Bull. Aust. Math. Soc. **84** (2011), 481–483
doi:10.1017/S0004972711002504

A RESULT ON MONOTONICALLY LINDELÖF GENERALIZED ORDERED SPACES

AI-JUN XU and WEI-XUE SHI ✉

(Received 26 March 2011)

Abstract

In this paper, we show that the character of any monotonically Lindelöf generalized ordered (GO) space is not greater than ω_1, which gives a negative answer to a question posed by Levy and Matveev ['Some questions on monotone Lindelöfness', *Questions Answers Gen. Topology* **26** (2008), 13–27, Question 51].

2010 *Mathematics subject classification*: primary 54F05; secondary 54D20.
Keywords and phrases: monotonically Lindelöf, generalized ordered spaces.

1. Introduction

A topological space X is *monotonically Lindelöf* if there is an operator r which assigns to every open cover $\mathcal{U}$ of X a countable open cover $r\mathcal{U}$ of X that refines $\mathcal{U}$ such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r\mathcal{U}$ refines $r\mathcal{V}$. In [1, Example 2.2, Corollary 2.4], Bennett *et al.* proved that a monotonically Lindelöf linearly ordered topological space (LOTS) need not be first countable and any monotonically Lindelöf compact LOTS is first countable. So, Levy and Matveev posed the following question.

QUESTION [4, Question 51]. Can the character of a monotonically Lindelöf generalized ordered space be greater than ω_1?

In this paper, we give a negative answer to this question.

For undefined terms and notation we refer to [2, 5, 7].

2. Main results

DEFINITION 2.1 [6]. Let L be a compact LOTS. For $x \in L$, put

$$0\text{-cf}(x) = \min\{|C| : C \text{ is a cofinal subset of } (\leftarrow, x)\}$$

and

$$1\text{-cf}(x) = \min\{|C| : C \text{ is a coinitial subset of } (x, \rightarrow)\};$$

0-cf(x) denotes the left cofinality of x, and 1-cf(x) denotes the right one of x.

This work is supported by NSFC (no:10971092), NJFU (no:163101088).

Note that 0-cf(x) and 1-cf(x) are dual notions, so we only discuss the left cofinality 0-cf(x) in the following. For a compact LOTS L, observe that 0-cf(x) = 0 if x is the left endpoint of L, 0-cf(x) = 1 if x has an immediate predecessor in L, and 0-cf(x) is a regular cardinal otherwise. It is well known that a GO-space X can be embedded as a dense subspace into the compact LOTS $l(X)$ that is called the minimal linearly ordered compactification of X (see [3]). For a GO-space X and $x \in X$, the left cofinality 0-cf(x) means the cofinality defined in its minimal linearly ordered compactification $l(X)$. By [3, Lemma 3.5], if 0-cf(x) $\geq \omega$, then there exists a cofinal increasing sequence $\{x_0(\alpha) \in X \mid \alpha < \text{0-cf}(x)\}$. In addition, x has an immediate predecessor or $[x, \rightarrow)$ is open in X if 0-cf(x) = 1.

The next definition was introduced by Matveev.

Definition 2.2. Let x be a point of a space X. Then X is said to be monotonically Lindelöf at x if there exists an operator r_x that assigns to every nonempty family $\mathcal{F}$ of neighborhoods of x a nonempty countable family $r_x\mathcal{F}$ of neighborhoods of x so that $r_x\mathcal{F}$ refines $\mathcal{F}$ and $r_x\mathcal{F}$ refines $r_x\mathcal{G}$ provided that $\mathcal{F}$ refines $\mathcal{G}$. In this case, r_x is called a monotone Lindelöf operator at the point x of X.

In [1, Example 2.3], Bennett *et al.* proved that $[0, \omega_1]$ considered as a LOTS is not monotonically Lindelöf. With a slight modification of the proof of [1, Example 2.3], we have the following lemma.

Lemma 2.3. *Suppose that $\kappa > \omega_1$ is a regular ordinal and S is a stationary subset of $[0, \kappa)$. Then $S \cup \{\kappa\}$ considered as a LOTS is not monotonically Lindelöf.*

Theorem 2.4. *Suppose that X is a GO-space. If X is monotonically Lindelöf, then both the left and right cofinalities at each point x of X are not greater than ω_1.*

Proof. We only prove the 'left' case; the other case can be proved similarly.

Suppose instead that the left cofinality 0-cf(x) of X at x is greater than ω_1. Take a cofinal increasing sequence $S_0(x) = \{x_0(\alpha) \in X \mid \alpha < \text{0-cf}(x)\}$. Without loss of generality, we may assume that $S_0(x)$ is closed in $(\leftarrow, x)$. Then $S_0(x)$ must be homeomorphic to a subspace H of $[0, \text{0-cf}(x))$. If H is a stationary subset, then $H \cup \{\text{0-cf}(x)\}$ is not monotonically Lindelöf by Lemma 2.3. Hence, $S_0(x) \cup \{x\}$ is not monotonically Lindelöf. However, this contradicts that $S_0(x) \cup \{x\}$ is a closed subspace of the monotonically Lindelöf space X. Therefore H is not stationary in $[0, \text{0-cf}(x))$. It follows that there exists a closed cofinal subset C of $[0, \text{0-cf}(x))$ such that $C \cap H = \emptyset$. Thus we conclude that H can be presented as a union of 0-cf(x) many pairwise disjoint open convex subsets of H and so can $S_0(x)$ in $(\leftarrow, x)$. Hence we may write

$$S_0(x) = \bigcup \{T_\alpha \mid \alpha < \text{0-cf}(x)\}$$

where T_α is convex and open, and if $x' \in T_\alpha$, $x'' \in T_\gamma$ for $\alpha < \gamma$, then $x' < x''$. Since 0-cf(x) $> \omega_1$, the set $\mathcal{U} = \{T_\alpha \mid \alpha < \omega_1\} \cup \{y \in S_0(x) \cup \{x\} \mid y \geq a \text{ for every } a \in T_{\omega_1}\}$ is an open cover of $S_0(x) \cup \{x\}$ that has no countable subcover. Thus $S_0(x) \cup \{x\}$ is not Lindelöf. We obtain a contradiction. □

https://doi.org/10.1017/S0004972711002504 Published online by Cambridge University Press

THEOREM 2.5. *The character of any monotonically Lindelöf GO-space* (X, τ) *is not greater than* ω_1.

PROOF. Suppose that (X, τ) is a monotonically Lindelöf GO-space. Then, for every $x \in X$, neither the left nor the right cofinalities of x is greater than ω_1 by Theorem 2.4. We only consider the case that both the left and the right cofinalities of x are ω_1. The other cases are easy. Let $\{a_\gamma \mid \gamma < \omega_1\}$ be a cofinal increasing sequence of $(\leftarrow, x)$ and $\{b_\gamma \mid \gamma < \omega_1\}$ a coinitial decreasing sequence of $(x, \rightarrow)$. Put

$$\mathcal{B}(x) = \{(a_\gamma, b_\gamma) \mid \gamma < \omega_1\}.$$

Then $\mathcal{B}(x)$ is a base for (X, τ) at the point x and $|\mathcal{B}(x)| \leq \omega_1$. Hence $\chi(x, (X, \tau)) \leq \omega_1$. In view of the arbitrariness of x, the character of GO-space (X, τ) is not greater than ω_1. □

Acknowledgement

The authors would like to express their thanks to the referee for valuable suggestions and comments that improved this paper.

References

[1] H. Bennett, D. Lutzer and M. Matveev, ‘The monotone Lindelöf property and separability in ordered spaces’, *Topology Appl.* **151** (2005), 180–186.
[2] R. Engleking, *General Topology*, revised and completed edition, Sigma Series in Pure Mathematics, 6 (Hedermann, Berlin, 1989).
[3] N. Kemoto, ‘Normality of products of GO-spaces and cardinals’, *Topology Proc.* **18** (1993), 133–142.
[4] R. Levy and M. Matveev, ‘Some questions on monotone Lindelöfness’, *Questions Answers Gen. Topology* **26** (2008), 13–27.
[5] D. Lutzer, ‘Ordered topological spaces’, in: *Surveys in General Topology* (ed. G. M. Reed) (Academic Press, New York, 1980), pp. 247–296.
[6] A.-J. Xu and W.-X. Shi, ‘Notes on monotone Lindelöf property’, *Czechoslovak Math. J.* **59**(134) (2009), 943–955.
[7] A.-J. Xu and W.-X. Shi, ‘Monotone Lindelöf property and linearly ordered extensions’, *Bull. Aust. Math. Soc.* **81** (2010), 418–424.

AI-JUN XU, School of Mathematical Sciences, Nanjing Normal University,
Nanjing 210046, PR China
e-mail: ajxu@njfu.edu.cn

WEI-XUE SHI, Department of Mathematics, Nanjing University,
Nanjing 210093, PR China
e-mail: wxshi@nju.edu.cn