A RECURRENCE RELATION GENERALIZING THOSE OF APÉRY

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Abstract

A three term recurrence relation is found for

$$g_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+a+d}{k+d} \binom{n+k+b+e}{k+e} \binom{n+k+c+f}{k+f}$$

when $a + d = b + c$. This includes the recurrence relations of Apéry associated with $\zeta(3)$, $\zeta(2)$ and $\log 2$ as special or limiting cases.

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In the fall of 1978 A. van der Poorten gave lectures on Apéry's proof of the irrationality of $\zeta(3)$ at a number of North American universities. The lecture was delightful. An approximation can be made by reading his published version of the lecture [14], but the lecture was even more exciting. See Mendes-France [6] for some comments similar to those van der Poorten left out of his published version. As a result of these lectures many people became aware that Apéry had shown that

$$(1) \qquad b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the three term recurrence relation

$$(2) \qquad n^3 b_n + (n-1)^3 b_{n-2} = (34n^3 - 51n^2 + 27n - 5) b_{n-1}$$

and that

(3)
$$c_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

satisfies

(4)
$$n^2 c_n - (n-1)^2 c_{n-2} = (11n^2 - 11n + 3)c_{n-1}.$$

The existence of one recurrence relation of this type could be an accident but the existence of two suggested to a number of people that there should be more. We received a few letters asking about the existence of three term recurrence relations for a number of sequences. The most common example was

(5)
$$d_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2.$$

It is easy to see that d_n satisfies a four term recurrence relation, and that

(6)
$$e_n = \sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}$$

satisfies a five term one. It is also very likely that they do not satisfy a linear recurrence relation of a lower order. This was not clear to the people we wrote to, and to be honest it is not clear when the sequences are written in this form. Fortunately, there is a much better way to write them, and in this other form it is clear that (5) is a little more complicated than (3) and that (6) is much more complicated than (1). The other form of writing these series is old, and has been studied by Euler, Pfaff, Gauss, Kummer, Jacobi and many others, so it should not be as unknown as it is. It is called *hypergeometric series*.

A generalized hypergeometric series, or better, a hypergeometric series, is a series Σc_k with c_{k+1}/c_k a rational function of k. This rational function is usually factored, so

(7)
$$\frac{c_{k+1}}{c_k} = \frac{(k+a_1) \cdots (k+a_p)x}{(k+b_1) \cdots (k+b_q)(k+1)},$$

and if $c_0 = 1$ this gives

(8)
$${}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}; x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k x^k}{(b_1)_k \cdots (b_q)_k k!}.$$

The shifted factorial $(a)_k$ is defined by

(9)
$$(a)_k := a(a+1) \cdots (a+k-1) = \Gamma(a+k)/\Gamma(a).$$

Gauss [4], defined two $_2F_1$ series to be *contiguous* if there is a change of 1 in one of the parameters and the others do not change. If F denotes $_2F_1(a, b; c; x)$ then $F(a +)$ will denote $_2F_1(a + 1, b; c, x)$, and obvious generalizations of this notation will be used in the following. Gauss showed that a $_2F_1$ and any two series

that are contiguous to it are linearly related, and explicitly gave the $\binom{6}{2} = 15$ identities. Since $_2F_1(a, b; c; x)$ is symmetric in a and b there are really nine different identities. Gauss took one of his identities, set $x = 1$, and it reduced to a two term recurrence relation. He iterated this and used it to prove that

$$(10) \qquad\qquad {}_2F_1\left({a, b \atop c}; 1\right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

The Gauss contiguous relations can be iterated, so $_2F_1(a, b; c; x)$ and any two $_2F_1$'s whose parameters differ by integers from those of this $_2F_1$ are linearly related. In particular if

$$(11) \qquad\qquad p_n = {}_2F_1(-n, n + a + b + 1; a + 1; t)$$

then

$$a_n p_{n+1} + b_n p_n + c_n p_{n-1} = 0, \qquad n = 1, 2, \ldots,$$

for some a_n, b_n, c_n, and they can be found explicitly. The above statements can be made much stronger, and bounds on the degree of the coefficients in any of the parameters can be given. Without this the results are obvious, since we could take $c_n = 0$, $b_n = -p_{n+1}$, $a_n = p_n$, which is a triviality. For the fundamental relations of Gauss the coefficients are linear in each parameter and in the variable x.

Since

$$\frac{d}{dx} {}_2F_1(a, b; c, x) = \frac{ab}{c} {}_2F_1(a + 1, b + 1; c + 1; x)$$

the relation connecting $_2F_1(a, b; c, x)$, $_2F_1(a + 1, b + 1; c + 1; x)$ and $_2F_1(a + 2, b + 2; c + 2; x)$ can be rewritten as a second order differential equation. It is well known that $_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ satisfies a differential equation of order $q + 1$ when $p \leqslant q + 1$. The condition $p \leqslant q + 1$ is necessary to obtain convergence of the series (8). It is not quite as well known that such a function and $q + 1$ of them that are contiguous to this function are linearly related. See Rainville [8, Chapter 14]. Kummer [5] considered both $_2F_1$ and $_1F_1$ in detail, and at the end of the second part of this paper he wrote that he had tried to study

$$_3F_2(a, b, c; d, e; x)$$

but only when $x = 1$ had he been able to do much. He only stated one transformation formula, but linear three term relations exist between a $_3F_2$ and any two series contiguous to it when the power series variable is one. Again these can be iterated, and a few of these relations have been recorded in print. The complete listing of these three term relations was only published in the late 1970's. One list is in Wilson [17], a second listing in very compact form is in Raynal [10].

At the $_4F_3$ level a function and any four functions contiguous to it are linearly related. When $x = 1$ it is usually (or always, we have not checked thoroughly)

true that a $_4F_3$ and three functions contiguous to it are linearly related. To find cases when a function and two contiguous series are linearly related we ask, by analogy with the $_3F_2$, $_2F_1$ cases, if there are any $_3F_2$'s that can be summed. Being able to sum a series that has a free parameter, or more, is usually an indication that a two term recurrence relation exists between contiguous series in this family, and what one hopes is that when new parameters are added the order of the recurrence or contiguous relations will not increase very much. Pfaff [7] summed the following series:

$$_3F_2\left(\begin{array}{c} -n, a, b \\ c, a + b + 1 - n - c \end{array} ; 1\right) = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}.$$

This series was evaluated again by Saalschütz, almost one hundred years after Pfaff, and since Pfaff's work was not well known this identity was known as Saalschütz's until recently. The condition the parameters satisfy is

$$-n + a + b + 1 = c + a + b + 1 - n - c,$$

or

$$(12) \qquad\qquad\qquad 1 + \sum a_i = \sum b_i.$$

If $p = q + 1$, if the series terminates, that is, if one of the a_i's is a negative integer, if the power variable is one, and if (12) holds the series is called *balanced*. The above analogy suggests that a balanced $_4F_3$ and any two series contiguous to it are likely related. This is not true, since a series contiguous to a balanced series is not balanced. For the sum of the top parameters will be equal to the sum of the bottom ones, or will be two more than the sum of the bottom ones. This suggests redefining contiguous for balanced series. Change two parameters by one each in such a way that the new series is still balanced. To see if this is reasonable, observe that b_n defined in (1) can be written as

$$(13) \qquad\qquad b_n = {_4F_3}\left(\begin{array}{c} -n, -n, n + 1, n + 1 \\ 1, 1, 1 \end{array} ; 1\right).$$

This is a balanced $_4F_3$, as is b_{n+1} and b_{n-1}. c_n from (3) is

$$(14) \qquad\qquad\qquad c_n = {_3F_2}\left(\begin{array}{c} -n, -n, n + 1 \\ 1, 1 \end{array} ; 1\right)$$

and so is a $_3F_2$ with power series variable equal to one. Thus both should satisfy three term recurrence relations. The other two sequences

$$(15) \qquad\qquad\qquad d_n = {_3F_2}\left(\begin{array}{c} -n, n + 1, n + 1 \\ 1, 1 \end{array} ; -1\right)$$

and

$$(16) \qquad\qquad\qquad e_n = {_4F_3}\left(\begin{array}{c} -n, -n, -n, n + 1 \\ 1, 1, 1 \end{array} ; -1\right)$$

do not satisfy the right type of restrictions. d_n satisfies a four term recurrence relation, and e_n satisfies a five term recurrence relation. One gets a three term relation for

(17) $$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}^2 (-1)^k = {}_3F_2\left(\begin{array}{c} -n, n+1, n+1 \\ 1,1,1 \end{array} ; 1\right)$$

The heuristic argument given above leads to the right conclusion. A balanced ${}_4F_3$ and any two balanced contiguous series to it are linearly related. These relations were first treated in detail by the same two who listed the ${}_3F_2$ relations. Wilson [18] gave a complete list, and Raynal [11] gave a method for deriving them and stated the fundamental ones. Wilson's list is more convenient to use, but it takes more space to print.

To see how to generalize (1) and (2) consider

(19) $$g_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+a+d}{k+d}\binom{n+k+b+e}{k+e}\binom{n+k+c+f}{k+f}.$$

We do not assume the parameters are integers, and define

(20) $$\binom{a}{b} = \frac{a!}{(a-b)!b!} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)}$$

if a is not a negative integer. If a is a negative integer then the sum (19) can still make sense if there is another gamma function that is in the denominator and each pole of $\Gamma(x)$ is cancelled by a zero of $[\Gamma(y)]^{-1}$. One has to be careful in taking limits, but this can usually be done. To avoid these problems assume they do not arise. The series for g_n can be rewritten as

(21) $$g_n = \binom{n+a+d}{d}\binom{n+b+e}{e}\binom{n+c+f}{f}$$
$${}_4F_3\left(\begin{array}{c} -n, -n-a, n+b+e+1, n+c+f+1 \\ d+1, e+1, f+1 \end{array} ; 1\right).$$

This ${}_4F_3$ is balanced when $b+c-a=d$ or $a+d=b+c$. Without a translation it would be hard (or impossible) to see why this is a natural condition to assume.

The series in (21) will be denoted by

(22) $$F_n = {}_4F_3\left(\begin{array}{c} -n, -n-a, n+b+e+1, n+c+f+1 \\ d+1, e+1, f+1 \end{array} ; 1\right).$$

If we show that

(23) $$A_n F_{n+1} + B_n F_n + C_n F_{n-1} = 0$$

then

(24) $$A'_n g_{n+1} + B'_n g_n + C'_n g_{n-1} = 0$$

with

$$A'_n = A_n \frac{(n + a + 1)(n + b + 1)(n + c + 1)}{(n + a + d + 1)(n + b + e + 1)(n + c + f + 1)},$$

$$B'_n = B_n,$$

$$C'_n = C_n \frac{(n + a + d)(n + b + e)(n + c + f)}{(n + a)(n + b)(n + c)}.$$

Note that F_n, F_{n-1} and F_{n+1} are balanced series related in the same way as are the series F, $F(a+, b+, c-, d-)$, and $F(a-, b-, c+, d+)$, where F is a balanced series

$$F = {}_4F_3\left(\begin{array}{l} a, b, c, d \\ e, f, g \end{array}; 1\right).$$

Note also that $F(a+, c-)$ is a balanced series contiguous to F and $F(a+, b+, c-, d-)$ and that $F(b-, d+)$ is contiguous to F and $F(a-, b-, c+, d+)$. The contiguous relation

(25)
$$\frac{(e - b)(f - b)(g - b)}{c(a - b + 1)}[F(a+, b-) - F]$$
$$- \frac{(e - d)(f - d)(g - d)}{a(c - d + 1)}[F(c+, d-) - F] + (d - b)F = 0$$

from [18] gives a relation connecting F, $F(a+, c-)$, and $F(b-, d+)$, another relation connecting F, $F(a+, b-)$, and $F(a+, b+, c-, d-)$, and a third connecting F, $F(b-, d+)$, and $F(a-, b-, c+, d+)$. Eliminating $F(a+, c-)$ and $F(b-, d+)$ from these three gives the desired three term relation. In terms of F_n it is formula (23) with

$$A_n = (n + d + 1)(n + e + 1)(n + f + 1)(n + a + d + 1)(n + a + e + 1)$$
$$\cdot (n + a + f + 1)(n + b + e + 1)(n + c + f + 1)/D_1,$$

$$C_n = (n + c + f - d)(n + c + f - e)(n + c)(n + b + e - d)(n + b)$$
$$\cdot (n + b + e - f)n(n + a)/D_2,$$

$$B_n = -(n + a + d + 1)(n + a + e + 1)(n + a + f + 1)(n + b + e + 1)$$
$$\cdot (n + c + f + 1)[(n + d + 1)(n + e + 1)(n + f + 1)$$
$$+ (n + a + 1)(2n + b + e + 2)(2n + a + c + f + 2)] \cdot D_1^{-1}$$
$$- (n + b + e - d)(n + b)(n + b + e - f)(n + a)n$$
$$\cdot [(n + c + f - d)(n + c + f - e)(n + c)$$
$$+ (2n + c + f)(n + b + e)(2n + a + c + f)]D_2^{-1}$$
$$- (2n + a + b + e + 1)n(n + c + f + 1),$$

$$D_1 = (n + d + 1)(n + e + 1)(n + f + 1)(n + b + e + 1)(2n + a + c + f + 2)$$
$$+ (n + c + f - d + 1)(n + c + f - e + 1)(n + c + 1)(n + a + 1)$$
$$\cdot (2n + b + e + 2) + (2n + c + f + 2)(n + a + 1)(2n + b + e + 2)$$
$$\cdot (n + b + e + 1)(2n + a + c + f + 2),$$

and

$$D_2 = (n + d)(n + e)(n + f)(n + b + e)(2n + a + c + f)$$
$$+ (n + c + f - d)(n + c + f - \cdot e)(n + c)(n + a)(2n + b + e)$$
$$+ (2n + c + f)(n + a)(2n + b + e)(n + b + e)(2n + a + c + f).$$

To find a recurrence relation for c_n, and more generally for

$$(26) \quad h_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n + a + d}{k + d} \binom{n + k + b + e}{k + b}$$
$$= \binom{n + a + d}{d} \binom{n + b + e}{e} {}_3F_2 \left(\begin{matrix} -n, -n - a, n + b + e + 1 \\ d + 1, e + 1 \end{matrix} ; 1 \right)$$

divide (23) by f and let $f \to \infty$. This gives a recurrence relation for the ${}_3F_2$ in (26), and so also for h_n. When this recurrence relation is written there is a c in it, but not in the ${}_3F_2$. Remove the c by using $a + d = b + c$.

If $d \to \infty$ in (22) with $a + d = b + c$ and b and c fixed, the resulting series is

$${}_3F_2 \left(\begin{matrix} -n, n + b + e + 1, n + c + f + 1 \\ e + 1, f + 1 \end{matrix} ; 1 \right)$$
$$= \left[\binom{n + b + e}{e} \binom{b + c + f}{f} \right]^{-1}$$
$$\cdot \sum_{k=0}^{n} \binom{n}{k} \binom{n + k + b + e}{k + e} \binom{n + k + c + f}{k + f} (-1)^k.$$

This gives a three term recurrence relation for the series defined in (17).

Similarly one can find a three term recurrence relation satisfied by

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{3n + k + 1}{k} (-1)^k = {}_4F_3 \left(\begin{matrix} -n, -n, -n, 3n + 2 \\ 1, 1, 1 \end{matrix} ; 1 \right)$$

and many other sequences which may have some arithmetic significance.

This is not the end of the three term recurrence relations. Van der Poorten [14] mentioned that T. Cusick had found two more.

$$(27) \qquad\qquad\qquad u_n = \sum_{k=0}^{n} \binom{n}{k}^4$$

satisfies

(28) $n^3 u_n = 2(2n - 1)(3n^2 - 3n + 1)u_{n-1} + (4n - 3)(4n - 4)(4n - 5)u_{n-2}$

and

(29) $$v_n = \sum_{k=0}^{n} \binom{n}{k}^3$$

satisfies

(30) $$n^2 v_n = (7n^2 - 7n + 2)v_{n-1} + 8(n - 1)^2 v_{n-2}.$$

These do not fit directly into the above pattern, since

(31) $$u_n = {}_4F_3\left(\begin{matrix} -n, -n, -n, -n \\ 1, 1, 1 \end{matrix}; 1 \right)$$

is not balanced and

(32) $$v_n = {}_3F_2\left(\begin{matrix} -n, -n, -n \\ 1, 1 \end{matrix}; -1 \right)$$

has the wrong power series variable.

However both series can be transformed to the appropriate type of series. To transform (32) recall a transformation of Whipple [16]

(33)
$$
{}_3F_2\left(\begin{matrix} a, b, c \\ a + 1 - b, a + 1 - c \end{matrix}; x \right)
$$
$$
= (1 - x)^{-a} {}_3F_2\left(\begin{matrix} a + 1 - b - c, a/2, (a + 1)/2 \\ a + 1 - b, a + 1 - c \end{matrix}; \frac{-4x}{(1 - x)^2} \right).
$$

When $x = -1$ this gives
(34)

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ a + 1 - b, a + 1 - c \end{matrix}; -1 \right) = 2^{-a} {}_3F_2\left(\begin{matrix} a + 1 - b - c, a/2, (a + 1)/2 \\ a + 1 - b, a + 1 - c \end{matrix}; 1 \right).
$$

The series on the left is said to be *well poised*, since the parameters can be paired, top ones with bottom ones, so that the sum is constant. The $n!$ in the denominator is considered as $(1)_n$. A well poised ${}_3F_2$ at $x = -1$ can be transformed to a ${}_3F_2$ at $x = 1$; and series whose parameters differ by integers correspond on the two sides. Thus Cusick's recurrence relation for v_n can be extended to a recurrence relation for

(35)
$$
\sum_{k=0}^{n} \binom{n}{k}\binom{n + 2a}{k + a}\binom{n + 2b}{k + b}
$$
$$
= \binom{n + 2a}{a}\binom{n + 2b}{b} {}_3F_2\left(\begin{matrix} -n, -n - a, -n - b \\ a + 1, b + 1 \end{matrix}; -1 \right)
$$

There is another way of pairing the terms to get a well poised series, so

(36)
$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+c}{k+a}\binom{n+c}{k+c-a}$$
$$= \binom{n+c}{a}\binom{n+c}{c-a} {}_3F_2\left(\begin{array}{c} -n, -n-c+a, -n-a \\ c-a+1, a+1 \end{array}; -1\right)$$

also satisfies a three term recurrence relation.

   Cusick's other sequence is well poised, but it is most easily treated as a special case of another class of well poised series, *very well poised* series. Whipple is also responsible for this transformation. He showed [15] that

$$_7F_6\left(\begin{array}{c} a, 1+a/2, b, c, d, e, f \\ a/2, a+1-b, a+1-c, a+1-d, a+1-e, a+1-f \end{array}; 1\right)$$
$$= \frac{\Gamma(a+1-d)\Gamma(a+1-e)\Gamma(a+1-f)\Gamma(a+1-d-e-f)}{\Gamma(a+1)\Gamma(a+1-d-e)\Gamma(a+1-d-f)\Gamma(a+1-e-f)}$$
$$\cdot {}_4F_3(a+1-b-c, d, e, f; \quad a+1-b, a+1-c, d+e+f-a; 1)$$

when the ${}_4F_3$ terminates and the ${}_7F_6$ series converges. The ${}_7F_6$ is well poised, but there is an added restriction. The factors

$$\frac{(a/2+1)_n}{(a/2)_n} = \frac{(2n+a)}{a}$$

introduce a linear factor in the series. This can be removed by taking $f = a/2$ or $b = a/2$. When this is done, say with $f = a/2$, the resulting identity is

$$_5F_4\left(\begin{array}{c} a, b, c, d, e \\ a+1-b, a+1-c, a+1-d, a+1-e \end{array}; 1\right)$$
$$= \frac{\Gamma(a+1-d)\Gamma(a+1-e)\Gamma(a/2+1)\Gamma(a/2+1-d-e)}{\Gamma(a+1)\Gamma(a+1-d-e)\Gamma(a/2+1-d)\Gamma(a/2+1-e)}$$
$$\cdot {}_4F_3\left(\begin{array}{c} a+1-b-c, d, e, a/2 \\ a+1-b, a+1-c, d+e-a/2 \end{array}; 1\right).$$

   The ${}_4F_3$ is balanced, but shifting a by one does not lead to a series whose parameters have been changed by one, because of the $a/2$. However take $e = (a+1)/2$ to get

$$_4F_3\left(\begin{array}{c} a, b, c, d \\ a+1-b, a+1-c, a+1-d \end{array}; 1\right)$$
$$= \frac{\Gamma(a+1-d)\Gamma((a+1)/2)\Gamma(a/2+1)\Gamma(1/2-d)}{\Gamma(a+1)\Gamma((a+1)/2-d)\Gamma(a/2+1-d)\Gamma(1/2)}$$
$$\cdot {}_4F_3\left(\begin{array}{c} a+1-b-c, b, a/2, (a+1)/2 \\ a+1-b, a+1-c, d+1/2 \end{array}; 1\right),$$

and now shifting any of the parameters by one leads to a series whose parameters have been shifted by one. Thus we can find the three term recurrence relation satisfied by

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+2a}{k+a} \binom{n+2b}{k+b} \binom{n+2c}{k+c}$$
$$= \binom{n+2a}{a} \binom{n+2b}{b} \binom{n+2c}{c} {}_4F_3\left(\begin{matrix} -n, n-a, -n-b, -n-c \\ a+1, b+1, c+1 \end{matrix} ; 1 \right).$$

This is not the end of three term recurrence relations. Wilson [18] showed that *very well poised two balanced* (sum of top $+2$ equals the sum of bottom) $_9F_8$'s satisfy three-term contiguity relations, where the definition of contiguous requires that both very well poised and two balanced be preserved. There are also basic hypergeometric series extensions of all these results. A series Σa_k is a basic hypergeometric series if a_{k+1}/a_k is a rational function of q^k for a fixed number q. Only one of the balanced $_4\varphi_3$ recurrence relations has been published so far. This is one that comes from a set of orthogonal polynomials [2]. The rest have been worked out by one of us (J.A.W.) and will be published in due course.

We were motivated to write this paper because of a note by Rieger [12]. He found four sequences that satisfy three term recurrence relations and derived the relations. The first two are special cases of the recurrence relation for Jacobi polynomials [13, (4.5.1)]. These are the polynomials defined in (11). The other two are special cases of the recurrence relation for h_n defined in (26).

The main interest in many of these recurrence relations is because of orthogonal polynomials. If $p_n(x)$ is a polynomial of degree n, $\{p_n(x)\}_{n=0}^{\infty}$ is a set of orthogonal polynomials if

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\, d\alpha(x) = 0, \qquad m \neq n,$$

for a positive measure $d\alpha(x)$. Such a polynomial set satisfies a three term recurrence relation

(38) $x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)$

with A_n, B_n, C_n real and $A_n C_{n+1} > 0$, $n = 0, 1, \ldots$. Conversely a set of polynomials that satisfies (38) with the appropriate positivity conditions is a set of orthogonal polynomials. See Szegö [13, Theorem 3.2.1]. At the $_4F_3$ level Wilson [17, 19] has shown that

$$p_n(x^2) = {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; 1 \right)$$

satisfies

$$\int_0^\infty p_n(x^2)p_m(x^2)\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 dx = 0,$$

$m \neq n$, when $a, b, c, d > 0$. He also found their recurrence relation. It was the existence of this relation that led to the complete list given in [18].

There are other applications of these recurrence relations. See [1] for an application of a three term recurrence relation for a $_3F_2$ with $x = 1$, and [3] for an application of a three term recurrence relation for what at the time was thought to be a messy double sum, but which in reality can be transformed into a two balanced very well poised $_9F_8$. See [9].

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