$$
$$0 \to \mathcal{E}' \to \mathcal{O}_X^{r'}(q') \to \mathcal{H} \to 0.$$

In particular, each of $\mathcal{E}$ and $\mathcal{E}'$ is torsion-free (or zero). If $\mathcal{E}$ is locally free, so is $\mathcal{E}'$.

Proof. Since $\mathcal{E}^\vee$ is coherent, there is a surjection $\mathcal{O}_X^r(-q) \twoheadrightarrow \mathcal{E}^\vee$ for some $q, r \gg 0$. We let $\mathcal{F}$ be the kernel of this surjection. Since $\mathcal{F}$ is necessarily coherent, we also have a surjection $\mathcal{O}_X^{r'}(-q') \twoheadrightarrow \mathcal{F}$ for some $q', r' \gg 0$. Letting $\mathcal{F}'$ be its kernel, we get short exact sequences of coherent sheaves

$$0 \to \mathcal{F} \to \mathcal{O}_X^r(-q) \to \mathcal{E}^\vee \to 0,$$
$$0 \to \mathcal{F}' \to \mathcal{O}_X^{r'}(-q') \to \mathcal{F} \to 0.$$

By dualizing, we get exact sequences

$$0 \to \mathcal{E} \to \mathcal{O}_X^r(q) \to \mathcal{F}^\vee \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X) \to 0, \tag{4.2}$$
$$0 \to \mathcal{F}^\vee \to \mathcal{O}_X^{r'}(q') \to \mathcal{F}'^\vee \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \to 0.$$

Letting $\mathcal{E}'$ be the cokernel of $\mathcal{E} \to \mathcal{O}_X^r(q)$, we get the two exact sequences (4.1). If $\mathcal{E}$ is locally free, then so is $\mathcal{E}^\vee$. In this case, $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X) = 0$ and $\mathcal{F}$ must also be locally free. This implies that $\mathcal{E}'$ is locally free. $\square$

4.2. The Hartogs lemma

We need to prove a version of the Hartogs lemma for formal schemes. Before we do this, we recall this result for the ordinary schemes.

Lemma 4.4. *Let A be a Noetherian integral domain such that $X = \mathrm{Spec}(A)$ is an S_2-scheme. Let $U \subset X$ be an open subscheme whose complement has codimension ≥ 2 in X.*

Let E be a finitely generated reflexive A-module and $\mathcal{E}$ the associated Zariski sheaf on X. Then the canonical map $E \to H^0(U, \mathcal{E}_U)$ is an isomorphism.

Proof. Since E is reflexive, we can find an exact sequence of A-modules

$$0 \to E \to A^r \to A^{r'} \tag{4.3}$$

for some $r, r' \geq 1$. This gives rise to a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
0 \longrightarrow & E & \longrightarrow A^r & \longrightarrow A^{r'} & \\
& \downarrow{\scriptstyle j^*} & \downarrow{\scriptstyle j^*} & \downarrow{\scriptstyle j^*} & \\
0 \to & H^0(U, \mathcal{E}_U) & \to \mathcal{O}(U)^r & \to \mathcal{O}(U)^{r'}, &
\end{array}
\tag{4.4}
$$

where $j : U \hookrightarrow X$ is the inclusion. A diagram chase shows that we can assume that $E = A$. But this case follows from Lemma 4.5. $\qquad\square$

The following lemma is well known:

Lemma 4.5. *Let A be a Noetherian integral domain with quotient field K which is an S_2-ring. Then $A = \bigcap\limits_{\mathrm{depth}\, A_{\mathfrak{p}} = 1} A_{\mathfrak{p}}$ inside K.*

Proof. The proof of this lemma is identical to that of [11, Lemma 031T(2)] once we observe that in a Noetherian integral domain which is an S_2-ring, a prime ideal has height 1 if and only if its depth is 1. We add a proof of the lemma for completeness.

Suppose $a, b \in A$ are two nonzero elements such that $a \in bA_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \subset A$ of depth 1. We need to show that $a \in (b)$.

Let $\mathrm{Ass}(b)$ denote the set of associated primes of (b). Since the integral domain A is an S_2-ring and $b \neq 0$, an easy consequence of [25, Propositions 16.4.6(ii), 16.4.10(i)] is that $A/(b)$ is an S_1-ring (see, for example, [20, Lemma 2.1]). In particular, all associated primes of $A/(b)$ have height 0. The latter statement is equivalent to saying that all associated primes of (b) are minimal, and hence have height 1 by Krull's principal ideal theorem [52, Theorem 13.5]. Since $1 \leq \mathrm{depth}\,(A_{\mathfrak{p}}) \leq \dim\,(A_{\mathfrak{p}}) \leq 1$, we conclude that $\mathrm{depth}\,(A_{\mathfrak{p}}) = 1$ for every $\mathfrak{p} \in \mathrm{Ass}(b)$. This implies by our hypothesis that $a \in bA_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Ass}(b)$. We are now done, because the canonical map

$$A/(b) \to \prod_{\mathfrak{p} \in \mathrm{Ass}(b)} A_{\mathfrak{p}}/bA_{\mathfrak{p}}$$

is injective (see, for example, [11, Lemma 0311]). $\qquad\square$

Lemma 4.6. *Let X be a Noetherian integral S_2-scheme and $j : U \hookrightarrow X$ an open immersion whose complement has codimension ≥ 2 in X. Then the unit-of-adjunction map $\mathcal{E} \to j_* \mathcal{E}_U$ is an isomorphism for every reflexive coherent Zariski sheaf $\mathcal{E}$ on X.*

Proof. This is a local question, and hence follows directly from Lemma 4.4. $\qquad\square$

Corollary 4.7. *Let $\mathcal{E}$ be as in Lemma 4.6. Then the restriction map $H^0(X, \mathcal{E}) \to H^0(U, \mathcal{E}_U)$ is an isomorphism.*

Proof. This is immediate from Lemma 4.6. □

Corollary 4.8. *Let $U \subset X$ be an open immersion as in Lemma 4.6 and $\mathcal{E}$ a reflexive coherent sheaf on U. Then $j_*\mathcal{E}$ is a reflexive coherent sheaf on X.*

Proof. By choosing a coherent extension of $\mathcal{E}$ on X and taking its double dual, we can find a reflexive coherent sheaf $\mathcal{E}'$ on X such that $j^*\mathcal{E}' \cong \mathcal{E}$. We now apply Lemma 4.6 to conclude the proof. □

4.3. The formal Hartogs lemma

Let X be a Noetherian scheme and $Y \subset X$ a closed subscheme. Let $\widehat{X}$ denote the formal completion of X along Y (see [21, Chapitre 0, §9] or [29, Chapter II, §10]). Let $\mathcal{I}_Y$ denote the sheaf of ideals on X defining Y. Let Y_m denote the closed subscheme of X defined by the ideal sheaf $\mathcal{I}_Y^m$. For any open subscheme $U \subset X$ such that $V := U \cap Y$ is dense in Y, we have a commutative diagram of Noetherian formal schemes

$$
\begin{array}{ccc}
V_m \xrightarrow{\iota_U} \widehat{U} \xrightarrow{c_U} U \\
{\scriptstyle j_Y}\downarrow \quad {\scriptstyle \widehat{j}}\downarrow \quad \quad {\scriptstyle j}\downarrow \\
Y_m \xrightarrow{\iota} \widehat{X} \xrightarrow{c_X} X,
\end{array}
\tag{4.5}
$$

where $V_m := U \cap Y_m$. In the right square, all arrows are flat morphisms (of locally ringed spaces), the vertical arrows are open immersions (see [11, §01HD] for the definition of open immersion of locally ringed spaces), and the horizontal arrows are the completion morphisms. In the left square, the vertical arrows are open immersions and the horizontal arrows are closed immersions. The two squares are Cartesian. The compositions of the two horizontal arrows are the given closed immersions $Y_m \hookrightarrow X$ and $V_m \hookrightarrow U$ of schemes.

For any quasicoherent sheaf $\mathcal{F}$ on X, we let $\widehat{\mathcal{F}}$ denote the pullback of $\mathcal{F}$ under c_X. Note that the canonical map $\widehat{\mathcal{F}} \to \varprojlim_m \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_m}$ is an isomorphism if $\mathcal{F}$ is coherent. We shall use this isomorphism later. For any morphism $f \colon X' \to X$ and quasicoherent sheaf $\mathcal{F}$ on X, we let $\mathcal{F}_{X'} = f^*(\mathcal{F})$. We shall write $\mathcal{F}_{Y_m}$ simply as $\mathcal{F}_m$ for $m \geq 1$. Our goal in this subsection is to prove a formal version of the Hartogs lemma.

Let A be an excellent normal integral domain and $J \subset A$ the radical ideal such that $V(J) = X_{\mathrm{sing}}$, where $X = \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be a complete intersection prime ideal such that $\mathrm{ht}(\mathfrak{p} + J) \geq \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(J)$ and $A/\mathfrak{p}$ is normal. Let $U \subset X$ be an open subscheme containing $V(\mathfrak{p}) \cap X^o$ whose complement has codimension ≥ 2 in X. Let $\widehat{A}$ be the $\mathfrak{p}$-adic completion of A and $\widehat{X} = \mathrm{Spf}\left(\widehat{A}\right)$, the formal spectrum of $\widehat{A}$. Let $\widehat{U}$ be the formal completion of U along $V(\mathfrak{p}) \cap U$. We let $Y = \mathrm{Spec}(A/\mathfrak{p})$ and $A_m := A/\mathfrak{p}^m$. Since A is excellent, so is U. It follows therefore from [22, Corollaire 6.5.4, Scholie 7.8.3(v)] that $\widehat{X}$ and $\widehat{U}$ are both normal integral formal schemes. Under these conditions, we have the following:

Lemma 4.9. *For any finitely generated reflexive A-module E with the associated Zariski sheaf $\mathcal{E}$ on X, the canonical map $\widehat{E} \to H^0\left(\widehat{U}, \widehat{\mathcal{E}}_U\right)$ is an isomorphism.*

Proof. Since E is reflexive, we can find an exact sequence of A-modules

$$0 \to E \to A^r \to A^{r'} \tag{4.6}$$

for some $r, r' \geq 1$. Since all arrows in the right square of diagram (4.5) are flat, we have an exact sequence of coherent sheaves

$$0 \to \widehat{\mathcal{E}} \to \mathcal{O}_{\widehat{X}}^r \to \mathcal{O}_{\widehat{X}}^{r'} \tag{4.7}$$

on $\widehat{X}$. This gives rise to a commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{E} & \longrightarrow & \widehat{A}^r & \longrightarrow & \widehat{A}^{r'} \\
 & & \widehat{j}^* \downarrow & & \widehat{j}^* \downarrow & & \widehat{j}^* \downarrow \\
0 & \to & H^0\left(\widehat{U}, \widehat{\mathcal{E}}_{\widehat{U}}\right) & \to & \mathcal{O}\left(\widehat{U}\right)^r & \to & \mathcal{O}\left(\widehat{U}\right)^{r'},
\end{array}
\tag{4.8}
$$

where $j \colon U \hookrightarrow X$ is the inclusion. A diagram chase shows that we can assume that $E = A$.

We now let $V = Y \cap U$. Let $j \colon U \hookrightarrow X$ and $\widehat{j} \colon \widehat{U} \hookrightarrow \widehat{X}$ denote the inclusion maps. We then have a commutative diagram

$$
\begin{array}{ccc}
\widehat{A} & \xrightarrow{\quad \cong \quad} & \varprojlim_{m \geq 1} A_m \\
\widehat{j}^* \downarrow & & \downarrow \widehat{j}^* \\
H^0\left(\widehat{U}, \mathcal{O}_{\widehat{U}}\right) & \xrightarrow{\cong} & \varprojlim_{n \geq 1} H^0\left(V_m, \mathcal{O}_{V_m}\right).
\end{array}
\tag{4.9}
$$

The top horizontal arrow is clearly an isomorphism, and the Milnor exact sequence for the cohomology of inverse limit sheaves (see, for example, [31, Proposition 1.6]) implies that the bottom horizontal arrow is also an isomorphism. It suffices therefore to show that the right vertical arrow is an isomorphism. We shall show by induction the stronger assertion that the restriction map $A_m \to H^0(V_m, \mathcal{O}_{V_m})$ is an isomorphism for all $m \geq 1$.

Let us first assume that $m = 1$. It follows from our assumption that Y is a Noetherian normal integral scheme in which the codimension of $Y \setminus V$ is ≥ 2. Therefore, the map $A_1 \to H^0(V, \mathcal{O}_V) = H^0(V_1, \mathcal{O}_{V_1})$ is an isomorphism by Lemma 4.4. We now prove the general case by induction on m. We consider the exact sequence of A_{m+1}-modules

$$0 \to \mathfrak{p}^m / \mathfrak{p}^{m+1} \to A_{m+1} \to A_m \to 0. \tag{4.10}$$

Since $\mathfrak{p} \subset A$ is a complete intersection, it follows that $\mathfrak{p}^m / \mathfrak{p}^{m+1} \cong S^m\left(\mathfrak{p}/\mathfrak{p}^2\right)$ is a free $A/\mathfrak{p}$-module. That is, $\mathfrak{p}^m / \mathfrak{p}^{m+1} \cong A_1^q$ for some $q \geq 1$. We thus get an exact sequence of A_{m+1}-modules

$$0 \to A_1^q \to A_{m+1} \to A_m \to 0. \tag{4.11}$$

This gives rise to the exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_Y^q \to \mathcal{O}_{Y_{m+1}} \to \mathcal{O}_{Y_m} \to 0. \tag{4.12}$$

Considering the cohomology groups and comparing them on X and U, we get a commutative diagram of exact sequences

$$0 \longrightarrow A_1^q \longrightarrow A_{m+1} \longrightarrow A_m \longrightarrow 0 \qquad (4.13)$$

$$0 \to H^0(V_1, \mathcal{O}_{V_1})^q \to H^0\left(V_{m+1}, \mathcal{O}_{V_{m+1}}\right) \to H^0(V_m, \mathcal{O}_{V_m}) \xrightarrow{\partial} H^1(V_1, \mathcal{O}_{V_1})^q.$$

The left and right vertical arrows are isomorphisms by induction on m. It follows from the five lemma that the middle vertical arrow must also be an isomorphism. This concludes the proof of the claim and the lemma. $\square$

Let X be an excellent normal integral scheme and $Y \subset X$ a local complete intersection closed subscheme which is integral and normal. Let $U \subset X$ be an open subscheme containing $Y \cap X^o$ such that $X \setminus U$ has codimension ≥ 2 in X. Under these hypotheses, we have the following 'formal Hartogs lemma':

Lemma 4.10. *Let $\mathcal{E}$ be a reflexive coherent sheaf on X. Then the unit-of-adjunction map $\widehat{\mathcal{E}} \to \widehat{j}_*\left(\widehat{\mathcal{E}_U}\right)$ is an isomorphism.*

Proof. Since this is a local question, we can assume that $X = \mathrm{Spec}(A)$ is affine and Y is a complete intersection on X. In this case, $\mathcal{E}$ is the Zariski sheaf associated to a finitely generated reflexive A-module E. Furthermore, the lemma is equivalent to showing that the canonical map $\widehat{E} \to H^0\left(\widehat{U}, \widehat{\mathcal{E}_U}\right)$ is an isomorphism. But this is the content of Lemma 4.9. $\square$

Corollary 4.11. *Let $\mathcal{E}$ be as in Lemma 4.10. Then the restriction map $H^0\left(\widehat{X}, \widehat{\mathcal{E}}\right) \to H^0\left(\widehat{U}, \widehat{\mathcal{E}_U}\right)$ is an isomorphism.*

Proof. This is immediate from Lemma 4.10. $\square$

4.4. The Lef(X, Y) condition

Let X be a Noetherian scheme and $Y \subset X$ a closed (resp., open) subscheme. We shall then say that (X, Y) is a closed (resp., open) pair. Recall from [23, §2, Exposé X] that a closed pair (X, Y) is said to satisfy the Lefschetz condition, and one says that Lef(X, Y) holds if, for any open subscheme $U \subset X$ containing Y and any coherent locally free sheaf $\mathcal{E}$ on U, the restriction map $H^0(U, \mathcal{E}) \to H^0\left(\widehat{X}, \widehat{\mathcal{E}}\right)$ is an isomorphism. We shall not recall the effective Lefschetz condition, because we do not need it.

We shall work with the following setup throughout this subsection. We fix a field k and let $X \hookrightarrow \mathbb{P}_k^N$ be an integral normal projective scheme over k of dimension $d \geq 3$. Suppose we are given a closed immersion $\iota: Y \hookrightarrow X$ of integral normal schemes such that $X^o \cap Y \subset Y^o$ and $2 \leq \dim(Y) \leq d-1$. We further assume that $Y \subset X$ is cut out by e general hypersurfaces in $\mathbb{P}_k^N$ such that Y is a complete intersection in X, where $e = \mathrm{codim}(Y, X)$. Let $U \subset X$ be an open subscheme containing $X^o \cap Y$ such that $X \setminus U$ has codimension ≥ 2 in X. Let $j: U \hookrightarrow X$ be the inclusion map. We shall let $\widehat{X}$ (resp., $\widehat{U}$)

denote the formal completion of X (resp., U) along Y (resp., $U \cap Y$). We shall continue to use the notations of diagram (4.5).

Lemma 4.12. *For any coherent reflexive sheaf $\mathcal{E}$ on X, the pullback map $H^0(X, \mathcal{E}) \to H^0\left(\widehat{X}, \widehat{\mathcal{E}}\right)$ is an isomorphism.*

Proof. Using the Milnor exact sequence for the cohomology of inverse limit sheaves, the lemma is equivalent to showing that the map $H^0(X, \mathcal{E}) \to \varprojlim_{m \geq 1} H^0(Y_m, \mathcal{E}_m)$ is an isomorphism. But this follows immediately from Lemma 4.2 and [11, Proposition 0EL1], since X is normal (hence S_2) and $d \geq 3$. $\qquad \square$

Lemma 4.13. *For any coherent reflexive sheaf $\mathcal{E}$ on U, the pullback map $H^0(U, \mathcal{E}) \to H^0\left(\widehat{U}, \widehat{\mathcal{E}}\right)$ is an isomorphism.*

Proof. We let $\mathcal{E}' = j_* \mathcal{E}$. It follows from Corollary 4.8 that $\mathcal{E}'$ is a reflexive coherent sheaf on X. We now consider the commutative diagram

$$
\begin{array}{ccc}
H^0(X, \mathcal{E}') & \xrightarrow{c_X^*} & H^0\left(\widehat{X}, \widehat{\mathcal{E}}'\right) \\
{\scriptstyle j^*} \downarrow & & \downarrow {\scriptstyle \widehat{j}^*} \\
H^0(U, \mathcal{E}) & \xrightarrow{c_U^*} & H^0\left(\widehat{U}, \widehat{\mathcal{E}}\right).
\end{array}
\tag{4.14}
$$

The top horizontal arrow is an isomorphism by Lemma 4.12. Corollary 4.7 implies that the left vertical arrow is an isomorphism. Since X is excellent and normal, Corollary 4.11 implies that the right vertical arrow is an isomorphism. A diagram chase now finishes the proof. $\qquad \square$

Lemma 4.14. *One has $Y^o = X^o \cap Y$.*

Proof. Since we are already given that $X^o \cap Y \subset Y^o$, the lemma follows because at any point $x \in X_{\mathrm{sing}} \cap Y$, the ideal of Y is defined by a regular sequence. $\qquad \square$

It follows from Lemma 4.14 that (X^o, Y^o) is a closed pair. We can now prove the following:

Proposition 4.15. *Assume that Y intersects X_{sing} properly in X. Then $\mathrm{Lef}(X^o, Y^o)$ holds.*

Proof. Let $U \subset X^o$ be an open subscheme containing Y^o. In view of Lemma 4.13, we only need to show that $X \setminus U$ has codimension ≥ 2 in X. Suppose to the contrary that there is an irreducible closed subscheme $Z \subset X \setminus U$ of dimension $d-1$. We must then have that the scheme theoretic intersections $Z \cap Y$ and $Z \cap (Y \cap X_{\mathrm{sing}})$ have same support, and therefore $\dim(Z \cap Y) \leq \dim(Y \cap X_{\mathrm{sing}})$. We also get

$$
\dim(Z \cap Y) \geq \dim(Z) - e = d - 1 - (d - \dim(Y)) = \dim(Y) - 1 > \dim(Y \cap X_{\mathrm{sing}}),
$$

and since $Y \subset X$ is cut out by e general hypersurfaces in $\mathbb{P}_k^N$, the intersection of Y and X_{sing} is proper and X is catenary. This is clearly a contradiction. $\qquad\square$

5. Lefschetz for étale fundamental group

In this section, we shall prove a Lefschetz hypersurface section theorem for the abelianized étale fundamental group of the regular locus of a normal projective scheme over a field under certain conditions. Our setup for this section is the following.

Let k be a field of exponential characteristic $p \geq 1$ and $X \hookrightarrow \mathbb{P}_k^N$ an integral projective scheme over k of dimension $d \geq 3$. We let $H \subset \mathbb{P}_k^N$ be a hypersurface such that the scheme-theoretic intersection $Y = X \cap H$ satisfies the following:

(1) Y is integral of dimension $d-1$.

(2) $X^o \cap Y$ is regular.

(3) Y is normal if X is so.

(4) Y does not contain any irreducible component of X_{sing}.

A hypersurface section Y satisfying these conditions will be called a 'good' hypersurface section. Note that if X is an R_a-scheme for some $a \geq 0$ and Y is good, then Y too is an R_a-scheme by (2) and (4). Since X_{sing} is reduced, it follows from (4) that the scheme-theoretic intersection $Y_s := Y \cap X_{\mathrm{sing}}$ is an effective Cartier divisor on X_{sing} [26, Lemma 3.3] and $(Y_s)_{\mathrm{red}} = Y_{\mathrm{sing}}$ (Lemma 4.14). We let

$$\iota\colon Y \hookrightarrow X,\ \iota^o\colon Y^o \hookrightarrow X^o,\ j\colon X^o \hookrightarrow X,\ \tilde{j}\colon Y^o \hookrightarrow Y,\ \tilde{\iota}\colon Y_s \hookrightarrow X_{\mathrm{sing}}$$

be the inclusions. We let $q = \dim(Y_s)$.

5.1. The Enriques–Severi–Zariski theorem

We shall need a version of the Enriques–Severi–Zariski theorem for some singular projective schemes and their regular loci. Suppose that $H \subset \mathbb{P}_k^N$ is a hypersurface of degree $m \geq 1$ such that $Y = X \cap H$ is good. Let $C_{Y/X}$ denote the conormal sheaf on Y associated to the regular embedding $\iota\colon Y \hookrightarrow X$ so that $C_{Y/X} = \mathcal{I}_Y / \mathcal{I}_Y^2 \cong \mathcal{O}_Y(-m)$, where $\mathcal{I}_Y$ is the sheaf of ideals on X defining Y. For any coherent sheaf $\mathcal{F}$ on Y and integer $n \geq 1$, we define $\mathcal{F}^{[n]} := \mathcal{F} \otimes_{\mathcal{O}_Y} S^n (C_{Y/X}) \cong \mathcal{F}(-nm)$. For any coherent sheaf $\mathcal{F}$ on Y^o and integer $n \geq 1$, we define $\mathcal{F}^{[n]} := \mathcal{F} \otimes_{\mathcal{O}_{Y^o}} S^n (C_{Y^o/X^o}) \cong \mathcal{F}(-nm)$.

Lemma 5.1. *Assume that X is an $(R_1 + S_b)$-scheme for some $b \geq 2$. Let $\mathcal{E}$ be a locally free sheaf on X. Then $H^i_{\mathrm{zar}}(X, \mathcal{E}(-j)) = 0$ for $i \leq b-1$ and $j \gg 0$.*

Proof. Let $\omega_{X/k}^\bullet$ denote the dualizing complex for X. Under the assumption of the lemma, it follows from [47, Lemma 4.27] that $\omega_{X/k}^\bullet \in D^{[-d,-b]}(X)$. Moreover, we have $\mathcal{H}^{-d}\left(\omega_{X/k}^\bullet\right) \cong j_*\left(\omega_{X^o/k}\right)$, where $\omega_{X^o/k}$ is the canonical sheaf of X^o. Since X is an $(R_1 + S_b)$-scheme for some $b \geq 2$, it is normal (see, for example, [52, Theorem 23.8]).

We can therefore apply the Grothendieck–Serre duality for normal projective schemes to get

$$H^i_{\mathrm{zar}}(X, \mathcal{E}(-j)) \cong \mathbb{H}^{-i}_{\mathrm{zar}}\left(X, \mathcal{E}^\vee \otimes^L_{\mathcal{O}_X} \omega^\bullet_{X/k}(j)\right).$$

The desired assertion now follows easily from the Serre vanishing theorem. $\square$

Lemma 5.2. *Assume that X is normal and let $\mathcal{E}$ be a coherent reflexive sheaf on Y^o. Then $H^0\left(Y^o, \mathcal{E}^{[n]}\right) = 0$ for $n \gg 0$.*

Proof. It follows from Corollary 4.8 that $\tilde{j}_* \mathcal{E}$ is a coherent reflexive sheaf on Y. We denote this extension by $\mathcal{E}$ itself. Using the exact sequences of Lemma 4.3 and tensoring them with $S^n\left(C_{Y/X}\right)$ (which is invertible on Y) and subsequently taking the cohomology, we get exact sequences

$$0 \to H^0\left(Z, \mathcal{E}^{[n]}\right) \to H^0\left(Z, \mathcal{O}^r_Z(q)^{[n]}\right) \to H^0\left(Z, \mathcal{E}'^{[n]}\right) \to H^1\left(Z, \mathcal{E}^{[n]}\right) \to H^1\left(Z, \mathcal{O}^r_Z(q)^{[n]}\right) \tag{5.1}$$

and

$$0 \to H^0\left(Z, \mathcal{E}'^{[n]}\right) \to H^0\left(Z, \mathcal{O}^{r'}_Z(q')^{[n]}\right), \tag{5.2}$$

where $Z \in \{Y, Y^o\}$. Corollary 4.7 and Lemma 5.1 together tell us that $H^0\left(Z, \mathcal{O}^r_Z(q)^{[n]}\right) = 0 = H^0\left(Z, \mathcal{O}^{r'}_Z(q')^{[n]}\right)$ for all $n \gg 0$. We conclude that $H^0\left(Z, \mathcal{E}^{[n]}\right) = 0$ and

$$H^1\left(Z, \mathcal{E}^{[n]}\right) \hookrightarrow H^1\left(Z, \mathcal{O}^r_Z(q)^{[n]}\right) \tag{5.3}$$

for all $n \gg 0$. $\square$

5.2. The Gysin map and Poincaré duality

We fix a prime-to-p integer n and let $\Lambda = \mathbb{Z}/n$ be the constant sheaf of rings on the étale site of $\mathbf{Sch}_k$. For any integer $m \in \mathbb{Z}$, we let $\Lambda(m)$ denote the usual Tate twist of the sheaf of nth roots of unity μ_n on the étale site of $\mathbf{Sch}_k$. For any étale sheaf of Λ-modules $\mathcal{F}$ on $\mathbf{Sch}_k$, we let $\mathcal{F}(m) = \mathcal{F} \otimes_\Lambda \Lambda(m)$. Let $D^+(X, \Lambda)$ denote the derived category of the bounded-below complexes of the sheaves of Λ-modules on the small étale site of X.

Let $\iota \colon Y \hookrightarrow X$ be as in §5.1. Recall from Gabber's construction [16] (see also [56, Definition 2.1]) that the regular closed embedding $\iota \colon Y \hookrightarrow X$ induces the Gysin homomorphism $\iota_* \colon H^i_{\text{ét}}(Y, \Lambda(m)) \to H^{i+2}_{\text{ét}}(X, \Lambda(m+1))$ for any pair of integers $i \geq 0$ and $m \in \mathbb{Z}$. This homomorphism is defined as follows. It follows from [14, §2] that the line bundle $\mathcal{O}_X(Y)$ (which we shall write for short as $\mathcal{O}(Y)$) on X has a canonical class $[\mathcal{O}(Y)] \in H^1_{Y, \text{ét}}(X, \mathbb{G}_m)$, and its image via the boundary map $H^1_{Y, \text{ét}}(X, \mathbb{G}_m) \to H^2_{Y, \text{ét}}(X, \Lambda)$ is Deligne's localized Chern class $c_1(Y)$. Here, $H^*_{Y, \text{ét}}(X, -)$ denotes the étale cohomology with support in Y.

At the level of the derived category $D^+(Y, \Lambda)$, this Chern class is given in terms of the map $c_1(Y) \colon \Lambda_Y \to \iota^! \Lambda_X(1)[2]$. Using this Chern class, Gabber's Gysin homomorphism

$\iota_*\colon H^*_{\acute{e}t}(Y,\Lambda(m))\to H^{*+2}_{\acute{e}t}(X,\Lambda(m+1))$ is the one induced on the cohomology by the composite map $\iota_*\colon \iota_*(\Lambda_Y)\to \iota_*\iota^!(\Lambda_X(1)[2])\to \Lambda_X(1)[2]$ in $D^+(X,\Lambda)$, where the second arrow is the counit-of-adjunction map.

The local complete intersection closed immersions $Y\hookrightarrow X$ and $Y_s\hookrightarrow X_{\mathrm{sing}}$ induce a diagram of distinguished triangles in $D^+(X,\Lambda)$ given by

$$\begin{array}{ccc}
\iota_*\circ\tilde{j}_!\left(\Lambda_{Y^o}\right) \longrightarrow \iota_*\left(\Lambda_Y\right) \longrightarrow \iota_*\circ\tilde{\iota}_*\left(\Lambda_{Y_s}\right) \\
\downarrow{\scriptstyle c_1(Y)} \qquad\qquad \downarrow{\scriptstyle c_1(Y_s)} \\
j_!\left(\Lambda_{X^o}\right)(1)[2] \to \Lambda_X(1)[2] \to \iota_*\left(\Lambda_{X_{\mathrm{sing}}}\right)(1)[2].
\end{array} \qquad (5.4)$$

The Cartesian square

$$\begin{array}{ccc}
Y_s & \xrightarrow{\tilde{\iota}} & X_{\mathrm{sing}} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\iota} & X
\end{array} \qquad (5.5)$$

and the functoriality of Deligne's localized Chern class imply (see [56, Corollary 2.12] or [16, Proposition 1.1.3]) that the right-side square in diagram (5.4) is commutative. It follows that there is a Gysin morphism $c_1^{j_!}(Y^o)\colon \iota_*\circ\tilde{j}_!\left(\Lambda_{Y^o}\right)\to j_!\left(\Lambda_{X^o}\right)(1)[2]$ in $D^+(X,\Lambda)$ such that the resulting left square in diagram (5.4) is commutative.

Applying the cohomology functor on $D^+(X,\Lambda)$, we get a commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
H^*_{c,\acute{e}t}(Y^o,\Lambda(m)) & \longrightarrow & H^*_{\acute{e}t}(Y,\Lambda(m)) & \longrightarrow & H^*_{\acute{e}t}(Y_s,\Lambda(m)) & \longrightarrow & H^{*+1}_{c,\acute{e}t}(Y^o,\Lambda(m)) \\
\downarrow{\scriptstyle \iota^o_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \tilde{\iota}_*} & & \downarrow{\scriptstyle \iota^o_*} \\
H^{*+2}_{c,\acute{e}t}(X^o,\Lambda(m+1)) \to & H^{*+2}_{\acute{e}t}(X,\Lambda(m+1)) \to & H^{*+2}_{\acute{e}t}(X_{\mathrm{sing}},\Lambda(m+1)) \to & H^{*+3}_{c,\acute{e}t}(X^o,\Lambda(m+1)),
\end{array}$$
$$(5.6)$$

where $H^*_{c,\acute{e}t}(-,\Lambda(m))$ denotes the étale cohomology with compact support [54, Chapter VI, §3] and the vertical arrows are the Gysin homomorphisms.

Assume now that k is either finite or algebraically closed. Recall in this case (see [32, Introduction] and [54, Chapter VI, §11], as well as [12, Corollary 4.2.3]) that there is a perfect pairing

$$H^i_{c,\acute{e}t}(X^o,\Lambda(m))\times H^{2d+e-i}_{\acute{e}t}(X^o,\Lambda(d-m))\to H^{2d+e}_{c,\acute{e}t}(X^o,\Lambda(d))\cong\Lambda \qquad (5.7)$$

for $m\in\mathbb{Z}$, where $e=1$ if k is finite and $e=0$ if k is algebraically closed. This pairing exists even if X^o is not smooth, but may not be perfect in the latter case. It is well known and follows from its construction (see [54, proof of Theorem VI.11.1] or directly use

[12, §3.3.13 and Remark 4.2.5]) that formula (5.7) is compatible with the closed immersion $\iota^o \colon Y^o \hookrightarrow X^o$ – that is, there is a commutative diagram

$$
\begin{array}{ccccc}
H^i_{c,\acute{e}t}(Y^o,\Lambda(m-1)) & \times & H^{2d-2+e-i}_{\acute{e}t}(Y^o,\Lambda(d-m)) & \to & H^{2d-2+e}_{c,\acute{e}t}(Y^o,\Lambda(d-1)) \\
\iota^o_* \downarrow & & \uparrow (\iota^o)^* & & \downarrow \iota^o_* \\
H^{i+2}_{c,\acute{e}t}(X^o,\Lambda(m)) & \times & H^{2d-2+e-i}_{\acute{e}t}(X^o,\Lambda(d-m)) & \longrightarrow & H^{2d+e}_{c,\acute{e}t}(X^o,\Lambda(d)).
\end{array}
\tag{5.8}
$$

5.3. A Lefschetz theorem for étale cohomology of X^o

Let $\iota \colon Y \hookrightarrow X$ be as in §5.1. We shall now prove a Lefschetz theorem for the étale cohomology of X^o. This is of independent interest in the study of singular varieties. In this paper, we shall use it in the proofs of the main results.

Proposition 5.3. *Assume that k is either finite or algebraically closed and X is an R_2-scheme. Then the pullback map $H^i_{\acute{e}t}(X^o,\Lambda) \to H^i_{\acute{e}t}(Y^o,\Lambda)$ is an isomorphism for $i \leq 1$ and injective for $i = 2$.*

Proof. Using diagram (5.8), the proposition is reduced to showing that the Gysin homomorphism

$$
\iota^o_* \colon H^i_{c,\acute{e}t}(Y^o,\Lambda(d-1)) \to H^{i+2}_{c,\acute{e}t}(X^o,\Lambda(d))
\tag{5.9}
$$

is an isomorphism for $i \geq 2d+e-3$ and surjective for $i = 2d+e-4$.

Using the long exact sequences of diagram (5.6) and the known bounds on the étale cohomological dimensions of X_{sing} and Y_s, the assertion that formula (5.9) is an isomorphism is equivalent to asserting that the Gysin homomorphism

$$
\iota_* \colon H^i_{\acute{e}t}(Y,\Lambda(d-1)) \to H^{i+2}_{\acute{e}t}(X,\Lambda(d))
\tag{5.10}
$$

is an isomorphism for $i \geq 2d+e-3$ and surjective for $i = 2d+e-4$. For $e = 0$, this follows from [71, Theorem 2.1]. We shall prove the case $e = 1$ using a similar strategy as follows.

We let $U = X \setminus Y$. The Leray spectral sequence for the inclusion $j' \colon U \hookrightarrow X$ yields a strongly convergent spectral sequence

$$
E^{a,b}_2 = H^a_{\acute{e}t}\big(X,R^b j'_*(\Lambda_U(d))\big) \Rightarrow H^{a+b}_{\acute{e}t}(U,\Lambda(d)).
\tag{5.11}
$$

We know that the canonical map $\Lambda_X(d) \xrightarrow{\cong} j'_*(\Lambda_U(d))$ is an isomorphism. Furthermore, the canonical map $R^b j'_*(\Lambda_U(d)) \to \iota_* \iota^* R^b j'_*(\Lambda_U(d))$ is an isomorphism for $b > 0$. We let $\mathcal{F}_b = \iota^* R^b j'_*(\Lambda_U(d))$. The sheaf exact sequence

$$
0 \to \iota_* \iota^!(\Lambda_X(d)) \to \Lambda_X(d) \to j'_*(\Lambda_U(d)) \to 0
$$

shows that the resulting boundary map $\partial' \colon \mathcal{F}_1 \to \iota^!(\Lambda_X(d)[2])$ is an isomorphism. Moreover, it follows from the construction of the spectral sequence (5.11) that the map on the cohomology groups $H^a_{\acute{e}t}(Y,\mathcal{F}_1) \xrightarrow{\partial'} H^{a+2}_{\acute{e}t}(X,\Lambda_X(d))$, induced by the composite map $\iota_*(\mathcal{F}_1) \to \iota_* \iota^!(\Lambda_X(d)[2]) \to \Lambda_X(d)[2]$, is the map

$$
E^{a,1}_2 = H^a_{\acute{e}t}(Y,\mathcal{F}_1) \xrightarrow{\partial} H^{a+2}_{\acute{e}t}(X,\Lambda_X(d)) = E^{a+2,0}_2.
$$

It follows that the composition

$$\mathbb{H}^a_{\acute{e}t}\left(X, \iota_*\iota^!(\Lambda_X(d)[2])\right) \xrightarrow{\partial'^{-1}} H^a_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\partial} H^{a+2}_{\acute{e}t}(X, \Lambda_X(d))$$

is induced by the canonical adjunction map $\iota_*\iota^!(\Lambda_X(d)[2]) \to \Lambda_X(d)[2]$. Precomposing the latter with the Gysin map (see §5.2) $\iota_*\Lambda_Y(d-1) \xrightarrow{c_1(Y)} \iota_*\iota^!(\Lambda_X(d)[2])$, we see that the composition

$$H^a_{\acute{e}t}(Y, \Lambda_Y(d-1)) \xrightarrow{\partial'^{-1}\circ c_1(Y)} H^a_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\partial} H^{a+2}_{\acute{e}t}(X, \Lambda_X(d)) \tag{5.12}$$

coincides with the Gysin homomorphism ι_* in diagram (5.6).

We next study the map $H^a_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\partial} H^{a+2}_{\acute{e}t}(X, \Lambda_X(d))$. Using the purity theorem for the closed pair (X^o, Y^o) of smooth schemes [54, Theorem VI.5.1], it follows that

$$\mathcal{F}_b|_{Y^o} \cong \begin{cases} \Lambda_{Y^o}(d) & \text{if } b = 1, \\ 0 & \text{if } b > 1. \end{cases} \tag{5.13}$$

On the other hand, the affine Lefschetz theorem [54, Theorem VI.7.3(d)] implies that the sheaf $R^b j'_*(\Lambda_U(d))$ is supported on a closed subscheme of Y whose dimension is bounded by $d - b$. We conclude that

$$H^a_{\acute{e}t}\left(X, R^b j'_*(\Lambda_U(d))\right) = 0 \text{ if } b \geq 2 \text{ and } a > \min\{2q+1, 2d-2b+1\}. \tag{5.14}$$

Using formulas (5.11), (5.13) and (5.14), we get exact sequences

$$H^{i-1}_{\acute{e}t}(U, \Lambda_U(d)) \to H^{i-2}_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\partial} H^i_{\acute{e}t}(X, \Lambda_X(d)) \to H^i_{\acute{e}t}(U, \Lambda_U(d)), \tag{5.15}$$

$$H^{2d-3}_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\partial} H^{2d-1}_{\acute{e}t}(X, \Lambda_X(d)) \to H^{2d-1}_{\acute{e}t}(U, \Lambda_U(d))$$

for $i \geq 2d$. Since $d \geq 3$ and the étale cohomological dimension of k is 1, another application of the affine Lefschetz theorem reduces these exact sequences to an isomorphism $H^{i-2}_{\acute{e}t}(Y, \mathcal{F}_1) \xrightarrow{\cong} H^i_{\acute{e}t}(X, \Lambda_X(d))$ for $i \geq 2d$ and a surjection $H^{2d-3}_{\acute{e}t}(Y, \mathcal{F}_1) \twoheadrightarrow H^{2d-1}_{\acute{e}t}(X, \Lambda_X(d))$.

Finally, we consider the long exact sequences

$$H^{i-1}_{\acute{e}t}(Y_s, \Lambda_{Y_s}(d-1)) \to H^i_{c,\acute{e}t}(Y^o, \Lambda_{Y^o}(d-1)) \to H^i_{\acute{e}t}(Y, \Lambda_Y(d-1)) \to H^i_{\acute{e}t}(Y_s, \Lambda_{Y_s}(d-1))$$

$$H^{i-1}_{\acute{e}t}(Y_s, \tilde{\iota}^*\mathcal{F}_1) \longrightarrow H^i_{c,\acute{e}t}(Y^o, \tilde{j}^*\mathcal{F}_1) \longrightarrow H^i_{\acute{e}t}(Y, \mathcal{F}_1) \longrightarrow H^i_{\acute{e}t}(Y_s, \tilde{\iota}^*\mathcal{F}_1)$$

$$\downarrow \partial$$

$$H^{i+2}_{\acute{e}t}(X, \Lambda_X(d)).$$

The left vertical isomorphism on the top is a consequence of formula (5.13). Using the bound on the cohomological dimension of Y_s, this diagram shows that the right vertical arrow is an isomorphism. Since the composite right vertical arrow is the Gysin

homomorphism as observed before, we conclude that the map ι_* in formula (5.10) is an isomorphism for $i \geq 2d-2$ and surjection for $i = 2d-3$. This finishes the proof. $\qquad\square$

5.4. A Lefschetz theorem for étale fundamental groups

Recall that the Lefschetz theorem for the étale fundamental groups of smooth projective schemes over a field was proven by Grothendieck [23, Exposé XII, Corollaire 3.5]. However, this is a very challenging problem for smooth nonprojective schemes, due to the presence of ramification when we extend étale covers to compactifications. We shall prove the following version of the Lefschetz theorem for the étale fundamental groups of smooth but nonprojective schemes:

Theorem 5.4. *Assume that k is either finite or algebraically closed and $X \subset \mathbb{P}^N_k$ is an $(R_3 + S_4)$-scheme. Let $Y \subset X$ be a good hypersurface section of degree $m \gg 0$. Then the induced map $\iota^o_*: \pi_1^{\mathrm{ab}}(Y^o) \to \pi_1^{\mathrm{ab}}(X^o)$ is an isomorphism of profinite topological abelian groups.*

Proof. Since $\pi_1^{\mathrm{ab}}(X^o) \xrightarrow{\cong} \varprojlim_{n \in \mathbb{Z}} \pi_1^{\mathrm{ab}}(X^o)/n$ and the same holds for Y^o, it suffices to show that the map $\pi_1^{\mathrm{ab}}(Y^o)/n \to \pi_1^{\mathrm{ab}}(X^o)/n$ is an isomorphism for every integer $n \in \mathbb{Z}$. Using the Pontryagin duality (see, for example, the middle of [62, proof of Lemma 1.9, p. 99]), $\left(\pi_1^{\mathrm{ab}}(Z)/n\right)^{\vee} \cong H^1_{\acute{e}t}(Z, \mathbb{Z}/n)$, for $Z \in \{X^o, Y^o\}$, we are reduced to showing that the pullback map

$$(\iota^o)^*: H^1_{\acute{e}t}(X^o, \mathbb{Z}/n) \to H^1_{\acute{e}t}(Y^o, \mathbb{Z}/n) \tag{5.16}$$

is an isomorphism for all $n \in \mathbb{Z}$. In view of Proposition 5.3, we can assume $n = p^r$, where $\mathrm{char}(k) = p > 0$. We shall prove this by induction on $r \geq 1$.

We have an exact sequence of constant étale sheaves on $\mathbf{Sch}_k$:

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^r \to \mathbb{Z}/p^{r-1} \to 0$$

for every $r \geq 2$, where $\mathbb{Z}/p \to \mathbb{Z}/p^r$ is multiplication by p^{r-1}. This yields a long exact sequence of étale cohomology groups

$$H^0_{\acute{e}t}(X^o, \mathbb{Z}/p^r) \to H^0_{\acute{e}t}\left(X^o, \mathbb{Z}/p^{r-1}\right) \to H^1_{\acute{e}t}(X^o, \mathbb{Z}/p) \to H^1_{\acute{e}t}(X^o, \mathbb{Z}/p^r)$$
$$\to H^1_{\acute{e}t}\left(X^o, \mathbb{Z}/p^{r-1}\right) \to H^2_{\acute{e}t}(X^o, \mathbb{Z}/p).$$

Since X^o is integral, the first arrow from the left in this exact sequence is surjective. The same applies to Y^o as well. We thus get a commutative diagram of exact sequences

$$0 \to H^1_{\acute{e}t}(X^o, \mathbb{Z}/p) \to H^1_{\acute{e}t}(X^o, \mathbb{Z}/p^r) \to H^1_{\acute{e}t}\left(X^o, \mathbb{Z}/p^{r-1}\right) \to H^2_{\acute{e}t}(X^o, \mathbb{Z}/p) \tag{5.17}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$0 \to H^1_{\acute{e}t}(Y^o, \mathbb{Z}/p) \to H^1_{\acute{e}t}(Y^o, \mathbb{Z}/p^r) \to H^1_{\acute{e}t}\left(Y^o, \mathbb{Z}/p^{r-1}\right) \to H^2_{\acute{e}t}(Y^o, \mathbb{Z}/p),$$

where the vertical arrows are the pullback maps. Using a diagram chase and an induction on r, we are reduced to showing that the pullback map

$$(\iota^o)^* \colon H^i_{\text{ét}}(X^o, \mathbb{Z}/p) \to H^i_{\text{ét}}(Y^o, \mathbb{Z}/p) \tag{5.18}$$

is an isomorphism for $i = 1$ and injective for $i = 2$.

We let $U = X \setminus Y$ so that $U^o = X^o \setminus Y^o$. We let $u \colon U \hookrightarrow X$ and $u^o \colon U^o \hookrightarrow X^o$ denote the inclusion maps. Then formula (5.18) is equivalent to the assertion that $H^i_{\text{ét}}(X^o, u^o_!(\mathbb{Z}/p|_{U^o})) = 0$ for $i = 1, 2$. Using the cohomology exact sequence associated to the relative Artin–Schreier sheaf exact sequence (where F is the Frobenius of $\mathcal{O}_{X^o}$)

$$0 \to u^o_!(\mathbb{Z}/p|_{U^o}) \to \mathcal{O}_{X^o}(-Y^o) \xrightarrow{1-F} \mathcal{O}_{X^o}(-Y^o) \to 0,$$

we get the long exact sequence

$$\cdots \to H^{i-1}_{\text{ét}}(X^o, \mathcal{O}_{X^o}(-Y^o)) \to H^i_{\text{ét}}(X^o, u^o_!(\mathbb{Z}/p|_{U^o})) \to H^i_{\text{ét}}(X^o, \mathcal{O}_{X^o}(-Y^o))$$
$$\xrightarrow{1-F} H^i_{\text{ét}}(X^o, \mathcal{O}_{X^o}(-Y^o)) \to \cdots$$

for $i \geq 1$. Using this, it suffices to show that $H^i_{\text{ét}}(X^o, \mathcal{O}_{X^o}(-Y^o)) = 0$ for $i \leq 2$. Equivalently, it suffices to show that $H^i_{\text{zar}}(X^o, \mathcal{O}_{X^o}(-Y^o)) = 0$ for $i \leq 2$.

We now consider the exact sequence of Zariski cohomology groups

$$\cdots \to H^{i-1}(X^o, \mathcal{O}_{X^o}(-Y^o)) \to H^i_{X_{\text{sing}}}(X, \mathcal{O}_X(-Y)) \to H^i(X, \mathcal{O}_X(-Y))$$
$$\to H^i(X^o, \mathcal{O}_{X^o}(-Y^o)) \to \cdots.$$

Since X is an $(R_3 + S_4)$-scheme, we have $\inf_{x \in X_{\text{sing}}} \{\text{depth}(\mathcal{O}_X(-Y)_x)\} \geq 4$. We conclude from [28, Theorem 3.8] and a spectral sequence argument that $H^i_{X_{\text{sing}}}(X, \mathcal{O}_X(-Y)) = 0$ for $i \leq 3$. The long exact sequence then tells us that the map $H^i(X, \mathcal{O}_X(-Y)) \to H^i(X^o, \mathcal{O}_{X^o}(-Y^o))$ of Zariski cohomology groups is an isomorphism for $i \leq 2$. The theorem therefore is finally reduced to showing that $H^i(X, \mathcal{O}_X(-Y)) = 0$ for $i \leq 2$. But this follows from Lemma 5.1, since the degree of the hypersurface H is very large. $\qquad\square$

Remark 5.5. If one knew that the eigenvalues of the Frobenius on $H^2(X^o, \mathcal{O}_{X^o}(-Y^o))$ were all different from 1, then the hypothesis of Theorem 5.4 (and hence Theorem 1.2) could be weakened.

6. Class field theory and applications

In this section, we shall study the class field theory of singular schemes over finite fields and its applications. In particular, we shall prove Theorems 1.2, 1.4 and 1.5.

6.1. Proof of Theorem 1.2

Let k be a finite field and $X \in \mathbf{Sch}_k$ an integral projective R_1-scheme of dimension $d \geq 1$ over k. It was shown in Corollary 3.3 that the Frobenius substitution associated to the regular closed points gives rise to a reciprocity homomorphism $\phi_X \colon \text{CH}_0^{LW}(X) \to \pi_1^{\text{ab}}(X^o)$. Also, the left vertical arrow of diagram (3.3) gives us the restriction map ϕ_X^0. What remains to show is that $\phi_X^0 \colon \text{CH}_0^{LW}(X)^0 \to \pi_1^{\text{ab}}(X^o)^0$ is surjective, and it is an

isomorphism of finite groups under either condition (1) or (2) of the theorem. We shall prove all of these by induction on d. Since the case $d \leq 2$ already follows from Theorem 3.6, we shall assume that $d \geq 3$.

We let $\pi \colon X_n \to X$ be the normalization morphism. Using Corollaries 2.6 and 2.7, the Zariski–Nagata purity theorem [24, Exposé X, Théorème 3.1] for $\pi_1((X_n)^o)$, and [62, Lemma 5.1(1)], we can assume that X is normal.

Part 1. We first consider the case when X has isolated singularities. Let $\alpha \in \mathrm{CH}_0^{LW}(X)$ be a 0-cycle such that $\phi_X(\alpha) = 0$. We now fix an embedding $X \hookrightarrow \mathbb{P}_k^N$ and apply [77, Theorem 3.1] to find a hypersurface $H \subset \mathbb{P}_k^N$ containing $\mathrm{Supp}(\alpha)$ such that the scheme-theoretic intersection $Y = X \cap H$ is smooth and does not meet X_{sing}. In particular, $Y \subset X^o$. We can then find a cycle $\alpha' \in \mathrm{CH}_0^F(Y)$ such that $\alpha = \iota_*(\alpha')$, where $\iota \colon Y \hookrightarrow X$ is the inclusion. Since X is an S_2-scheme of dimension ≥ 3 and $Y \subset X^o$, it follows from [23, Exposé XII, Corollaire 3.5] that Y is connected (hence integral) and the map $\pi_1(Y) \to \varprojlim_W \pi_1(W)$ is an isomorphism, where the limit is taken over all open neighborhoods of Y in X.

Since $Y \subset X^o$, there is a factorization

$$\pi_1(Y) \to \varprojlim_U \pi_1(U) \to \varprojlim_W \pi_1(W), \tag{6.1}$$

where the first limit is over all open neighborhoods of Y contained in X^o. Note that the second arrow is an isomorphism, and therefore so is the first arrow. On the other hand, we showed in the proof of Lemma 4.15 that for any open $U \subset X^o$ containing Y, the codimension of $X^o \setminus U$ is ≥ 2. We conclude from the Zariski–Nagata purity theorem that the first limit in formula (6.1) is $\pi_1(X^o)$. It follows that the map $\pi_1(Y) \to \pi_1(X^o)$ is an isomorphism. We therefore conclude that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^F(Y) & \xrightarrow{\phi_Y} & \pi_1^{\mathrm{ab}}(Y) \\ {\scriptstyle\iota_*}\downarrow & & \cong\,\downarrow{\scriptstyle\iota_*} \\ \mathrm{CH}_0^{LW}(X) & \xrightarrow{\phi_X} & \pi_1^{\mathrm{ab}}(X^o) \end{array} \tag{6.2}$$

such that the right vertical arrow is an isomorphism and $\alpha' \in \mathrm{CH}_0^F(Y)$.

It follows that $\iota_* \circ \phi_Y(\alpha') = 0$. Since ϕ_Y is injective by induction, we get $\alpha' = 0$. In particular, we get $\alpha = \iota_*(\alpha') = 0$. This shows that ϕ_X is injective. The surjectivity of ϕ_X^0 and finiteness of $\mathrm{CH}_0^{LW}(X)^0$ and $\pi_1^{\mathrm{ab}}(X^o)^0$ also follow from diagram (6.2) and induction on d. We have thus finished the proof of Theorem 1.2 when X has isolated singularities.

Part 2. We now assume that X is an $(R_3 + S_4)$-scheme. In particular, it is normal. Moreover, it is regular if $d \leq 3$, in which case the theorem is due to Kato and Saito [34]. If $d = 4$, then X has isolated singularities, in which case we have already proven the theorem in part 1. We can therefore assume that $d \geq 5$. As before, we let $\alpha \in \mathrm{CH}_0^{LW}(X)$ be a 0-cycle such that $\phi_X(\alpha) = 0$.

We can now apply [20, Theorem 6.3], which says that for every integer $m \gg 1$ there exists a hypersurface $H \subset \mathbb{P}_k^N$ of degree m containing $\mathrm{Supp}(\alpha)$ such that the hypersurface section $Y = X \cap H$ satisfies the following:

(1) $Y \cap X^o = Y^o$ is regular.

(2) Y is an $(R_3 + S_4)$-scheme.

(3) Y contains no irreducible component of X_{sing}.

Since X is normal and integral, it follows from [23, Exposé XII, Corollaire 3.5] that Y is connected. Since (2) implies that Y is also normal, it follows that it must be integral. In particular, it is good (see the beginning of §5). Moreover, there is a 0-cycle $\alpha' \in \text{CH}_0^{LW}(Y)$ such that $\alpha = \iota_*(\alpha')$ if we let $\iota \colon Y \hookrightarrow X$ be the inclusion.

Using Corollary 2.7 and [62, Lemma 5.1(1)], we get a commutative diagram

$$
\begin{array}{ccc}
\text{CH}_0^{LW}(Y) & \xrightarrow{\phi_Y} & \pi_1^{\text{ab}}(Y^o) \\
{\scriptstyle \iota_*}\downarrow & & \cong \downarrow {\scriptstyle \iota_*} \\
\text{CH}_0^{LW}(X) & \xrightarrow{\phi_X} & \pi_1^{\text{ab}}(X^o).
\end{array}
\tag{6.3}
$$

The right vertical arrow is an isomorphism by Theorem 5.4, since $m \gg 0$. It follows that $\phi_Y(\alpha') = 0$. We conclude by induction that $\alpha' = 0$. The surjectivity of ϕ_X and finiteness of $\pi_1^{\text{ab}}(X^o)^0$ also follow from the diagram, because ϕ_Y is surjective and $\pi_1^{\text{ab}}(Y^o)^0$ is finite by induction. This finishes the proof of the theorem, except that we still need to show that ϕ_X^0 is surjective without condition (1) or (2) of the theorem.

As $d \geq 3$ and X is normal (equivalently, $(R_1 + S_2)$), we can find a hypersurface section $Y = X \cap H$ which satisfies conditions (1) and (3) in part 2, and it is an $(R_1 + S_2)$-scheme. We already observed that Y is then an integral normal scheme. Now, the map $\iota_* \colon \pi_1^{\text{ab}}(Y^o) \to \pi_1^{\text{ab}}(X^o)$ is surjective by Proposition 4.15 and [23, Exposé X, Corollaire 2.6]. Since the map ϕ_Y in diagram (6.3) is surjective on degree 0 subgroups by induction, we conclude that ϕ_X^0 is surjective. $\qquad\square$

6.2. Proof of Theorem 1.4

Let X be as in Theorem 1.4. We need some preparation before we prove the theorem.

By [62, Theorem 6.2], the norm maps $N_x \colon K_0(k(x)) \to K_0(k)$ for $x \in X_{(0)}^o$ induce a natural homomorphism $\deg \colon H_{\text{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right) \to \mathbb{Z}$. Since these norms are multiplication by the degrees of the field extensions $k(x)/k$, we see that the composition $\mathcal{Z}_0(X^o) \xrightarrow{\text{cyc}_X} H_{\text{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right) \xrightarrow{\deg} \mathbb{Z}$ is the degree homomorphism. We let $H_{\text{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right)^0$ be the kernel of this map. We then get a commutative diagram

$$
\begin{array}{ccccccc}
0 \to & H_{\text{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right)^0 & \to & H_{\text{nis}}^d\left(X, \mathcal{K}_{d,X}^M\right) & \xrightarrow{\deg} & \mathbb{Z}, & \\
& {\scriptstyle \rho_X^0}\downarrow & & {\scriptstyle \rho_X}\downarrow & & \downarrow & \\
0 \longrightarrow & \pi_1^{\text{ab}}(X^o)^0 & \longrightarrow & \pi_1^{\text{ab}}(X^o) & \longrightarrow & \widehat{\mathbb{Z}}, &
\end{array}
\tag{6.4}
$$

where the right vertical arrow is the canonical inclusion into the profinite completion and $\pi_1^{\text{ab}}(X^o) \to \widehat{\mathbb{Z}}$ is the push-forward map induced by the structure map of X^o.

Since the last term of the top row of diagram (6.4) is torsion free, we have an exact sequence of inverse limits

$$0 \to \varprojlim_m H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)^0 / m \to \varprojlim_m H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)/m \to \widehat{\mathbb{Z}}. \tag{6.5}$$

Lemma 6.1. *The map* $\rho_X \colon H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right) \to \pi^{\mathrm{ab}}_1(X^o)$ *is injective.*

Proof. If $\pi \colon X_n \to X$ denotes the normalization of X, then the maps $\pi^* \colon H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right) \to H^d_{\mathrm{nis}}\left(X_n, \mathcal{K}^M_{d,X_n}\right)$ and $\pi^* \colon \pi^{\mathrm{ab}}_1(X^o) \to \pi^{\mathrm{ab}}_1(X^o_n)$ are isomorphisms. The first isomorphism holds for dimension reasons and by the exactness of π_* on Nisnevich sheaves. The second isomorphism holds by the Zariski–Nagata purity theorem [24, Exposé X, Théorème 3.1]. We can therefore assume that X is normal. By diagram (6.4), it suffices to show that ρ^0_X is injective.

We assume first that X is geometrically connected. Since $H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)^0$ is a finite group (hence profinite complete) by [62, Theorem 6.2(1)] (take $I = \mathcal{O}_X$ and $T = X$) and $\pi^{\mathrm{ab}}_1(X^o)$ is profinite complete, diagram (6.4) and formula (6.5) give rise to a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 \to & H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)^0 & \to & \varprojlim_m H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)/m & \stackrel{\deg}{\to} & \widehat{\mathbb{Z}} & \\
& \rho^0_X \downarrow & & \widehat{\rho}_X \downarrow & & \| & \\
0 \longrightarrow & \pi^{\mathrm{ab}}_1(X^o)^0 & \longrightarrow & \pi^{\mathrm{ab}}_1(X^o) & \longrightarrow & \widehat{\mathbb{Z}}. &
\end{array} \tag{6.6}$$

The middle vertical arrow is an isomorphism by [35, Theorem 9.1(3)]. It follows that the left vertical arrow must also be an isomorphism.

Suppose now that X is not geometrically connected. Since $\widehat{\rho}_X$ is always an isomorphism by [35, Theorem 9.1(3)], it suffices to show, using formula (6.5), that $H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)^0$ is profinite complete. To show the latter, it suffices to prove the stronger claim that $H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)^0$ is torsion of bounded exponent. Since the image of ρ^0_X is finite by Theorem 1.2, the claim is equivalent to proving that the kernel of ρ^0_X is torsion of bounded exponent. Since $\widehat{\rho}_X$ is an isomorphism, it is enough to show that the kernel of the completion map $\alpha_X \colon H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right) \to \varprojlim_m H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)/m$ is torsion of bounded exponent.

Now, we know that there exists a finite field extension k'/k such that $X_{k'}$ is a disjoint union of geometrically connected integral normal schemes. We pick any irreducible component Y of $X_{k'}$ and let $\pi \colon Y \to X$ be the projection map. Then π is a finite étale morphism of normal schemes whose degree divides $[k' : k]$. Since we have shown that α_Y is injective, it suffices to show that the kernel of the map $\pi^* \colon H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right) \to H^d_{\mathrm{nis}}\left(Y, \mathcal{K}^M_{d,Y}\right)$ is torsion of bounded exponent. But this follows directly from [35, Lemma 4.5(1)]. $\qquad \square$

We now prove Theorem 1.4. The claim that the cycle class map $\mathrm{cyc}_X \colon \mathcal{Z}_0(X^o) \to H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)$ factors through $\mathrm{CH}^{LW}_0(X)$ is a direct consequence of Lemmas 3.1 and 6.1, and Corollary 3.3. The resulting map $\mathrm{cyc}_X \colon \mathrm{CH}^{LW}_0(X) \to H^d_{\mathrm{nis}}\left(X, \mathcal{K}^M_{d,X}\right)$ is surjective by [35, Theorem 2.5]. It is injective because $\phi_X = \rho_X \circ \mathrm{cyc}_X$ is injective by Theorem 1.2.

6.3. Proof of Theorem 1.5

Let X be as in Theorem 1.5 and let $f \colon \widetilde{X} \to X$ be a resolution of singularities with reduced exceptional divisor E. By Corollary 2.6, we can assume that X is normal. Since $f^* \colon \mathrm{CH}^{LW}_0(X) \to \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ is clearly surjective for all $m \geq 1$, we only need to show that this map is injective for all $m \gg 1$. The latter is equivalent to showing that the map $f^* \colon \mathrm{CH}^{LW}_0(X)^0 \to \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)^0$ is injective for all $m \gg 1$.

We let $C(X^o) = \varprojlim_m \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ and let $C(X^o)^0$ denote the kernel of the degree map $C(X^o) \to \mathbb{Z}$. It was shown in [38, Proposition 3.2] that the Frobenius substitution associated to closed points in X^o defines a reciprocity map $\phi_{X^o} \colon C(X^o) \to \pi^{\mathrm{ab}}_1(X^o)$ such that one has a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^{LW}_0(X) & \xrightarrow{\phi_X} & \pi^{\mathrm{ab}}_1(X^o) \\
{\scriptstyle f^*}\downarrow & & \| \\
C(X^o) & \xrightarrow{\phi_{X^o}} & \pi^{\mathrm{ab}}_1(X^o).
\end{array} \tag{6.7}$$

If we restrict this diagram to the degree 0 subgroups, then Theorem 1.2 says that the top horizontal arrow is an isomorphism. On the other hand, [38, Theorem III] (if $\mathrm{char}(k) \neq 2$) and [5, Theorem 8.5] (in general) say that the bottom horizontal arrow is an isomorphism. It follows that the map $f^* \colon \mathrm{CH}^{LW}_0(X)^0 \to C(X^o)^0$ is an isomorphism. In particular, $C(X^o)^0$ is finite.

Since $C(X^o) \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)$ for every $m \geq 1$, it follows that $C(X^o)^0 \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)^0$ for every $m \geq 1$. We conclude that $\left\{\mathrm{CH}_0\left(\widetilde{X} \mid mE\right)^0\right\}_{m \geq 1}$ is an inverse system of abelian groups whose transition maps are all surjective and whose limit $C(X^o)^0$ is finite. But this implies that this inverse system is eventually constant. That is, the map $C(X^o)^0 \to \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)^0$ is an isomorphism for all $m \gg 1$. It follows that the map $f^* \colon \mathrm{CH}^{LW}_0(X)^0 \to \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)^0$ is an isomorphism for all $m \gg 1$.

6.4. Necessity of R_1-condition

We show by an example that it is necessary to assume the R_1-condition in Theorem 1.2. Let C be the projective plane curve over a finite field k which has a simple cusp along the origin and is regular elsewhere. Its local ring at the singular point is analytically isomorphic to $k\left[\left[t^2, t^3\right]\right]$, which is canonically a subring of its normalization $k[[t]]$. Let

$\pi\colon \mathbb{P}^1_k \to C$ denote the normalization map. Let $S \cong \mathrm{Spec}\left(k\left[t^2,t^3\right]/\left(t^2,t^3\right)\right)$ denote the reduced conductor and $\widetilde{S} \cong \mathrm{Spec}\left(k[t]/\left(t^2\right)\right)$ its scheme-theoretic inverse image in $\mathbb{P}^1_k$. We have a commutative diagram with exact rows:

$$0 \longrightarrow \varprojlim_m \mathcal{O}^\times(mS)/k^\times \longrightarrow \varprojlim_m K_0(C,mS) \longrightarrow \mathrm{Pic}(C) \longrightarrow 0 \qquad (6.8)$$

$$0 \longrightarrow \varprojlim_m \mathcal{O}^\times\left(m\widetilde{S}\right)/k^\times \longrightarrow \varprojlim_m K_0\left(\mathbb{P}^1_k,m\widetilde{S}\right) \longrightarrow \mathrm{Pic}\left(\mathbb{P}^1_k\right) \longrightarrow 0.$$

The isomorphism of the middle vertical map follows from the known result that the double relative K-groups $K_0\left(C,\mathbb{P}^1_k,mS\right)$ and $K_{-1}\left(C,\mathbb{P}^1_k,mS\right)$ vanish.

It is easy to check from the K-theory localization sequence that $\mathrm{Pic}\left(\mathbb{P}^1_k,m\widetilde{S}\right) \xrightarrow{\cong} K_0\left(\mathbb{P}^1_k,m\widetilde{S}\right)$. On the other hand, the known class field theory for curves (with modulus) tells us that there is a canonical isomorphism $\varprojlim_m \mathrm{Pic}^0\left(\mathbb{P}^1_k,m\widetilde{S}\right) \xrightarrow{\cong} \pi^{\mathrm{ab}}(C^o)^0$. It follows that there are isomorphisms $(1+tk[[t]])^\times \xrightarrow{\cong} \mathbb{W}(k) \xrightarrow{\cong} \pi^{\mathrm{ab}}(C^o)^0$. On the other hand, $\mathrm{CH}_0^{LW}(C)^0 \cong \mathrm{Pic}^0(C) \cong k$. This shows that there is no reciprocity map $\mathrm{CH}_0^{LW}(C)^0 \to \pi^{\mathrm{ab}}(C^o)^0$, and the two can not be isomorphic.

7. Lefschetz for generalized Albanese variety

In order to prove the remaining of our main results, we need to use a Lefschetz hypersurface theorem for the generalized Albanese variety of smooth quasiprojective schemes over algebraically closed fields. The goal of this section to establish such a Lefschetz theorem.

We assume in this section that k is an algebraically closed field of characteristic $p > 0$. Recall from [68] that to any quasiprojective scheme V over k, there is associated a semi-abelian variety $\mathrm{Alb}_S(V)$ over k together with a morphism $alb_V\colon V \to \mathrm{Alb}_S(V)$ which have the universal property that given any semi-abelian variety A over k and a morphism $f\colon V \to A$, there exists a unique affine morphism $\tilde{f}\colon \mathrm{Alb}_S(V) \to A$ such that $f = \tilde{f} \circ alb_V$. Recall here that an affine morphism between two commutative group schemes over k is the composition of a group homomorphism with a translation of the target scheme.

The assignment $V \mapsto \mathrm{Alb}_S(V)$ is a covariant functor for arbitrary morphisms of quasiprojective schemes. If V is smooth and projective, then $\mathrm{Alb}_S(V)$ is the Albanese variety in the classical sense. If V is a smooth curve, then $\mathrm{Alb}_S(V)$ coincides with Rosenlicht's generalized Jacobian [64] or Serre's generalized Jacobian with modulus [70]. We shall call $\mathrm{Alb}_S(V)$ the 'generalized Albanese variety' of V.

For any quasiprojective scheme V over k, let $\mathrm{Alb}_W(V)$ denote the Albanese variety of V which is universal for rational maps from V to abelian varieties over k (see [76] or [46, Chapter II, §3]). Let $\mathrm{Cl}(V)$ denote the divisor class group of V and $\mathrm{Cl}^0(V)$ the subgroup of $\mathrm{Cl}(V)$ consisting of Weil divisors which are algebraically equivalent to zero in the sense of [17, Chapter 19]. If V is projective and R_1, then we recall from [76] (see also

[46, Chapter IV, §4]) that there is an abelian variety $\mathrm{Pic}_W(V)$ over k, known as the Weil–Picard variety of V, such that $\mathrm{Pic}_W(V)(k) \cong \mathrm{Cl}^0(V)$. Moreover, $\mathrm{Alb}_W(V) \cong \mathrm{Alb}_W(V^o)$ is the dual of $\mathrm{Pic}_W(V)$ [46, Chapter VI, p. 152]. We shall therefore refer to $\mathrm{Alb}_W(V)$ as the 'Weil–Albanese variety' of V.

7.1. Generalized Albanese of a smooth variety

Let $X \in \mathbf{Sch}_k$ be an integral projective R_1-scheme of dimension $d \geq 1$. Let $U \subset X^o$ be a nonempty open subscheme and set $Z = X \setminus U$, endowed with the reduced induced closed subscheme structure. In this case, Serre gave an explicit description of $\mathrm{Alb}_S(U)$ in [69]. We recall this description. We remark here that even if we do not assume X to be smooth, the exposition of [69] remains valid in the present case with no modification.

Let $\mathrm{Div}(X)$ denote the free abelian group of Weil divisors on X. Let $\Lambda^1_U(X)$ denote the image of the push-forward map $\mathcal{Z}_{d-1}(Z) \to \mathcal{Z}_{d-1}(X) = \mathrm{Div}(X)$. There is thus a canonical homomorphism $\iota_U : \Lambda^1_U(X) \to \frac{\mathrm{Cl}(X)}{\mathrm{Cl}^0(X)} = \mathrm{NS}(X)$, where $\mathrm{NS}(X)$ is the Néron–Severi group of X. Let $\Lambda_U(X)$ denote the kernel of the canonical map $\Lambda^1_U(X) \xrightarrow{\iota_U} \mathrm{NS}(X)$ so that the quotient $\mathrm{Div}(X) \twoheadrightarrow \mathrm{Cl}(X)$ induces a homomorphism $\Lambda_U(X) \to \mathrm{Cl}^0(X)$. It was shown by Serre [69] that $\mathrm{Alb}_S(U)$ is the Cartier dual of the 1-motive $[\Lambda_U(X) \to \mathrm{Pic}_W(X)]$ (see [13] for the definitions of 1-motives and their Cartier duals).

We thus have a canonical exact sequence of algebraic groups

$$0 \to \Lambda_U(X)^\vee \to \mathrm{Alb}_S(U) \to \mathrm{Alb}_W(X) \to 0, \tag{7.1}$$

where $\Lambda_U(X)^\vee$ is the Cartier dual of the constant group scheme over k associated to the lattice $\Lambda_U(X)$. In particular, $\Lambda_U(X)^\vee$ is a split torus of rank equal to the rank of the lattice $\Lambda_U(X)$.

7.2. A Lefschetz theorem for $\mathrm{Alb}_S(U)$

We shall now prove a Lefschetz theorem for the generalized Albanese variety. We let $X \subset \mathbb{P}^N_k$ be an integral normal projective scheme of dimension $d \geq 3$ over k which is an R_2-scheme. Let $U \subset X^o$ be a nonempty open subscheme. We let $Z = X \setminus U$ with the reduced closed subscheme structure. We let $H \subset \mathbb{P}^N_k$ be a hypersurface and $Y = X \cap H$ the scheme theoretic intersection. We shall say that Y is 'Z-admissible' if the following hold:

(1) Y is good (see §5).
(2) For every irreducible component Z' of Z of dimension $d-1$, the scheme-theoretic intersection $Y \cap Z'$ is integral of dimension $d-2$.

Let $\iota : Y \hookrightarrow X$ be the inclusion of a Z-admissible hypersurface section of X. Then the construction of the pullback map on algebraic cycles in [17, Chapter 2, §4] yields a homomorphism $\iota^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$. Furthermore, it easily follows from the proof of [17, Corollary 2.4.1] that it induces the pullback maps $\iota^* : \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ and $\iota^* : \mathrm{Cl}^0(X) \to \mathrm{Cl}^0(X)$. Taking the quotients, we get a pullback map $\iota^* : \mathrm{NS}(X) \to \mathrm{NS}(Y)$. We shall follow the notations of §5.

Lemma 7.1. *Assume that X is normal, H is a hypersurface of degree $m \gg 0$, and $Y = X \cap H$ is good. Then the map $\iota^*\colon \mathrm{NS}(X)_{\mathrm{tor}} \to \mathrm{NS}(Y)_{\mathrm{tor}}$ is injective.*

Proof. Since X is normal and Y is good, the latter is also normal. It follows therefore from [17, Example 10.3.4] that the pullback maps $j^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X^o)$ and $\tilde{j}^*\colon \mathrm{NS}(Y) \to \mathrm{NS}(Y^o)$ are isomorphisms. Hence, the lemma is equivalent to the statement that the map $(\iota^o)^*\colon \mathrm{NS}(X^o)_{\mathrm{tor}} \to \mathrm{NS}(Y^o)_{\mathrm{tor}}$ is injective.

Since $\mathrm{Pic}_W(X)(k)$ and $\mathrm{Pic}_W(Y)(k)$ are divisible, there is a commutative diagram of short exact sequences

$$0 \to \mathrm{Cl}^0(X)_{\mathrm{tor}} \to \mathrm{Pic}(X^o)_{\mathrm{tor}} \to \mathrm{NS}(X^o)_{\mathrm{tor}} \to 0 \qquad (7.2)$$

$$0 \to \mathrm{Cl}^0(Y)_{\mathrm{tor}} \to \mathrm{Pic}(Y^o)_{\mathrm{tor}} \to \mathrm{NS}(Y^o)_{\mathrm{tor}} \to 0.$$

Since $Y \subset X$ is a general hypersurface section, it follows from the Lefschetz theorem for the Weil–Albanese variety [46, Chapter VII, Theorem 5] that the canonical map $\mathrm{Alb}_W(Y) \to \mathrm{Alb}_W(X)$ is an isogeny of abelian varieties whose kernel is isomorphic to the finite infinitesimal group scheme α_{p^r} for some $r \geq 0$. Considering the induced map between the dual abelian varieties, we see that the pullback morphism $\mathrm{Pic}_W(X) \to \mathrm{Pic}_W(Y)$ is an isogeny of abelian varieties [10, Theorem 11.1]. It is then an easy exercise to show that the map $\mathrm{Cl}^0(X)_{\mathrm{tor}} \to \mathrm{Cl}^0(Y)_{\mathrm{tor}}$ is surjective. Using a diagram chase in diagram (7.2), the lemma is now reduced to showing that the map $\mathrm{Pic}(X^o)_{\mathrm{tor}} \to \mathrm{Pic}(Y^o)_{\mathrm{tor}}$ is injective.

We first fix a prime-to-p integer n. Since k is algebraically closed, we can identify μ_n with $\mathbb{Z}/n$. Since $H^0_{\text{ét}}\left(X^o, \mathcal{O}_{X^o}^\times\right) \cong k^\times$ (because X is R_1) and the latter is a divisible group, one observes using the Kummer sequence that $_n\mathrm{Pic}(X^o) \cong H^1_{\text{ét}}(X^o, \mathbb{Z}/n)$. By the same token, we have $_n\mathrm{Pic}(Y^o) \cong H^1_{\text{ét}}(Y^o, \mathbb{Z}/n)$. It follows therefore from Proposition 5.3 that the map $_n\mathrm{Pic}(X^o) \to {}_n\mathrm{Pic}(Y^o)$ is injective. We note here that this part of Proposition 5.3 does not require X to be R_2.

We now let $n = p^r$ for some $r \geq 1$. Using the short exact sequence of étale sheaves

$$0 \to \mathcal{O}_{X^o}^\times \xrightarrow{n} \mathcal{O}_{X^o}^\times \to \mathcal{O}_{X^o}^\times/n \to 0, \qquad (7.3)$$

we see that $_n\mathrm{Pic}(X^o) \cong H^0_{\text{ét}}\left(X^o, \mathcal{O}_{X^o}^\times/n\right)$. Using the similar isomorphism for Y^o, we need to show that the map $H^0_{\text{ét}}\left(X^o, \mathcal{O}_{X^o}^\times/n\right) \to H^0_{\text{ét}}\left(Y^o, \mathcal{O}_{Y^o}^\times/n\right)$ is injective. Comparing the exact sequence (7.3) with the similar sequence for Y^o, this injectivity is equivalent to showing that $H^0_{\text{ét}}\left(X^o, \mathcal{K}_{1,X^o|Y^o}/n\right) = 0$, where we let $\mathcal{K}_{1,X^o|Y^o} = \mathrm{Ker}\left(\mathcal{O}_{X^o}^\times \twoheadrightarrow (\iota^o)^*\left(\mathcal{O}_{Y^o}^\times\right)\right)$. Note here that $\mathcal{K}_{1,X^o|Y^o}/n = \mathrm{Ker}\left(\mathcal{K}_{1,X^o}/n \to (\iota^o)^*\left(\mathcal{O}_{Y^o}^\times/n\right)\right)$, since $\mathcal{O}_{Y^o}^\times$ is p-torsion free.

We let $W_r\Omega^\bullet_{X^o}$ be the p-typical de Rham–Witt complex of X^o (see, for example, [30]) and let $W_r\Omega^i_{X^o,\log}$ be the image of the Bloch–Gabber–Kato homomorphism $\mathrm{dlog}\colon \mathcal{K}^M_{i,X^o} \to W_r\Omega^i_{X^o}$. This map is given by $\mathrm{dlog}(\{x_1, \ldots, x_i\}) = \mathrm{dlog}[x_1]_r \wedge \cdots \wedge \mathrm{dlog}[x_i]_r$, where $[\cdot]_r$ denotes the Teichmüller homomorphism $[\cdot]_r\colon \mathcal{O}_{X^o}^\times \to (W_r\mathcal{O}_{X^o})^\times$. The Bloch–Gabber–Kato homomorphism induces an isomorphism $\mathrm{dlog}\colon \mathcal{K}^M_{i,X^o}/n \xrightarrow{\cong} W_r\Omega^i_{X^o,\log}$.

We let $W_r\Omega^1_{X^o/Y^o,\log}$ denote the image of dlog: $\mathcal{K}_{1,X^o|Y^o}/n \hookrightarrow W_r\Omega^1_{X^o}$. It suffices to show that $H^0_{\text{ét}}\left(X^o, W_r\Omega^1_{X^o|Y^o,\log}\right) = 0$. Using the short exact sequence [32, Theorem 1.1.6]

$$0 \to W_{r-1}\Omega^1_{X^o|Y^o,\log} \xrightarrow{p} W_r\Omega^1_{X^o|Y^o,\log} \to W_1\Omega^1_{X^o|Y^o,\log} \to 0 \tag{7.4}$$

and induction on r, it suffices to show that $H^0_{\text{ét}}\left(X^o, W_1\Omega^1_{X^o|Y^o,\log}\right) = 0$.

One easily checks that the image of the composite inclusion

$$\mathcal{K}_{1,X^o|Y^o}/p \hookrightarrow \Omega^1_{X^o} \hookrightarrow \Omega^1_{X^o}(\log Y^o)$$

lies in the $\mathcal{O}_{X^o}$-submodule $\Omega^1_{X^o|Y^o} := \Omega^1_{X^o}(\log Y^o)(-Y^o)$ [32, Theorem 1.2.1]. Hence, it suffices to show that $H^0_{\text{ét}}\left(X^o, \Omega^1_{X^o|Y^o}\right) = 0$.

To show this, we use the exact sequence

$$0 \to \Omega^1_{X^o}(-Y^o) \to \Omega^1_{X^o|Y^o} \xrightarrow{\text{Res}} \mathcal{O}_{Y^o}(-Y^o) \to 0, \tag{7.5}$$

where Res is the Poincaré residue map twisted by $\mathcal{O}_{X^o}(-Y^o)$. It suffices therefore to show that the left and the right terms of this sequence have no global sections. Since $\text{char}(k) = p > 0$, we are finally reduced to showing that

$$H^0_{\text{zar}}(Y^o, \mathcal{O}_{Y^o}(-Y^o)) = H^0_{\text{zar}}\left(X^o, \Omega^1_{X^o/k}(-Y^o)\right) = 0, \tag{7.6}$$

where we note that these Zariski cohomologies coincide with the corresponding étale cohomologies.

Now, we first note that $\mathcal{O}_Y(Y)$ is very ample on Y. This already implies that $H^0_{\text{zar}}(Y, \mathcal{O}_Y(-Y)) = 0$ [29, Exercise III.7.1]. Since Y is normal, we conclude from Corollary 4.7 that $H^0_{\text{zar}}(Y^o, \mathcal{O}_{Y^o}(-Y^o)) = 0$. On the other hand, since $\Omega^1_{X^o/k}$ is locally free and $m \gg 0$, it follows from Lemma 5.2 that $H^0_{\text{zar}}\left(X^o, \Omega^1_{X^o/k}(-Y^o)\right) = 0$. This concludes the proof of the lemma. $\qquad\square$

Proposition 7.2. *Assume that X is an $(R_2 + S_2)$-scheme, H is a hypersurface of degree $m \gg 0$, and $Y = X \cap H$ is good. Then the map $\iota^*\colon \text{NS}(X) \to \text{NS}(Y)$ is injective.*

Proof. Since $\text{Cl}^0(X)$ is divisible, the map $\text{Pic}(X^o)/n \to \text{NS}(X)/n$ is an isomorphism for every integer $n \neq 0$. It follows from [57, Théorème 2] that $\text{NS}(X)$ is a finitely generated abelian group. Hence, there exists a short exact sequence

$$0 \to \text{NS}(X)_{\text{tor}} \to \text{NS}(X) \to \text{NS}(X)_{\text{free}} \to 0, \tag{7.7}$$

where the first group is finite and the last group is free of finite rank (called the Weil–Picard rank of X). We can therefore find a prime number $\ell \neq p$ such that the map $\text{NS}(X)/\ell^r \to \text{NS}(X)_{\text{free}}/\ell^r$ is an isomorphism for all $r \geq 1$. It follows now from the Kummer sequence that there is a series of homomorphisms

$$\varprojlim_{r \geq 1} \text{Pic}(X^o)/\ell^r \xrightarrow{\cong} \varprojlim_{r \geq 1} \text{NS}(X)/\ell^r \xrightarrow{\cong} \varprojlim_{r \geq 1} \text{NS}(X)_{\text{free}}/\ell^r \hookrightarrow \varprojlim_{r \geq 1} H^2_{\text{ét}}(X^o, \mathbb{Z}/\ell^r) \cong H^2_{\text{ét}}(X^o, \mathbb{Z}_\ell).$$

Comparing with the similar maps for Y, we obtain a commutative diagram

$$\text{NS}(X)_{\text{free}} \hookrightarrow \widehat{\text{NS}(X)}_\ell \hookrightarrow H^2_{\acute{e}t}(X^o, \mathbb{Z}_\ell) \qquad (7.8)$$

$$\downarrow \iota^* \qquad \downarrow \iota^* \qquad \downarrow (\iota^o)^*$$

$$\text{NS}(Y)_{\text{free}} \hookrightarrow \widehat{\text{NS}(Y)}_\ell \hookrightarrow H^2_{\acute{e}t}(Y^o, \mathbb{Z}_\ell),$$

where $\widehat{A}_\ell$ denotes the ℓ-adic completion of an abelian group A.

Using Lemma 7.1, the exact sequence (7.7), and diagram (7.8), we reduce the proposition to showing that the pullback map

$$H^2_{\acute{e}t}(X^o, \mathbb{Z}/\ell^r) \to H^2_{\acute{e}t}(Y^o, \mathbb{Z}/\ell^r) \qquad (7.9)$$

is injective for all $r \geq 1$. But this follows from Proposition 5.3. $\qquad\square$

We can now prove our Lefschetz theorem for the generalized Albanese variety.

Theorem 7.3. *Let $X \subset \mathbb{P}^N_k$ be an integral projective scheme of dimension $d \geq 3$ over an algebraically closed field k of characteristic $p > 0$. Let $U \subset X^o$ be a dense open subscheme and $Z = X \setminus U$. Assume that X is an $(R_2 + S_2)$-scheme and $H \subset \mathbb{P}^N_k$ is a hypersurface of degree $m \gg 0$ such that $Y = X \cap H$ is Z-admissible. Then the map $\text{Alb}_S(U \cap Y)(k) \to \text{Alb}_S(U)(k)$ is an isomorphism.*

Proof. We let $V = U \cap Y$ and consider the commutative diagram of the short exact sequences of abelian groups (see formula (7.1))

$$0 \to \Lambda_V(Y)^\vee(k) \to \text{Alb}_S(V)(k) \to \text{Alb}_W(Y)(k) \to 0 \qquad (7.10)$$

$$\downarrow \alpha \qquad\qquad \downarrow \beta \qquad\qquad \downarrow \iota_*$$

$$0 \to \Lambda_U(X)^\vee(k) \to \text{Alb}_S(U)(k) \to \text{Alb}_W(X)(k) \to 0,$$

where the vertical arrows are the canonical maps induced by the inclusion $Y \hookrightarrow X$. We have seen in the proof of Lemma 7.1 that the right vertical arrow ι_* is an isomorphism. So we need to show that α is an isomorphism to prove the theorem.

Since Y is Z-admissible, we see that the homomorphism $\iota^*\colon \Lambda^1_U(X) \to \Lambda^1_V(Y)$ is bijective. But this implies by virtue of Proposition 7.2 that the homomorphism $\iota^*\colon \Lambda_U(X) \to \Lambda_V(Y)$ is also bijective. Taking the Cartier duals of these groups, we conclude that α is an isomorphism. $\qquad\square$

8. The Suslin homology

The goal of this section is to prove Theorem 1.6, which identifies the Levine–Weibel Chow group of a projective R_1-scheme over an algebraically closed field with the Suslin homology of its regular locus. We begin by recalling the definition of Suslin homology of smooth schemes.

8.1. Recollection of Suslin homology

Let k be any field. Let Δ_i denote the algebraic i-simplex – that is, the spectrum of the ring $k[x_0,\ldots,x_i]/(x_0+\cdots+x_i-1)$. Set $X \in \mathbf{Sch}_k$. Recall from [73] that the Suslin homology $H_i^S(X,A)$ of X with coefficients in an abelian group A is defined to be the ith homology of the complex $(C_*(X) \otimes_{\mathbb{Z}} A, \partial)$, where $C_i(X)$ is the free abelian group on the set of integral closed subschemes of $X \times \Delta_i$ which are finite and surjective over Δ_i. The boundary map is given by the alternating sum

$$\partial = \sum_{j=0}^{i} (-1)^j \delta_j^* : C_i(X) \to C_{i-1}(X),$$

where δ_j^* is the pullback map between the cycle groups induced by the inclusion $\delta_j : X \times \Delta_{i-1} \hookrightarrow X \times \Delta_i$, given by $x_j = 0$. Note that the finiteness and surjectivity conditions on the cycles over Δ_i insure that this pullback is defined.

As explained in [73], $H_*^S(X,A)$ is an algebraic analogue of the singular homology of topological spaces. We shall write $H_*^S(X,\mathbb{Z})$ as $H_*^S(X)$ for short. One easily checks from the definition that $H_*^S(-,A)$ is a covariant functor on $\mathbf{Sch}_k$. By [53, Proposition 14.18] and [15, Chapter 4, §9], the Suslin homology is also a part of the motivic homology and cohomology theories of algebraic varieties in the sense of $\mathbb{A}^1$-homotopy theory.

It is easy to see from the definition that the identity map $C_0(X) \to \mathcal{Z}_0(X)$ induces a surjective homomorphism $H_0^S(X) \twoheadrightarrow \mathrm{CH}_0^F(X)$. This is an isomorphism if X is complete. Otherwise, $H_0^S(X)$ carries more information about X than its Chow group. We shall be interested in the group $H_0^S(X,A)$. In this case, the universal coefficient theorem implies that there is a functorial isomorphism $H_0^S(X)/n \xrightarrow{\cong} H_0^S(X,\mathbb{Z}/n)$ for any integer $n \in \mathbb{Z}$.

In this paper, we shall use the following description of $H_0^S(X)$ due to Schmidt [66, Theorem 5.1]. Assume that X is a reduced scheme which is dense open in a projective scheme $\overline{X}$. Let $\nu \colon C \to \overline{X}$ be a finite morphism from a regular projective integral curve whose image is not contained in $\overline{X} \setminus X$. Let $f \in k(C)^\times$ be such that it is regular in a neighborhood of $\nu^{-1}(\overline{X} \setminus X)$ and $f(x) = 1$ for every $x \in \nu^{-1}(\overline{X} \setminus X)$. Then the identity map $C_0(X) \to \mathcal{Z}_0(X)$ induces an isomorphism between $H_0^S(X)$ and the quotient of $\mathcal{Z}_0(X)$ by the subgroup generated by $\nu_*(\mathrm{div}(f))$, where the (C,f) runs through the collection of all curves C and $f \in k(C)^\times$ as before. We shall let $\mathcal{R}_0^S(X)$ denote this subgroup.

8.2. Chow group with modulus and Suslin homology

One of the key steps in proving Theorem 1.6 is to show that the Suslin homology coincides with the Chow group of 0-cycles with modulus (see §3.3 for the definition of the latter) in certain cases. We shall prove this result of independent interest in this subsection. We expect this to have many applications in the theory of 0-cycles with modulus.

Let X be a regular projective scheme over a field k and $D \subset X$ an effective Cartier divisor. It is then an easy exercise to show, using Schmidt's description of Suslin homology, that the identity map of $\mathcal{Z}_0(X \setminus D)$ induces a surjection $\mathrm{CH}_0(X \mid D) \twoheadrightarrow H_0^S(X \setminus D)$. We let Λ be $\mathbb{Z}\left[\frac{1}{p}\right]$ if $\mathrm{char}(k) = p > 0$ and $\mathbb{Z}/n$, where n is any nonzero integer, if $\mathrm{char}(k) = 0$. The following result was obtained by the second author in a joint work with F. Binda [4]. Since the paper is not yet published, we present a proof.

Proposition 8.1. *Let X be a regular projective scheme over a field k and $D \subset X$ a reduced effective Cartier divisor whose all irreducible components are regular. Then $\mathrm{CH}_0(X \mid D)_\Lambda \twoheadrightarrow H_0^S(X \setminus D)_\Lambda$ is an isomorphism.*

Proof. We can assume that X is connected. We let $U = X \setminus D$. We need to show that $\mathcal{R}_0^S(U)$ dies in $\mathrm{CH}_0(X \mid D)_\Lambda$. So we let $\nu \colon C \to X$ be a finite morphism from a regular integral projective curve whose image is not contained in D. We let $E = \nu^{-1}(D)$ and let $f \in \mathcal{O}_{C,E}^\times$ be such that either $E = \emptyset$ or $f(x) = 1$ for all $x \in E$. Our assertion is immediate if $E = \emptyset$, and we therefore assume that this is not the case.

Since ν is a finite morphism of regular schemes, we can find a factorization $C \xrightarrow{\nu'} \mathbb{P}_X^n \xrightarrow{\pi} X$ of ν such that the first map is a closed immersion and the second map is the canonical projection. Since π is smooth, it follows that $\pi^*(D)$ is reduced with regular irreducible components. There is a push-forward map $\pi_* \colon \mathcal{Z}_0(\mathbb{P}_U^n) \to \mathcal{Z}_0(U)$ such that $\pi_*(\mathcal{R}_0(\mathbb{P}_X^n \mid \mathbb{P}_D^n)) \subset \mathcal{R}_0(X \mid D)$ (see [6] or [43, §2]). Since $\nu_*(\mathrm{div}(f)) = \pi_* \circ \nu'_*(\mathrm{div}(f))$, it suffices to show that $\nu'_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0(\mathbb{P}_X^n \mid \mathbb{P}_D^n)_\Lambda$. We can therefore assume that $\nu \colon C \hookrightarrow X$ is a closed immersion.

Since D is reduced with regular irreducible components, we can apply [63, Proposition A.6] to find a finite sequence of blowups $\pi \colon X' \to X$ along the closed points lying over D such that the scheme-theoretic inverse image $D' := X' \times_X D$ satisfies the following:

(1) The irreducible components of D'_{red} are regular.
(2) The strict transform C' of C is regular.
(3) C' intersects D'_{red} only in the regular locus of D'_{red} and transversely.

Since π is proper, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Z}_0(X' \setminus D') & \longrightarrow & \mathrm{CH}_0(X' \mid D') \\
\Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi_*} \\
\mathcal{Z}_0(X \setminus D) & \longrightarrow & \mathrm{CH}_0(X \mid D),
\end{array}
\tag{8.1}
$$

where π_* is the push-forward map between the 0-cycle groups. Since C is regular, the map $\pi \colon C' \to C$ is an isomorphism and hence $f \in k(C')^\times$ such that $\mathrm{div}(f)_C = \pi_*(\mathrm{div}(f)_{C'})$. Moreover, f is a regular invertible function in a neighborhood of $D' \cap C'$ with $f(x) = 1$ for every $x \in D' \cap C'$.

It follows from diagram (8.1) that $\mathrm{div}(f)_C$ will die in $\mathrm{CH}_0(X \mid D)_\Lambda$ if we can show that $\mathrm{div}(f)_{C'}$ dies in $\mathrm{CH}_0(X' \mid D')_\Lambda$. Equivalently, $\mathrm{div}(f)_{C'}$ dies in $\mathrm{CH}_0(X' \mid D'_{\mathrm{red}})_\Lambda$ by [55, Theorem 1.3]. We can therefore assume that our original curve $C \subset X$ has the property that it is regular and intersects D transversely in the regular locus of D. But in this case, it is easy to see that $f \in \mathcal{O}_{C,E}^\times$ and $f(x) = 1$ for all $x \in E$ if and only if $f \in \mathrm{Ker}\left(\mathcal{O}_{C,E}^\times \to \mathcal{O}_E^\times\right)$. This concludes the proof. $\qquad\square$

8.3. Relation with the Levine–Weibel Chow group

Let k be any field and X an integral projective R_1-scheme of dimension $d \geq 1$ over k. We first define a canonical homomorphism from the Levine–Weibel Chow group of X to the Suslin homology of X^o.

Lemma 8.2. *There is an inclusion of subgroups $\mathcal{R}_0^{LW}(X) \subseteq \mathcal{R}_0^S(X^o)$ inside $\mathcal{Z}_0(X^o)$. In other words, the identity map of $\mathcal{Z}_0(X^o)$ defines a canonical surjection*

$$\theta_X \colon \mathrm{CH}_0^{LW}(X) \twoheadrightarrow H_0^S(X^o).$$

Proof. By Lemma 2.4, we can replace $\mathcal{R}_0^{LW}(X)$ by $\mathcal{R}_0^{LW}(X, X_{\mathrm{sing}})$. We now let $C \subset X$ be an integral curve with $C \cap X_{\mathrm{sing}} = \emptyset$ and set $f \in k(C)^{\times}$. Since C is closed in the projective scheme X which does not meet X_{sing}, it is clear that the pair (C_n, f) defines a relation in $\mathcal{R}_0^S(X^o)$ according to Schmidt's description of $H_0^S(X^o)$. $\qquad\square$

Lemma 8.3. *Assume that $\mathrm{char}(k) = p > 0$ and $d = 2$. Then the kernel of $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow H_0^S(X^o)$ is a p-primary torsion group of bounded exponent.*

Proof. Let $\pi \colon \widetilde{X} \to X$ be a resolution of singularities of X such that the reduced exceptional divisor $E \subset \widetilde{X}$ has strict normal crossings (see [51] for the existence of $\widetilde{X}$). For an integer $m \geq 1$, let $mE \hookrightarrow \widetilde{X}$ denote the infinitesimal thickening of E in $\widetilde{X}$ of order m.

It is clear from the definitions of $\mathrm{CH}_0^{LW}(X)$, $\mathrm{CH}_0\left(\widetilde{X} \mid D\right)$, and $H_0^S(X^o)$ that the identity map of $\mathcal{Z}_0(X^o)$ defines, by the pullback via π^*, the canonical surjective maps

$$\mathrm{CH}_0^{LW}(X) \xrightarrow{\pi^*} \mathrm{CH}_0\left(\widetilde{X} \mid mE\right) \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid E\right) \twoheadrightarrow H_0^S(X^o) \qquad (8.2)$$

for every integer $m \geq 1$ such that the composite map is θ_X. The first arrow from the left is an isomorphism for all $m \gg 1$ by Theorem 1.5, and the third arrow is an isomorphism after inverting p by Proposition 8.1. We thus have to show that the kernel of the surjection $\mathrm{CH}_0\left(\widetilde{X} \mid mE\right) \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid E\right)$ is a p-group of bounded exponent[4] if $m \gg 1$. But this follows by comparing the map (see diagram (3.6)) $cyc_{\widetilde{X}\mid mE} \colon \mathrm{CH}_0\left(\widetilde{X} \mid mE\right) \to F^2 K_0\left(\widetilde{X}, mE\right)$ with $cyc_{\widetilde{X}\mid E} \colon \mathrm{CH}_0\left(\widetilde{X} \mid E\right) \to F^2 K_0\left(\widetilde{X}, E\right)$ for $m \gg 1$, and applying Proposition 3.5 in combination with [42, Lemma 3.4]. $\qquad\square$

We shall now generalize Lemma 8.3 to arbitrary dimension.

Theorem 8.4. *Let k be a perfect field of characteristic $p > 0$ and X an integral projective R_1-scheme of dimension $d \geq 2$ over k. Then the kernel of the canonical surjection $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow H_0^S(X^o)$ is a p-primary torsion group. Equivalently, the map*

$$\theta_X \colon \mathrm{CH}_0^{LW}(X)\left[\tfrac{1}{p}\right] \to H_0^S(X^o)\left[\tfrac{1}{p}\right]$$

is an isomorphism.

---

[4]If we do not insist on bounded exponent, then we can directly apply [55, Theorem 1.3].

Proof. We shall prove the theorem by induction on d. The case $d = 2$ follows from Lemma 8.3. So we can assume $d \geq 3$. Let $\nu \colon C \to X$ be a finite morphism from a regular integral projective curve whose image is not contained in X_{sing} and set $f \in \mathrm{Ker}\left(\mathcal{O}_{C,E}^{\times} \twoheadrightarrow \mathcal{O}_E^{\times}\right)$, where $E = \nu^{-1}\left(X_{\mathrm{sing}}\right)$ with the reduced closed subscheme structure. We need to show that $\nu_*(\mathrm{div}(f)) \in \mathrm{CH}_0^{LW}(X)$ is killed by a power of p.

We can get a factorization $C \overset{\nu'}{\hookrightarrow} \mathbb{P}_X^n \overset{\pi}{\to} X$ of ν, where ν' is a closed immersion and π is the canonical projection. Since the singular locus of $\mathbb{P}_X^n$ coincides with $\mathbb{P}_{X_{\mathrm{sing}}}^n$, it is clear that $\nu'_*(\mathrm{div}(f)) \in \mathcal{R}_0^S\left(\mathbb{P}_{X^o}^n\right)$ and $\nu_*(\mathrm{div}(f)) = \pi_*\left(\nu'_*(\mathrm{div}(f))\right)$. Using Corollary 2.7, it suffices therefore to show that $\nu'_*(\mathrm{div}(f))$ is killed by some power of p in $\mathrm{CH}_0^{LW}\left(\mathbb{P}_X^n\right)$. We can thus assume that $\nu \colon C \to X$ is a closed immersion.

We now fix a closed embedding $X \hookrightarrow \mathbb{P}_k^N$. We let $C' = C \cap X^o$. Since X^o and C' are smooth (this uses the perfectness of k) and $d \geq 3$, we can use [1, Theorem 7] (for k infinite) and [77, Theorem 3.1] (for k finite) to find a hypersurface $H \subset \mathbb{P}_k^N$ containing C and not containing X, such that the scheme-theoretic intersection $X \cap H$ has the property that it contains no irreducible component of X_{sing} and $H \cap X^o$ is smooth.

We let W be the connected component of $H \cap X^o$ which contains C' and $Y \subset X$ the closure of W with the reduced closed subscheme structure. Then $Y \subset X$ is an integral closed subscheme of dimension $d - 1$ containing C which satisfies the following properties:

(1) $Y \cap X^o$ is smooth.
(2) $\dim\left(Y \cap X_{\mathrm{sing}}\right) \leq \dim\left(X_{\mathrm{sing}}\right) - 1 \leq d - 3 = \dim(Y) - 2$. In particular, Y is an R_1-scheme.

We let $A = Y \cap X_{\mathrm{sing}}$ and $U = Y \cap X^o$ so that $Y_{\mathrm{sing}} \subset A$ and $U \subset Y^o$. Let $C \overset{\nu'}{\hookrightarrow} Y \overset{\iota}{\hookrightarrow} X$ be the factorization of ν. It follows from the choice of Y that $\nu'_*(\mathrm{div}(f)) \in \mathcal{R}_0^S(U) \subset \mathcal{R}_0^S(Y^o)$. On the other hand, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{CH}_0^{LW}(Y,A) & \overset{\cong}{\to} & \mathrm{CH}_0^{LW}(Y) \\
\downarrow & & \downarrow \\
H_0^S(U) & \longrightarrow & H_0^S(Y^o),
\end{array}
\tag{8.3}
$$

where the existence of the left vertical arrow follows directly from the proof of Lemma 8.2 and the top horizontal arrow is an isomorphism by Lemma 2.4.

Since $\nu'_*(\mathrm{div}(f)) \in \mathrm{CH}_0^{LW}(Y,A)$, it follows from this diagram and by induction that $\nu'_*(\mathrm{div}(f))$ is killed by a power of p in $\mathrm{CH}_0^{LW}(Y)$. Equivalently, $\nu'_*(\mathrm{div}(f))$ is killed by a power of p in $\mathrm{CH}_0^{LW}(Y,A)$. The push-forward map $\iota_* \colon \mathcal{Z}_0(Y \setminus A) \to \mathcal{Z}_0(X^o)$ and Corollary 2.7 together imply that $\nu_*(\mathrm{div}(f)) = \iota_* \circ \nu'_*(\mathrm{div}(f))$ is killed by a power of p in $\mathrm{CH}_0^{LW}(X)$. This concludes the proof. $\qquad\square$

8.4. The Albanese homomorphism

We now assume that k is an algebraically closed field. Let X be a connected smooth quasiprojective scheme of dimension $d \geq 1$ over k. The covariance of Suslin homology defines the push-forward map $\deg_X \colon H_0^S(X) \to H_0^S(k) \cong \mathbb{Z}$. This is also called the degree

map, since $\deg_X([x]) = [k(x) : k]$ for a closed point $x \in X$. We let $H_0^S(X)^0 = \mathrm{Ker}(\deg_X)$. Let $\mathrm{Alb}_S(X)$ be the generalized Albanese variety of X (see §7).

Let $\vartheta_X \colon \mathcal{Z}_0(X)^0 \to \mathrm{Alb}_S(X)(k)$ be given by $\vartheta_X(\sum_i n_i[x_i]) = \sum_i n_i(alb_X(x_i))$. It was shown in [72, Lemma 3.1] that this map factors through the quotient by $\mathcal{R}_0^S(X)$ to yield the Albanese homomorphism

$$\vartheta_X \colon H_0^S(X)^0 \to \mathrm{Alb}_S(X)(k). \tag{8.4}$$

Furthermore, ϑ_X defines a natural transformation of covariant functors from $\mathbf{Sm}_k$ to abelian groups as X varies. The map ϑ_X was in fact discovered by Ramachandran [61], who showed more generally that there exists an Albanese group scheme $\mathcal{A}lb_S(X)$ and an Albanese homomorphism $\vartheta_X \colon H_0^S(X) \to \mathcal{A}lb_S(X)(k)$ such that $\mathrm{Alb}_S(X)$ is the identity component of $\mathcal{A}lb_S(X)$ and formula (8.4) is the induced map on the degree 0 part. If X is projective over k, then ϑ_X coincides with the classical Albanese homomorphism from the degree 0 Chow group of 0-cycles on X.

Suppose now that $X \in \mathbf{Sch}_k$ is an integral projective R_1-scheme of dimension $d \geq 1$. Recall from §7 that the universal rational map $alb_X^w \colon X_n \dashrightarrow \mathrm{Alb}_W(X)$ extends to a regular morphism $alb_X^w \colon X^o \to \mathrm{Alb}_W(X)$. Moreover, the universal property of $\mathrm{Alb}_S(X^o)$ shows that this map is the composition $X^o \xrightarrow{alb_{X^o}} \mathrm{Alb}_S(X^o) \to \mathrm{Alb}_W(X)$.

Recall from [44, §7] that if X is normal, then $alb_X^w \colon X^o \to \mathrm{Alb}_W(X)$ gives rise to the Albanese homomorphism $\alpha_X \colon \mathrm{CH}_0^{LW}(X)^0 \to \mathrm{Alb}_W(X)(k)$. The construction of this homomorphism is identical to that of ϑ_{X^o}. If X is smooth, this is the classical Albanese homomorphism for $\mathrm{CH}_0^F(X)^0$. The main result of [44, §7] is that α_X is an isomorphism between the torsion subgroups, extending the famous Bloch–Roitman–Milne torsion theorem for smooth projective schemes. Using Corollary 2.6 and the birational invariance of $\mathrm{Alb}_W(X)$, this result immediately extends to R_1-schemes:

Proposition 8.5. *Let $X \in \mathbf{Sch}_k$ be an integral projective R_1-scheme of dimension $d \geq 1$. Then the Albanese map $alb_X^w \colon X^o \to \mathrm{Alb}_W(X)$ induces a homomorphism*

$$\alpha_X \colon \mathrm{CH}_0^{LW}(X)^0 \to \mathrm{Alb}_W(X)(k),$$

which is an isomorphism on the torsion subgroups.

8.5. Proof of Theorem 1.6

We shall now prove Theorem 1.6. We let k be an algebraically closed field and $X \in \mathbf{Sch}_k$ an integral projective R_1-scheme of dimension $d \geq 1$. We assume first that $\mathrm{char}(k) = p > 0$. We need to show in this case that the map $\theta_X \colon \mathrm{CH}_0^{LW}(X) \to H_0^S(X^o)$ is an isomorphism.

By Theorem 8.4, we only have to show that $\mathrm{Ker}(\theta_X)\{p\} = 0$. For this, we consider the diagram

$$
\begin{array}{ccc}
\mathrm{CH}_0^{LW}(X)^0 & \xrightarrow{\theta_X} & H_0^S(X^o)^0 \\
{\scriptstyle \alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta_{X^o}} \\
\mathrm{Alb}_W(X)(k) & \xleftarrow{\cong} & \mathrm{Alb}_S(X^o)(k),
\end{array}
\tag{8.5}
$$

where the bottom horizontal arrow is an isomorphism by formula (7.1). It follows from the construction of various maps that this diagram is commutative.

If $a \in \mathrm{Ker}(\theta_X)\{p\}$, then it must lie in $\mathrm{CH}_0^{LW}(X)^0$. Moreover, its image under θ_X will die in $H_0^S(X^o)^0$. This in turn implies by diagram (8.5) that $\alpha_X(a) = 0$. Proposition 8.5 implies that $a = 0$.

We now assume that $\mathrm{char}(k) = 0$. In this case, we have to show that the map $\theta_X \colon \mathrm{CH}_0^{LW}(X)/n \to H_0^S(X^o)/n$ is an isomorphism for all integers $n \neq 0$.

We let $\pi \colon X' \to X$ be the normalization map. We let $A = \pi^{-1}(X_{\mathrm{sing}})$ and $U = X' \setminus A$. We then have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{CH}_0^{LW}(X')/n & \xleftarrow{\cong} & \mathrm{CH}_0^{LW}(X',A)/n & \xrightarrow[\cong]{\pi_*} & \mathrm{CH}_0^{LW}(X)/n \\
\theta_{X'} \downarrow\downarrow & & \theta_{X'} \downarrow\downarrow & & \theta_X \downarrow\downarrow \\
H_0^S(X'^o)/n & \longleftarrow & H_0^S(U)/n & \xrightarrow[\cong]{\pi_*} & H_0^S(X^o)/n.
\end{array}
\tag{8.6}
$$

The top horizontal arrows are isomorphisms by Lemma 2.4 and Corollary 2.6. Suppose that the left vertical arrow is an isomorphism. Then the middle and the right vertical arrows also are isomorphisms. We can therefore assume that X is normal.

Let $\pi \colon \widetilde{X} \to X$ be a resolution of singularities of X such that the reduced exceptional divisor $E \subset \widetilde{X}$ has strict normal crossings. As in formula (3.4), there are canonical surjections

$$
\mathrm{CH}_0^{LW}(X)/n \xtwoheadrightarrow{\pi^*} \mathrm{CH}_0\left(\widetilde{X} \mid mE\right)/n \twoheadrightarrow \mathrm{CH}_0\left(\widetilde{X} \mid E\right)/n \twoheadrightarrow H_0^S(X^o)/n.
\tag{8.7}
$$

The first arrow from the left is an isomorphism for all $m \gg 1$ by [26, Theorem 1.8], and the third arrow is an isomorphism by Proposition 8.1. We thus have to show that $\mathrm{CH}_0\left(\widetilde{X} \mid mE\right)/n \to \mathrm{CH}_0\left(\widetilde{X} \mid E\right)/n$ is an isomorphism for all $m \geq 1$. But this follows from [55, Theorem 1.3(2)]. This concludes the proof of Theorem 1.6. □

8.6. Class field theory with finite coefficients

We shall now prove Theorem 1.3 as an application of Theorem 8.4. We restate it for convenience.

Theorem 8.6. *Let X be an integral projective R_1-scheme of dimension $d \geq 1$ over a finite field. Let n be any integer prime to $\mathrm{char}(k)$. Then the reciprocity map*

$$
\phi_X \colon \mathrm{CH}_0^{LW}(X)/n \to \pi_1^{\mathrm{ab}}(X^o)/n
$$

is an isomorphism of finite abelian groups.

Proof. By an argument identical to the one in diagram (8.6), we can assume that X is normal. We shall first show by induction on d that ϕ_X is surjective. This is clear for $d \leq 2$ by Theorem 1.2. We assume therefore that $d \geq 3$. We fix an integer n prime to $\mathrm{char}(k)$. We fix an embedding $X \subset \mathbb{P}_k^N$ and apply [20, Theorem 6.3] to find a hypersurface $H \subset \mathbb{P}_k^N$ such that the scheme-theoretic intersection $Y = X \cap H$ is normal and smooth along X^o

and intersects X_{sing} properly. We argued in the proof of Theorem 1.2(2) that Y must be integral in this case.

By Corollary 2.7, we get a commutative diagram

$$
\begin{array}{ccc}
\text{CH}_0^{LW}(Y)/n & \xrightarrow{\phi_Y} & \pi_1^{\text{ab}}(Y^o)/n \\
\downarrow{\scriptstyle\iota_*} & & \downarrow{\scriptstyle\iota_*} \\
\text{CH}_0^{LW}(X)/n & \xrightarrow{\phi_X} & \pi_1^{\text{ab}}(X^o)/n,
\end{array}
\tag{8.8}
$$

where $\iota\colon Y \hookrightarrow X$ is the inclusion.

It follows from Proposition 4.15 and [23, Exposé X, Corollaire 2.6] that the right vertical arrow is surjective. The top horizontal arrow is surjective by induction. We conclude that ϕ_X is surjective. To finish the proof of the theorem, it suffices now to show that $\text{CH}_0^{LW}(X)/n$ and $\pi_1^{\text{ab}}(X^o)/n$ are both finite abelian groups of the same cardinality.

By Theorem 8.4, we can replace $\text{CH}_0^{LW}(X)/n$ by $H_0^S(X^o)/n \cong H_0^S(X^o, \mathbb{Z}/n)$. Similarly, we can replace $\pi_1^{\text{ab}}(X^o)/n$ by $H_{\text{ét}}^1(X^o, \mathbb{Z}/n)^* := \text{Hom}_{\mathbb{Z}/n}\big(H_{\text{ét}}^1(X^o, \mathbb{Z}/n), \mathbb{Z}/n\big)$. On the other hand, [36, Corollary 7.1] implies that $H_0^S(X^o, \mathbb{Z}/n) \cong H_{\text{ét}}^1(X^o, \mathbb{Z}/n)^*$ and [19, Theorem 4.1] says that $H_{\text{ét}}^1(X^o, \mathbb{Z}/n)^*$ is finite. It follows that $\text{CH}_0^{LW}(X)/n$ and $\pi_1^{\text{ab}}(X^o)/n$ are finite and have the same cardinality. $\qquad\square$

8.7. Chow group versus Suslin homology over finite fields

We shall now show that the assumption that k is algebraically closed in Theorem 1.6 is essential. Assume that k is a finite field and $X \in \mathbf{Sch}_k$ satisfies one of the two conditions of Theorem 1.2. Let $\pi_1^{t,\text{ab}}(X^o)$ be the abelianized tame fundamental group of X^o [66] which describes the finite étale covers of X^o which are tamely ramified along X_{sing}. We then have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \text{CH}_0^{LW}(X) & \xrightarrow{\phi_X} & \pi_1^{\text{ab}}(X^o) & \longrightarrow & \widehat{\mathbb{Z}}/\mathbb{Z} & \to & 0 \\
& & \downarrow{\scriptstyle\theta_X} & & \downarrow & & \| & & \\
0 & \longrightarrow & H_0^S(X^o) & \xrightarrow{\phi_{X^o}} & \pi_1^{t,\text{ab}}(X^o) & \longrightarrow & \widehat{\mathbb{Z}}/\mathbb{Z} & \to & 0,
\end{array}
\tag{8.9}
$$

whose rows are exact. The top row is given by Theorem 1.6 and the bottom row is given by [66, Theorem 8.7]. It is clear that the middle vertical arrow may not be injective in general. This implies that θ_X is not injective in general.

9. The Roitman torsion theorem

Let k be an algebraically closed field of arbitrary characteristic. Let X be a smooth quasiprojective scheme over k which admits an open embedding $X \hookrightarrow \overline{X}$, where $\overline{X}$ is smooth and projective over k. Then Spieß and Szamuely [72] showed that the Albanese homomorphism ϑ_X (see formula (8.4)) is an isomorphism on prime-to-p torsion subgroups, where p is the exponential characteristic of k. This was an important development, as it provided a crucial breakthrough in eliminating the projectivity

hypothesis from the famous Roitman torsion theorem [63]. Geisser [18] subsequently showed that the prime-to-p condition in the torsion theorem of Spieß and Szamuely could be eliminated if one assumed resolution of singularities.

The goal of this section is prove Theorem 1.7, which eliminates the prime-to-p condition from the torsion theorem of [72] without assuming resolution of singularities.

9.1. Some preliminaries

We shall use the following results in our proof.

Lemma 9.1. *Let $X \subset \mathbb{P}_k^N$ be an integral Cohen–Macaulay closed subscheme of dimension $d \geq 2$ and let $C \subset \mathbb{P}_k^N$ be a closed subscheme such that the scheme-theoretic intersection $C \cap X$ has codimension ≥ 2 in X. Then for all $m \gg 0$, a general hypersurface $H \subset \mathbb{P}_k^N$ of degree m containing C has the property that $X \cap H$ is an integral scheme of dimension $d-1$.*

Proof. Since $\mathrm{codim}(C \cap X, X) \geq 2$, by [1, Theorem 1] a general hypersurface $H \subset \mathbb{P}_k^N$ of any degree $m \gg 0$ containing C has the property that $X \cap H$ is irreducible of dimension $d-1$ and smooth along $X_{\mathrm{reg}} \setminus C$. In particular, it is generically smooth. That is, $X \cap H$ satisfies Serre's R_0 condition.

Since X is Cohen–Macaulay, any hypersurface $H \subset \mathbb{P}_k^N$ containing C has the property that $X \cap H$ is Cohen–Macaulay. In particular, it satisfies Serre's S_1 condition. But it is classical that a Noetherian scheme is reduced if and only if it satisfies R_0 and S_1 conditions. We therefore conclude that a general hypersurface $H \subset \mathbb{P}_k^N$ of any degree $m \gg 0$ containing C has the property that $X \cap H$ is reduced and irreducible, hence integral of dimension $d-1$. $\qquad\square$

Lemma 9.2. *Let $X \subset \mathbb{P}_k^N$ be a smooth and connected projective scheme of dimension $d \geq 3$. Let $Z \subset X$ be a nowhere dense reduced closed subscheme with $(d-1)$-dimensional irreducible components $Z_1, \ldots, Z_r$. Let $C \subset X$ be a reduced curve with no components lying in Z. Assume that the embedding dimension of C at each of its closed points is at most 2. Then for all $m \gg 0$, a general hypersurface $H \subset \mathbb{P}_k^N$ of degree m containing C has the property that $X \cap H$ is Z-admissible.*

Proof. Since each Z_i is a Cartier divisor on a smooth scheme, it is Cohen–Macaulay of dimension $d-1 \geq 2$. Furthermore, our hypothesis implies that $C \cap Z_i$ has codimension ≥ 2 in Z_i for each i. Since $\mathrm{edim}(C \cap X^o) < 3$, it follows from [1, Theorem 7] that a general hypersurface $H \subset \mathbb{P}_k^N$ of any degree $m \gg 0$ containing C has the property that $X \cap H$ is smooth. We combine this with Lemma 9.1 to conclude the proof. $\qquad\square$

We shall also need the following result on the invariance of the p-primary torsion subgroup of the generalized Albanese variety under monoidal transformations.

Lemma 9.3. *Assume that $\mathrm{char}(k) = p > 0$ and let U be a smooth quasiprojective scheme of dimension $d \geq 1$ over k. Suppose that there exists an open immersion $U \subset X$ such that X is a smooth projective scheme. Let $\pi \colon \widetilde{X} \to X$ be the morphism obtained by a successive blowups along closed points. Then the induced homomorphism $\pi_* \colon \mathrm{Alb}_S\left(\pi^{-1}(U)\right)(k) \to \mathrm{Alb}_S(U)(k)$ is an isomorphism on the p-primary torsion subgroups.*

Proof. We let $\widetilde{U} = \pi^{-1}(U)$. It follows from formula (7.1) that there is a commutative diagram of exact sequences of abelian groups

$$0 \to \Lambda_{\widetilde{U}}\left(\widetilde{X}\right)^{\vee}(k) \to \mathrm{Alb}_S\left(\widetilde{U}\right)(k) \to \mathrm{Alb}_W\left(\widetilde{X}\right)(k) \to 0 \qquad (9.1)$$

$$0 \longrightarrow \Lambda_U(X)^{\vee}(k) \longrightarrow \mathrm{Alb}_S(U)(k) \longrightarrow \mathrm{Alb}_W(X)(k) \longrightarrow 0.$$

with vertical maps π_*.

Since $T(k)$ is divisible and $T(k)\{p\} = 0$ for an algebraic torus T over k, we see that the maps $\mathrm{Alb}_S\left(\widetilde{U}\right)(k)\{p\} \to \mathrm{Alb}_W\left(\widetilde{X}\right)(k)\{p\}$ and $\mathrm{Alb}_S(U)(k)\{p\} \to \mathrm{Alb}_W(X)(k)\{p\}$ are isomorphisms. On the other hand, one knows that the Weil–Albanese variety of a smooth scheme is a birational invariant. This implies that the right vertical arrow in the diagram is an isomorphism. We conclude that the middle vertical arrow is an isomorphism on the p-primary torsion subgroups. $\qquad\square$

9.2. Proof of Theorem 1.7

We shall now prove Theorem 1.7. We let U be a smooth quasiprojective scheme of dimension $d \geq 1$ over k with an open immersion $U \subset X$ such that X is smooth and projective over k. We have to show that the Albanese homomorphism $\vartheta_U \colon H_0^S(U)_{\mathrm{tor}} \to \mathrm{Alb}_S(U)(k)_{\mathrm{tor}}$ is an isomorphism.

We can assume X to be integral. We can also assume that $\mathrm{char}(k) = p > 0$. We shall prove the theorem by induction on d. The case $d \leq 2$ follows from [18, Theorem 1.1]. We therefore assume $d \geq 3$.

We fix a closed embedding $X \hookrightarrow \mathbb{P}_k^N$ and let $Z = X \setminus U$ with reduced structure. Let $H \subset \mathbb{P}_k^N$ be a hypersurface such that the scheme-theoretic intersection $Y = X \cap H$ satisfies the condition of Lemma 9.2. Using the covariance of the Albanese homomorphism (see the beginning of §8.4), we get a commutative diagram

$$\begin{array}{ccc} H_0^S(Y \cap U)^0 & \xrightarrow{\vartheta_{Y \cap U}} & \mathrm{Alb}_S(Y \cap U)(k) \\ \downarrow{\iota_*} & & \downarrow{\iota_*} \\ H_0^S(U)^0 & \xrightarrow{\vartheta_U} & \mathrm{Alb}_S(U)(k), \end{array} \qquad (9.2)$$

where $\iota \colon Y \hookrightarrow X$ is the inclusion. Using Theorem 7.3, the known case $d \leq 2$, and an induction on d, we see that ϑ_U is surjective on the torsion subgroups. In the rest of the proof, we shall show that this map is injective too.

We shall prove the injectivity in several steps. We fix an element $\alpha \in \mathcal{Z}_0(U)$ such that $\alpha \neq 0$ in $H_0^S(U)$ but $n\alpha = 0$ in $H_0^S(U)$ for some integer $n \geq 2$. By the torsion theorem of Spieß and Szamuely [72], we can assume $n = p^a$, where a is a positive integer. We must then have $\alpha \in \mathcal{Z}_0(U)^0$. We shall show that $\vartheta_U(\alpha) \neq 0$. This will finish the proof.

Since $n\alpha = 0$ in $H_0^S(U)$, we can find a finite collection of distinct integral normal curves $\{C_1, \ldots, C_m\}$ with finite maps $\nu_i \colon C_i \to X$, none of whose images is contained in Z, and elements $f_i \in \mathcal{O}_{C_i, E_i}^{\times}$ such that $f_i(x) = 1$ for every $x \in E_i$ and $n\alpha = \sum_i (\nu_i)_*(\mathrm{div}(f_i))$.

Here, $E_i = \nu_i^{-1}(Z)$. We let $C_i' = \nu_i(C_i)$ and $C' = \bigcup_i C_i' \subset X$. Since we cannot always find a hypersurface section of X which is smooth along U and contains C', we have to go through some monoidal transformations of X.

Step 1. We can find a morphism $\pi \colon \widetilde{X} \to X$ which is a composition of monoidal transformations whose centers are closed points such that the following hold:

(1) The strict transform D_i' of each C_i' is smooth so that $D_i' \cong C_i$.
(2) $D_i' \cap D_j' = \emptyset$ for $i \neq j$.
(3) Each $D_i' \subset \widetilde{X}$ intersects the exceptional divisor E (which is reduced) transversely.

It is clear that there exists a set of distinct blown-up closed points $T \subset X$ such that $\pi \colon \pi^{-1}(X \setminus T) \to X \setminus T$ is an isomorphism. Set $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{Z} = \pi^{-1}(Z)$ with reduced structure. We shall identify D_i' with C_i and the composite map $C_i \xrightarrow{\cong} D_i' \xrightarrow{\pi} C_i'$ with ν_i. Let C denote the strict transform of C' with irreducible components $\{C_1, \ldots, C_m\}$. We then have $E_i = \nu_i^{-1}(Z) = \widetilde{Z} \cap C_i$. Since $\mathrm{Supp}(\alpha) \subset C' \cap U$, we can find $\alpha' \in \mathcal{Z}_0(\widetilde{U})$ supported on C such that $\pi_*(\alpha') = \alpha$. This implies that $\pi_*(n\alpha' - \sum_i \mathrm{div}(f_i)) = 0$. Setting $\beta = n\alpha' - \sum_i \mathrm{div}(f_i)$, we get $\pi_*(\beta) = 0$ in the cycle group $\mathcal{Z}_0(U)$.

Step 2. We let $T' = T \cap U = \{y_1, \ldots, y_s\}$. We can then write $\beta = \sum\limits_{i=0}^{s} \beta_i$, where β_i is a 0-cycle on $\widetilde{U}$ supported on $\pi^{-1}(y_i)$ for $1 \leq i \leq s$ and β_0 is supported on $\widetilde{U} \setminus E$. We then get $\sum\limits_{i=0}^{s} \pi_*(\beta_i) = 0$ in $\mathcal{Z}_0(U) \subseteq \mathcal{Z}_0(X)$. Since all closed points of T are distinct and the support of $\pi_*(\beta_0)$ is disjoint from T', and hence from T, one easily checks that we must have $\pi_*(\beta_i) = 0$ for all $0 \leq i \leq s$. Since π is an isomorphism away from T, we must have $\beta_0 = 0$. We can therefore assume that β is a 0-cycle on $E \cap \widetilde{U}$.

We now note that each $\pi^{-1}(\{y_i\})$ is a $(d-1)$-dimensional projective scheme whose irreducible components are successive point blowups of $\mathbb{P}_k^{d-1}$. Moreover, we have $\pi_*(\beta_i) = 0$ under the push-forward map $\pi_* \colon \mathcal{Z}_0(\pi^{-1}(\{y_i\})) \to \mathbb{Z}$, induced by the maps $\pi \colon \pi^{-1}(\{y_i\}) \to \mathrm{Spec}(k(y_i)) \xrightarrow{\cong} \mathrm{Spec}(k)$. But this means that $\deg(\beta_i) = 0$. Taking the sum, we get $\deg(\beta) = \sum\limits_{i=1}^{s} \deg(\beta_i) = 0$. We can therefore find finitely many smooth projective rational curves $L_1, \ldots, L_{m'}$ on $E \cap \widetilde{U}$ and rational functions $f_j' \in k(L_j)$ such that $\beta = \sum\limits_{j=1}^{m'} \mathrm{div}(f_j')_{L_j}$ [42, Lemma 6.3].

Step 3. Using an argument of Bloch [9, Lemma 5.2], after possibly further blowup of $\widetilde{X}$ along the closed points of $E \cap \widetilde{U}$, we can assume that no more than two L_js meet at a point, and they intersect C transversely (note that C is smooth along E). In particular, in combination with items (1)–(3) of step 1, this implies that $D := C \cup (\cup_j L_j)$ is a reduced curve with the following properties (see four lines up from the bottom of [9, p. 5.2]):

a) Each component of D is smooth (note that $D = C$ away from $(\cup_j L_j)$).
b) D is smooth along $\widetilde{X} \setminus \widetilde{U}$.

c) $D \cap \widetilde{U}$ has only ordinary double point singularities – that is, exactly two components of $D \cap \widetilde{U}$ meet at any of its singular points with distinct tangent directions.

In particular, the embedding dimension of D at each of its closed points is at most 2. Furthermore, we have

$$n\alpha' = \sum_{i=1}^{m} \operatorname{div}(f_i) + \beta = \sum_{i=1}^{m} \operatorname{div}(f_i) + \sum_{j=1}^{m'} \operatorname{div}\left(f_j'\right). \tag{9.3}$$

Since $L_j \cap \widetilde{Z} = \emptyset$ for each j, it follows that $n\alpha' \in \mathcal{R}_0^S\left(\widetilde{U}\right)$. Note also that $\widetilde{X}$ is an integral smooth projective scheme.

Step 4. Let $\left\{\widetilde{Z}_1, \ldots, \widetilde{Z}_r\right\}$ be the set of irreducible components of $\widetilde{Z}$ of dimension $d-1$ with integral closed subscheme structure on each $\widetilde{Z}_i$. We fix a closed embedding $\widetilde{X} \hookrightarrow \mathbb{P}_k^M$. It follows from Lemma 9.2 that for all $q \gg 0$, a general hypersurface $H \subset \mathbb{P}_k^M$ of degree q containing D has the property that the scheme-theoretic intersection $Y = X \cap H$ is $\widetilde{Z}$-admissible. Since $q \gg 0$, we can also ensure using the Enriques–Severi–Zariski vanishing theorem that $H^0\left(\widetilde{X}, \Omega^1_{\widetilde{X}/k}(-Y)\right) = 0$. We choose such a hypersurface H and let $\iota \colon Y \hookrightarrow \widetilde{X}$ denote the inclusion. We let $V = Y \cap \widetilde{U}$.

Step 5. It follows from equation (9.3) and step 4 that $\alpha' \in \mathcal{Z}_0(V)$ and $n\alpha' \in \mathcal{R}_0^S(V)$ – that is, $n\alpha' = 0$ in $H_0^S(V)$. Note that $\alpha' \neq 0$ in $H_0^S(V)^0$, since $\pi_*(\alpha') = \alpha$ is not zero in $H_0^S(U)_{\mathrm{tor}}^0$. Since the Albanese homomorphism is a natural transformation between two functors on $\mathbf{Sm}_k$ (see formula (8.4)), there is a commutative diagram

$$\begin{array}{ccc}
H_0^S(V)_{\mathrm{tor}}^0 & \xrightarrow{\vartheta_V} & \mathrm{Alb}_S(V)(k)_{\mathrm{tor}} \\
{\scriptstyle \iota_*}\downarrow & & \downarrow{\scriptstyle \iota_*} \\
H_0^S\left(\widetilde{U}\right)_{\mathrm{tor}}^0 & \xrightarrow{\vartheta_{\widetilde{U}}} & \mathrm{Alb}_S\left(\widetilde{U}\right)(k)_{\mathrm{tor}}.
\end{array} \tag{9.4}$$

By the choice of H and Theorem 7.3, the right vertical arrow is an isomorphism. Since $\alpha' \in H_0^S(V)_{\mathrm{tor}}^0$, it follows by induction on d that $\vartheta_V(\alpha') \neq 0$. Hence, we get

$$\vartheta_{\widetilde{U}}(\alpha') = \vartheta_{\widetilde{U}} \circ \iota_*(\alpha') = \iota_* \circ \vartheta_V(\alpha') \neq 0. \tag{9.5}$$

We now consider another commutative diagram,

$$\begin{array}{ccc}
H_0^S\left(\widetilde{U}\right)^0 \{p\} & \xrightarrow{\vartheta_{\widetilde{U}}} & \mathrm{Alb}_S\left(\widetilde{U}\right)(k)\{p\} \\
{\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\
H_0^S(U)^0\{p\} & \xrightarrow{\vartheta_U} & \mathrm{Alb}_S(U)(k)\{p\}.
\end{array} \tag{9.6}$$

Using this diagram, we get

$$\vartheta_U(\alpha) = \vartheta_U \circ \pi_*(\alpha') = \pi_* \circ \vartheta_{\widetilde{U}}(\alpha').$$

Since $\vartheta_{\widetilde{U}}(\alpha') \neq 0$ by diagram (9.5), we conclude from Lemma 9.3 that $\vartheta_U(\alpha) \neq 0$. This concludes the proof of Theorem 1.7.

Competing Interest. None.

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