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On groups with extremal blocks

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Let G be a finite group. It is shown that G is 2-closed if and only if

(a) every 2-block of G has full defect, and

(b) every Sylow 2-intersection is centralized by a Sylow 2-subgroup of G.

As a consequence it is shown that G is a TI-group if and only if every 2-block of G has either full defect or defect zero and (b) holds. This result and a theorem of Kwok yield complete characterizations of finite groups with certain relations being satisfied by every nonprincipal irreducible character.

1. Introduction

Let G be a finite group. It is well known that if G is 2-closed (has a normal Sylow 2-subgroup), then every 2-block of G has full defect. It is also well known [9] that if G is a TI-group (the intersection of distinct Sylow 2-subgroups of G is the identity), then every 2-block of G has either full defect or defect zero. Since the Mathieu groups M_{22} and M_{24} have only one 2-block, the converses of the above statements are false.

In Section 2, 2-closure and the TI-property are characterized by means of the defects of 2-blocks and the following property:

CI: every 2-Sylow intersection in G is centralized by a Sylow 2-subgroup of G.

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A 2-Sylow intersection is an intersection of two *distinct* Sylow 2-subgroups of G . Theorem 1 generalizes a theorem by Harada [6] concerning the 2-closure property of groups with abelian Sylow 2-subgroups.

In Section 3, the results of Section 2 are applied to the following problem: suppose that $g \in G^{\#}$, k is a complex number and the following property holds:

CD: $X(1) - X(g) = k$ for all nonprincipal irreducible characters X of G .

It was shown by Kwok [7] that groups with the CD-property have elementary abelian Sylow 2-subgroups. In Theorem 3, groups with the CD-property are completely characterized.

2. On 2-closed groups and *TI*-groups

The main result of this section is the following:

THEOREM 1. *Let G be a finite group. Then G is 2-closed if and only if the following conditions are satisfied:*

(a) every 2-block of G has full defect; and

(b) G has the CI-*property.*

Proof. If G is 2-closed, then *(a)* follows and *(b)* is void.

So suppose that G is of minimal order among those groups which satisfy *(a)* and *(b)*, but are not 2-closed.

Case 1. G is a *TI*-group with a cyclic or generalized quaternion Sylow 2-subgroup. By [8], Theorem 3, G contains a normal subgroup N of odd order such that G/N is 2-closed and contains a single involution. Let x be an involution in G and let $H = \langle x \rangle N$. Then H is normal in G and contains all involutions of G . Suppose that the inertia group $T_H(\theta)$ of every irreducible character θ of N in H is H . Then, by [2], (9.10) and (9.12), if X is an irreducible character of H , then X_N is an irreducible character of N . Thus, by [4], Lemma (3D), $H = \langle x \rangle \times N$, a contradiction, since G is a *TI*-group, but not a 2-closed one. Thus let θ be an irreducible character of N satisfying

$T_H(\theta) = N$. Let X be an irreducible constituent of θ^G ; then, by [2], (9.11), X belongs to a block of defect zero, a contradiction.

Case 2. G is a *TI*-group not covered by Case 1. Again by [8], Theorem 2, G has a single chief factor of even order which is isomorphic to a simple group with an irreducible character of order $|G|_2$. Thus, by Clifford's Theorem, G has an irreducible character divisible by $|G|_2$, in contradiction to *(a)*.

Case 3. G is not a *TI* -group. Let x be an involution belonging to distinct Sylow 2-subgroups of G . By *(b)* and by [5], Lemmas 3.1 and 3.3, $H = C_G(x)$ is not 2-closed and satisfies *(b)*. Thus either $G = H$ or H does not satisfy *(a)*.

If $G = H$, consider $N = G/\langle x \rangle$. By [5], Lemma 3.3, N satisfies *(b)* and by [3], Chapter V, (4.5), N satisfies *(a)*. Thus, by induction, N is 2-closed and hence so is G , a contradiction.

So suppose, finally, that H does not satisfy *(a)*. Let D be a defect group of a 2-block in H satisfying $2^d = |D| < |G|_2$. Then, by [9], D is a Sylow 2-intersection, which implies, in view of *(b)*, that $C_H(D)\cdot D = C_H(D) = C_G(D)$. Hence, by [1], (5A), $C_G(D)$ has a block of defect d , and again by [1], (5C), it follows in view of *(b)* that G has a block of defect d , a final contradiction.

Theorem 1 yields rather easily the following:

THEOREM 2. *Let G be a finite group. Then G is a TI-group if and only if the following conditions are satisfied:*

(a) every 2-block of G has either full defect or defect zero; and

(b) G has the CI-*property.*

Proof. If G is a *TI*-group, then *(b)* is trivial and *(a)* holds by [9].

So suppose that G satisfies *(a)* and *(b)*, but it is not a *TI*-group.

Let u be an involution belonging to distinct Sylow 2-subgroups of G. Denote $C_G(u)$ by H; then *(b)* implies $|H|_2 = |G|_2$ and by [5], Lemma 3.1, H is not 2-closed. Suppose that D is a defect group of a 2-block of H satisfying $|D| = 2^d < |G|_2$. Then, as in the final argument of the proof of Theorem 1, G has a 2-block of defect d. By *(a)*, $d = 0$, in contradiction to the fact that $u \in D$. Therefore H has blocks of full defect only, and by Theorem 1, H is 2-closed, a final contradiction.

3. Characterization of CD-groups

Theorem 2 and [7] yield the following:

THEOREM 3. *Suppose that* g *is a nonidentity element of a finite group* G, *and let* k *be a complex number. Then the equality*

$$X(1) - X(g) = k \tag{1}$$

holds for all nonprincipal irreducible characters X *of* G *over the complex field if and only if* g *is an involution,* $k = |G|_2 = 2^n$, *and either*

(i) $G = \langle g \rangle N$, *with* N *an abelian normal subgroup of* G *of odd order and* $\langle g \rangle = C_G(g)$, *or*

(ii) $G \cong \mathrm{PSL}(2, 2^n)$, $n \geq 2$.

Proof. If g is an involution and either *(i)* or *(ii)* holds, then it is easy to check that (1) is satisfied with $k = |G|_2$.

So suppose that (1) is satisfied. Then by [7], g is an involution, a Sylow 2-subgroup S of G is elementary abelian, $k = |S| = 2^n$ and G has one conjugacy class of involutions. In addition, if G is simple, then by [7] either $|G| = 2$ or $G \cong \mathrm{PSL}(2, 2^n)$.

Since g is an involution and G possesses only one conjugacy class of those, it follows by (1) and the fact that $k = |G|_2$ that G has no proper normal subgroups of even order. Thus $G/O(G)$ is a simple group satisfying (1) with respect to $gO(G)$. Consequently, either $|G/O(G)| = 2$

or $G/O(G) \cong \mathrm{PSL}(2, 2^n)$.

In the former case $k = 2$ and by (1), $G' = O(G)$. It follows also by (1) that if $X(1) > 1$, then $X(g) \geq 0$. Thus the nonprincipal linear character is the only one which takes a negative value -1 on g . The orthogonality relations hence imply that $X(g) = 0$ for every nonlinear character X of G . Thus $C_G(g) = 2$ and G is of type *(i)*.

It remains to deal with the case $G/O(G) \cong \mathrm{PSL}(2, 2^n)$. Suppose that G has a 2-block B of defect a , $0 < a < n$. Then by [3], Chapter IV, (3.14),

$$\sum X(1)X(g) = 0 ,$$

where the summation ranges over all $X \in B$. Since $0 < a$, it follows by (1) that $X(g) \neq 0$ for every $X \in B$. Thus there exists $Y \in B$ such that $Y(g) < 0$. However, since G has elementary abelian Sylow 2-subgroups and one conjugacy class of involutions, it follows that

$$Y(1) + (k-1)Y(g) \geq 0 .$$

This inequality, together with (1), yields

$$k + kY(g) \geq 0 ;$$

hence $Y(g) = -1$ and $Y(1) = k - 1$. This is a contradiction since $Y(1)$ is odd and $a < n$. Thus G satisfies conditions *(a)* and *(b)* of Theorem 2 and consequently G is a *TI*-group. It follows by [8], Theorem 6 and remarks, in view of the fact that G has no proper normal subgroup of even order, that $G \cong \mathrm{PSL}(2, 2^n)$, as required.

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