the rectangle under the segments (internal or external) of AB is to be equal.

Case 1. Bisect AB in D; draw DE perpendicular to AB and equal to C. Produce ED to F so that EF be equal to AD or DB; with centre E and radius EF describe a circle cutting AB in G.

G is the required point of division.

Case 2. Bisect AB in D; draw BE perpendicular to AB and equal to C. Join DE; with centre D and radius DE describe a circle cutting AB produced in G.

G is the required point of division.

Kötter's synthetic geometry of algebraic curves.

By Norman Fraser, B.D.

In the following paper I propose to give a short account of Dr Ernst Kötter's purely geometrical theory of the algebraic plane curves. This theory is developed in a treatise* which, in 1886, gained the prize of the Berlin Royal Academy; but the contents of my paper are also partly drawn from a course of lectures delivered by Dr Kötter in the University of Berlin, W.S. 1887-88.

The method followed is that of involutions, and we might pass at once to a discussion of these; but it may be of interest to call attention, first of all, to one of the most remarkable features of the theory—Dr Kötter's treatment of imaginary points and lines. He undertakes to show that the projective relationship between two aggregates may be extended so as to embrace their imaginary elements, and, hence, that imaginary elements may take their place with real elements in the theory of involutions. One method of establishing this result has been given by von Staudt;† but it travels beyond the limits of Plane Geometry. Many writers on the subject simply assume the point at issue; a course which has its advantages, and may be recommended to any reader who should chance to find

* *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven.* (*Transactions of the Royal Academy of Science.* Berlin, 1887. Also published separately in the same year.)

† *Beiträge zur Geometrie der Lage.* Nürnberg, 1856-60.

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the following condensed and necessarily imperfect account of Dr Kötter's method unintelligible.

Imaginary Elements.

An elliptic involution has, we know, no real double elements. Imaginary points in a line are defined as the double points of the various elliptic involutions which may be formed of the real points of the line. Each involution thus yields a pair, an imaginary point and its conjugate, which, like the involution itself, are completely determined when two real pairs of the involution, say AA′ and BB′, are given. These pairs will divide one another, B and B′ lying the one in the finite, and the other in the infinite line AA′. We denote the two imaginary points which they determine by writing them down in order, as ABA′B′. It remains to distinguish between the two imaginary points thus denoted, and, in doing so, we follow von Staudt. To prepare the way for this distinction let us glance at the case of a hyperbolic involution. Here, also, we may determine the two real double points by two pairs of the involution, AA′ and BB′. Now, of these double points, one lies to the *right*, and the other to the *left* of the centre of the involution, and, to give effect to this distinction, we may indicate the former by naming our four points *from left to right*, say ABB′A′, and the latter by naming them from *right to left*, AA′B′B, or A′B′BA. This is what we agree to do in the case of the elliptic involution; we assign to each imaginary double point a particular direction, which is indicated by the order in which we name the four points which determine the involution; and if we denote the one by ABA′B′, we denote the other by AB′A′B, or, which is the same thing, by B′A′BA. Similarly, if CC′, DD′ be any other pairs of the same involution, out imaginary points may equally well be denoted by CDC′D′ and CD′C′D.

In exactly the same way imaginary lines, through a real point, are defined as the double rays of an elliptic involution in a pencil, and are denoted by *aba′b′*, *ab′a′b*, respectively.

An imaginary line is said to contain an imaginary point, if it be possible to represent the one by *aba′b′* and the other by ABA′B′, where *a*, *b*, *a′* and *b′* are the lines joining A, B, A′ and B′ to O, the real point of the imaginary line. (In this case *ab′a′b* will evidently contain AB′A′B.)

To find the imaginary line joining two imaginary points which lie in two real lines, we find two representations of them, ABA′B′ and CDC′D′, which are in perspective; that is, such that CA, DB, C′A′, and D′B′ meet in one point. This point will be the real point of the imaginary line required. It can be proved that only one such point can be found, so that the problem is a completely determinate one. Similarly, two imaginary lines intersect in a perfectly definite imaginary point.

From any imaginary point outside a real line we may project its imaginary points—of which there is a twofold infinity—into the real points of the plane. We are thus able to represent the points, real and imaginary, of a line by the real points of the plane: the imaginary points being represented by points outside of the line, while the real points of the line represent themselves. Again, the points of another line may be represented by a different arrangement of the points of the plane, or, which is the same thing, by the points of a superimposed plane; while the real and imaginary rays of a pencil may be represented by the real lines of a plane. We are now able to investigate the nature of the projective relationship between imaginary elements in two aggregates by observing the real points and lines which represent these imaginary elements. In the case of imaginary, as in that of real elements, Dr Kötter defines the projective relationship as the result of a series of perspective relationships: that is, two aggregates are projective when they can be connected by a series of aggregates, of which any consecutive pair are in perspective. Accordingly our task reduces itself to that of observing the relation subsisting between the points and lines of two planes, when the aggregates of imaginary elements, which they represent, are in perspective.

This relation is somewhat complicated, and is only loosely described when we say, that when a point or line in the one plane describes or envelopes a conic, the corresponding point or line in the other plane describes or envelopes a corresponding conic; but it establishes conclusively that in the case of imaginary, as in that of real elements, the projective relationship is *continuous, unambiguous*, and *determined by three correspondences.*

We are thus able to carry over to imaginary elements all the known projective properties of real elements. Hereafter, when we speak of ranges or pencils, these are to be understood as containing both real and imaginary elements.

Involutions of Various Degrees.

Involutions of the Second Degree.

If, in two projective aggregates on the same base, every element has the same correspondent, whether considered as belonging to the first or the second aggregate, then each element makes, with its correspondent, a pair of what is termed an involution of the second degree. For example, if

$$A_1B_1C_1A_2B_2C_2 \ \ldots \ \barwedge A_2B_2C_2A_1B_1C_1 \ \ldots$$

then A_1A_2, B_1B_2, C_1C_2 are pairs of an involution. It is easy to prove that, if the required property holds for any one element, say C_1, it will hold for all; that is, that if we can prove

$$A_1B_1C_1C_2 \ \ldots \ \barwedge A_2B_2C_2C_1 \ \ldots$$

then A_1A_2, B_1B_2, C_1C_2 will be an involution as above.

A second degree involution may be generated in the following way:—Let A_1A_2, B_1B_2, C_1C_2 be an involution. Then

$$\begin{aligned} A_1B_1C_1C_2 &\barwedge A_2B_2C_2C_1 \ \ldots \\ &\barwedge B_2A_2C_1C_2 \ \ldots \end{aligned}$$

that is, C_1 and C_2 are double elements of a duplex aggregate

$$A_1B_1X \ \ldots \barwedge B_2A_2Y \ \ldots$$

Again, if E_1E_2 are another pair of the involution,

$$\begin{aligned} A_1B_1E_1E_2 &\barwedge A_2B_2E_2E_1 \ \ldots \\ &\barwedge B_2A_2E_1E_2 \ \ldots \end{aligned}$$

that is E_1 and E_2 are double elements of another duplex aggregate

$$A_1B_1X' \ \ldots \barwedge B_2A_2Y' \ \ldots$$

Similarly any other pair of our involution can be generated as double elements of a duplex aggregate of the form

$$A_1B_1 \ \ldots \barwedge B_2A_2 \ \ldots$$

So we may generate our involution by taking

$$A_1B_1M \ \ldots \barwedge B_2A_2N \ \ldots$$

where M is fixed while N varies, and by taking the double elements of the various duplex aggregates so formed. (To give the pair

A_1A_2, N must be A_2; and to give B_1B_2, N must be B_2.) Hence corresponding to successive pairs

$$\begin{array}{lcccccc} & A_1A_2 & B_1B_2 & C_1C_2 & D_1D_2 & E_1E_2 & \ldots \\ \text{we have a series of values of N :—} & A_2 & B_2 & N_1 & N_2 & N_3 & \ldots \\ \text{Similarly, if instead of } M \text{ we take } M'\text{, we get for N :—} & A_2 & B_2 & N_1' & N_2' & N_3' & \ldots \\ \text{and if we take } M'' \text{ we get :—} & A_2 & B_2 & N_1'' & N_2'' & N_3'' & \ldots \end{array}$$

All these aggregates are called "characteristic" ranges or pencils of the involution. It can be proved that they are all projective to one another for a given involution, and we are thus entitled to call the involution itself projective to any of them, and write

$$\begin{array}{cccccccc} & A_1A_2, & B_1B_2, & C_1C_2, & D_1D_2, & E_1E_2 & \ldots \\ \barwedge & A_2 & B_2 & N_1 & N_2 & N_3 & \ldots \end{array}$$

Further, if two involutions have projective characteristic ranges, they are said to be projective to one another.

Involutions of the Third Degree.

Suppose we take an involution of the second degree,

$$\begin{array}{cccc} A_1A_2, & B_1B_2, & M_1M_2 & \ldots \end{array}$$

and set it projective to a range on the same base $B_3 \quad A \quad N \quad \ldots$ (where N is variable); then points which are common to the involution and range for a given position of N, make up a group of what is termed an involution of the third degree. If, when N is N_1, our involution takes the form A_1A_2, B_1B_2, M_1M_2, P_1P_2, Q_1Q_2, R_1R_2 ... and our range takes the form $B_3 \quad A_3 \quad N_1 \quad P_2 \quad Q_2 \quad R_2$ ... then P_2 Q_2 and R_2 are members of the particular group of the third degree involution corresponding to N_1. It can be proved that a second degree involution and a projective range have always three points common; thus our third degree involution is made up of groups of *three* elements. As before, the range described by N, while M_1M_2 are fixed, is called the characteristic range (or pencil, when we are dealing with pencils), and we have

$$\begin{array}{cccccc} & A_1A_2A_3, & B_1B_2B_3, & C_1C_2C_3, & D_1D_2D_3 & \ldots \\ \barwedge & A_3 & B_3 & N_1 & N_2 & \ldots \\ \barwedge & A_3 & B_3 & N_1' & N_2' & \ldots \text{ \&c.} \end{array}$$

The A's and B's have no properties not shared by other groups of the involution: if $K_1K_2K_3$, $L_1L_2L_3$ are any two groups, the involution may be generated from K_1K_2, L_1L_2 ... $\barwedge$ L_3 K_3 ...

If two groups contain the same element it will be common to all groups.

If an involution should contain two threefold elements (that is, two groups of the form DDD, D′D′D′), all other groups will be regular. If it should contain one threefold element, there will be two double elements (two groups of the form PPP_1, QQQ_1); if no threefold element, there will be four double elements.

Involutions of the n^{th} Degree.

We pass now to involutions of the n^{th} degree. If we take two projective involutions of the $\overline{n-m}$ and m^{th} degree respectively, viz. :—

$$\begin{array}{llllll} & A_1A_2A_3 & \ldots & A_{n-m}, & B_1B_2B_3 & \ldots B_{n-m}, \ldots \\ \barwedge & B_{n-m+1} & \ldots & B_n\ \ , & A_{n-m+1} & \ldots A_n\ \ , \ldots \end{array}$$

and make a variable group of the latter correspond to a fixed group of the former, we shall obtain a series of groups of n elements common to the two involutions, which will together make up what is termed an involution of the n^{th} degree. This involution will be projective to the m^{th} degree involution traced out by the variable group.

The involution is determined by any two groups. A given element of the base belongs to one group of the involution or else to all; in which last case the involution may be broken up into that element and an involution of the $\overline{n-1}^{st}$ degree.

If an involution of the n^{th} degree contains two n^{fold} elements, all other groups are regular; if one n^{fold} element, at most $n-1$ groups will contain double elements. (We make use of this in the theory of polars.)

Rank in Involutions.

Before introducing a new distinction, that of "rank," we must define "involution-pencils" and "nets." Suppose we have an involution of the n^{th} degree $V_1, V_2, V_3 \ldots$ and a fixed group U which does not belong to it, then the involution U, V will, while V traces out the involution $V_1, V_2, V_3 \ldots$, trace out an "involution-pencil" with centre U. We denote this pencil by $U(V_1V_2V_3 \ldots)$, and we say that it is in perspective with the involution $V_1, V_2, V_3 \ldots$. All involution-pencils which are in perspective with the same or projective involutions are said to be projective to one another.

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Any three groups U V W which do not belong to the same involution determine an "involution-net" of the second dimension, which is made up of all the groups of all the involutions of the pencil U(V,W ...). This net contains every involution which is determined by any two of its groups. The net of the second dimension is strictly analogous to a plane, an involution corresponding to a line, and a group to a point. Any four groups U V W X which do not belong to the same net of the second dimension, determine a net of the third dimension, which is made up of all the groups of all the involutions determined by U and each of the groups of the second dimension net VWX. It corresponds to space of three dimensions. Generally, any $\mu+1$ groups (of the same degree, of course) which do not belong to the same net of the $\overline{p-1}^{st}$ dimension, determine a net of the μ^{th} dimension. It may be defined as follows: Let U,V,W be an involution, U being a fixed group; let V describe a net of the $\overline{u-1}^{st}$ dimension, then W will describe a net of the μ^{th} dimension. This net contains entirely all nets of lower dimensions determined by any of its groups. All contained nets of the $\overline{\mu-1}^{st}$ dimensions which have in common a given net of the $\overline{\mu-2}^{nd}$ dimension make up what is termed a net-pencil.

Involutions of the Second Rank.

Now let us take in the same net of the second dimension two projective, but not perspective involution-pencils with centres at U and V, and let corresponding involutions of the pencils have in common the groups W_1, W_2, W_3 ... ; then $U(W_1W_2W_3 ...)$ $\barwedge$ $V(W_1W_2W_3 ...)$ generate what is termed an involution of the second rank. Evidently such an involution (to be distinguished from all involutions hitherto discussed, which were of first rank) corresponds exactly to a conic in plane geometry; it will contain U and V, and be determined by any five groups. Moreover, it has two groups in common with any first rank involution in the same second dimension net.

Involutions of the Third Rank.

Again, if, in a net of the third dimension, we take three projective net-pencils whose axes are the involutions U_1, U_2; U_2, U_3; U_3, U_4; viz.,

$$U_1U_2(U_4U_5U_6 \ldots U)$$
$$\barwedge U_2U_3(U_4U_5U_6 \ldots U)$$
$$\barwedge U_3U_1(U_4U_5U_6 \ldots U);$$

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then pencils will intersect in a series of groups $U_4\ U_5\ U_6\ \ldots$ which make up what is termed an involution of the third rank. As we see, it is determined by six groups. An involution of the third rank has three groups in common with any second dimension net contained in the same third dimension net.

Involutions of the μ^{th} Rank.

Generally, an involution of the μ^{th} rank is determined by $\mu+3$ groups, which all lie in the same net of the μ^{th} dimension, and is generated by the intersection of corresponding nets of μ projective $\overline{\mu-1}^{st}$ dimension net-pencils :—

$$
\begin{array}{l}
U_2U_3\ \ldots\ U_\mu\quad(U_{\mu+1,}\ U_{\mu+2,}\ U_{\mu+3}\ \ldots\ U)\\
\barwedge\ U_3U_4\ \ldots\ U_\mu U_1(U_{\mu+1,}\ U_{\mu+2,}\ U_{\mu+3}\ \ldots\ U)\\
\barwedge\ U_4U_5\ \ldots\ U_1\,U_2(U_{\mu+1,}\ U_{\mu+2,}\ U_{\mu+3}\ \ldots\ U)\\
\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
\barwedge\ U_1U_2\ \ldots\ U_{\mu-1}\ (U_{\mu+1,}\ U_{\mu+2,}\ U_{\mu+3}\ \ldots\ U)
\end{array}
$$

The rank of an involution cannot be higher than its degree. We shall call an involution of the m^{th} degree and μ^{th} rank an (m,μ) involution. A simple range or pencil may be looked upon as a $(1, 1)$ involution.

An (m,μ) involution has with a projective (n,ν) involution on the same base $m\nu+n\mu$ elements in common.

Theory of Curves.

Conics.

We begin with the conic. A conic is generated by the intersection of corresponding rays of two projective pencils with centres at real or imaginary points; that is, if we have two pencils $P(R\ S\ T\ \ldots)$ $\barwedge\ Q(R\ S\ T\ \ldots)$ where R S and T are not in the same straight line, then corresponding rays will meet in a conic. A conic is thus seen to be determined by five points, and the same conic is generated whatever points upon it be taken as the centres of the generating pencils. The tangent at P is the ray in the pencil P which corresponds to QP in Q; similiarly the tangent at Q is the ray in Q which corresponds to PQ in P.

Conic-Pencils.

Two conics meet in four points, through which an infinite number of conics may be drawn, constituting what is termed a conic-pencil. This, however, is otherwise defined. Let two conics, K_1^2

and K_2^2, have a common point at P.* Let Q be a point on K_1^2 and R a point on K_2^2. Then K_1^2 may be generated by pencils at P and Q :— $p_1p_2p_3 \ldots \barwedge q_1q_2q_3 \ldots$
and K_2^2 by pencils at P and R :— $p_1p_2p_3 \ldots \barwedge r_1r_2r_3 \ldots$
Then $q_1q_2q_3 \ldots \barwedge r_1r_2r_3 \ldots$ will generate another conic K^2, which will pass through Q and R; also through all the intersections of K_1^2 and K_2^2 except P, since at each of these intersections corresponding rays p_n and q_n, p_n and r_n and therefore q_n and r_n meet. Now take on K^2 a series of points S T, &c.; then K^2 may be generated by any two of the pencils

$$\begin{array}{ll} q_1q_2q_3 \ldots & (\text{where } q_1r_1s_1t_1 \ldots \\ \barwedge r_1r_2r_3 \ldots & \quad q_2r_2s_2t_2 \ldots \\ \barwedge s_1s_2s_3 \ldots & \quad q_3 \ldots \quad \ldots \\ \barwedge t_1t_2t_3 \ldots & \text{are evidently also projective pencils).} \\ \ldots \quad \ldots \quad \ldots & \end{array}$$

Now the pencil $p_1p_2p_3 \ldots$ will generate along with any one of these pencils a conic which contains P and all the other intersections of K_1^2 and K_2^2; and all such conics together constitute a "conic-pencil." We call the points common to the conics the "ground-points" of the pencil. We say that the conic-pencil is projective to the ray-pencil made up of the tangents at any of the ground-points to the various conics of the pencil. It is also projective to any of the ray-pencils $q_1r_1s_1t_1 \ldots$
$q_2r_2s_2t_2 \ldots$ &c.

Any straight line meets a conic-pencil in an involution of the second degree. For, take any straight line l; it will meet the pencil in points formed by taking points common to the range

$$\underline{l(p_1p_2p_3 \ldots)}$$

and the various projective ranges

$$\begin{array}{l} l(q_1q_2q_3 \ldots) \\ l(r_1r_2r_3 \ldots) \\ l(s_1s_2s_3 \ldots) \\ \ldots \quad \ldots \quad \ldots \end{array}$$

The p's and q's will give a pair Q_1Q_2; the p's and r's R_1R_2, and so on; and from the theory of involutions it follows that Q_1Q_2 R_1R S_1S_2 T_1T_2 are pairs of an involution of the second degree, of which each of the ranges $l(q_1r_1s_1 \ldots)$, &c., is a characteristic range. l thus meets the conic-pencil in a projective involution.

* The diagram, which is very simple, may be supplied by the reader.

Similarly any conic will meet the conic-pencil in a projective involution of the fourth degree.

Through a given point (not one of the ground-points) one and only one conic of the pencil passes. For let P be the point: any line l through P will meet the pencil in a second degree involution, in which there will be one point P^1 conjugate to P. Then as l revolves round P, P^1 will describe the conic of the pencil which passes through P.

We may define "conic-nets" just as in the case of involutions. Any three conics which do not belong to the same pencil determine a net of the second dimension. For example, all conics which pass through three given points make up a conic-net of the second dimension. Similiarly, all which pass through two points make up a net of the third dimension, and so on; though it does not follow that all the conics of nets of the second or third dimension pass through three or two points respectively.

Generation of Conics by Involutions.

So far we have looked on conics as generated by two pencils, or (for we may so express it) by a pencil and a (1, 1) ray-involution. They may also be generated by the aid of involutions of higher degree and rank. If we cut a conic by the rays of a pencil from a point P which does not lie upon the conic, the intersections of each ray will be projected from a point Q on the conic into a pair of a ray-involution of the second degree with centre Q. If Q be not on the conic, the involution will be of the second degree and second rank. In either case the involution at Q will be projective to the pencil at P. So we can generate a conic by means of a simple pencil and a projective (2, 1) or (2, 2) involution. From either of these modes of generation we can ascertain the number of points of intersection of two conics. Take the former. Let K_1^2 be generated in the ordinary way by two pencils at R and S; K_2^2 may be generated by the pencil R and a (2, 1) involution at one of its points Q; also by the pencil S and another (2, 1) involution at Q projective to the first. Evidently any ray QX common to these two involutions will meet the corresponding rays of R and S at a point X which is common to both conics. But two projective (2, 1) involutions have four common elements; hence two conics intersect in four points, real or imaginary.

Cubics.

Suppose we have a fixed point P and through it a pencil $p_1p_2p_3$ ... ; further, four points $P_1P_2P_3P_4$ and through them a projective conic-pencil L_1^2, L_2^2, L_3^2 ... then the intersections of corresponding members of $p_1\ p_2\ p_3\ \ldots \barwedge L_1^2, L_2^2, L_3^2 \ldots$ generate a "cubic" or curve of the third degree. Any line l will meet this curve in points common to the range and involution

$l(p_1p_2p_3 \ldots) \barwedge l(L_1^2, L_2^2, L_3^2 \ldots)$ that is in a group of three points.

P is evidently a point on the curve. Let L^2 be the conic of the pencil which passes through P, then the corresponding ray p will be the tangent at P to the cubic.

Cubic-Pencils.

Suppose we take two cubics K_1^3 and K_2^3. Let P be any common point, and let $p_1p_2p_3$ ... be a pencil at P.

Then K_1^3 may be generated by $p_1p_2p_3 \ldots \barwedge L_1^2,\ L_2^2,\ L_3^2 \ldots$
and K_2^3 may be generated by $p_1p_2p_3 \ldots \barwedge M_1^2, M_2^2, M_3^2 \ldots$
These two conic-pencils determine an "array" (Schaar) of conics made up of projective pencils

$$\begin{array}{lll} L_1^2, & L_2^2, & L_3^2 \ldots \\ M_1^2, & M_2^2, & M_3^2 \ldots \\ N_1^2, & N_2^2, & N_3^2 \ldots \\ Q_1^2, & Q_2^2, & Q_3^2 \ldots \\ \ldots & \ldots & \ldots \end{array}$$

where $L_1^2, M_1^2, N_1^2, Q_1^2$ ... belong to a pencil which is projective to $L_2^2, M_2^2, N_2^2, Q_2^2$ and so on.

Then the cubic generated by $p_1p_2p_3$ ... and the various pencils of our array constitute what is termed a "cubic-pencil." Any line will meet this pencil in an involution of the third degree. Any point in the plane determines one and only one cubic of the pencil.

Cubics may also be generated by a pencil and a (3, 3) ray-involution.

Curves of the Fourth and Higher Degrees.

A curve of the fourth degree may be generated by a pencil and a projective cubic-pencil ; also by two projective conic-pencils.

And generally, a curve of the n^{th} degree may be generated by two projective curve-pencils of the $\overline{n-m}^{th}$ and m^{th} degree respectively.

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That is, K^n is generated by $K_1^{n-m}, K_2^{n-m}, K_3^{n-m} \ldots$
$\barwedge K_1^m, K_2^m, K_3^m \ldots$

It will meet any line in the plane in general in n points.

Generation of Curves by Involutions.

A curve of the n^{th} degree may also be generated by the aid of involutions. Suppose we have a pencil P and an (m, μ) ray-involution at a point Q. Then, if we set

$$p_1 p_2 p_3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots$$

the intersections of each ray p_r with the corresponding group of rays q_m^r will develope a curve of the m^{th} degree. It can be shown, however, that the locus developed in this way includes besides our required curve the μ^{fold} line PQ (compare the familiar fact that two perspective pencils develope not only the base of perspective, but also the line joining the two centres). This has to be subtracted in order to give us our curve K^m. Thus, to find the number of intersections of any line with our curve. Let l be any line in the plane. It may be generated by $p_1 p_2 p_3 \ldots$ and a perspective pencil $q_1^1 q_1^2 q_1^3 \ldots$ at Q. Then all intersections of l with K^m will be given by rays common to

$$q_1^1 q_1^2 q_1^3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots$$

But these are (1, 1) and (m, μ) involutions; hence they have $m + \mu$ common rays. Of these, however, μ coincide in PQ, and we are left with the result that l meets K^m in m points.

It can be shown that Q is an $\overline{m - \mu}^{fold}$ point on K^m: if $\mu = m$, Q does not lie on the curve.

Points common to two Curves.

To find the number of points common to K^m and K^n. Let K^m be generated by $p_1 p_2 p_3 \ldots \barwedge q_m^1 q_m^2 q_m^3 \ldots \quad (m, \mu)$
and K^n by $p_1 p_2 p_3 \ldots \barwedge q_n^1 q_n^2 q_n^3 \ldots \quad (n, \nu)$
then points common to K^m and K^n are projected into rays common to $q_m^1 q_m^2 q_m^3 \ldots \barwedge q_n^1 q_n^2 q_n^3$. Of these there are $m\nu + n\mu$; but $\mu\nu$ of them are accounted for by the coincidence of the μ^{fold} ray QP with the ν^{fold} ray QP. Hence K^m and K^n have in common, outside of Q, $m\nu + n\mu - \mu\nu$ points. In the general case when Q does not lie on either curve, $m = \mu$ and $n = \nu$; thus in general two curves of the m^{th} and n^{th} degree have mn points in common. They cannot have more unless K^m and K^n have in common some curve of a lower degree, and take the form $K^r K^{m-r}$ and $K^r K^{n-r}$

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Propositions on Common Points.

If, of the p^2 intersections of two curves, K_1^p and K_2^p, pm lie on K^m, the remaining p^2-pm shall lie on some curve K^{p-m}. Take the pencil K_1^p, K_2^p and find the curve K_3^p in it which holds an additional point Q of K^m. Then K_3^p and K^m have $pm+1$ points in common (namely, pm of the ground-points of the pencil, and Q), and, therefore, must have in common some curve of lower degree. Let K_3^p be of the form K^rK^{p-r} and K^m of the form K^rK^{m-r}. (If $r=m$ our proposition is proved). Again, K^rK^{p-r} has with K_1^p p^2 common points, of which pm lie in K^rK^{m-r}. Therefore K^{p-r} has with K_1^p p^2-pr common points, of which $pm-pr$ lie on K^{m-r}. That is, K^{p-r} and K^{m-r} have p^2-pr instead of $(p-r)(m-r)$ points in common, and hence must be of the form K^sK^{p-r-s} and K^sK^{m-r-s}. And so we can proceed until $r+s+\ \dots\ =m$, proving that K_3^p is of the form K^mK^{p-m}. That is, the p^2-pm ground-points of the pencil which do not lie on K^m lie on a curve K^{p-m}. Q.E.D.

We can prove the more general proposition: if K^m and K^n have mn points in common, of which pn lie on K^p, the remaining $(m-p)n$ shall lie on some curve K^{m-p}.

All curves of the p^{th} degree which contain $3p-1$ fixed points belonging to a cubic, shall also contain another point dependent on these. For, let K_1^p and K_2^p have in common $(3p-1)$ points of K^3. K_1^p and K_2^p will each cut K^3 in one additional point: let these be P and Q. We wish to prove that P and Q are identical.

Through P take any line a, cutting K^3 in P_1 and P_2, then K_2^pa (a curve of the $\overline{p+1}^{st}$ degree) cuts K^3 in $3p+3$ points: $3p$ of these (viz., the $3p-1$ fixed points and P) lie on K_1^p; therefore, by the proposition enunciated above, the remaining three, Q P_1 and P_2, lie in a straight line. But this can only be the line a, and as a meets K^3 in three points only, P and Q must be identical. Hence every curve of the p^{th} degree, which passes through the $3p-1$ points, passes through P. Q.E.D.

When $p=3$ we get the following proposition:—Of the nine ground-points of a cubic-pencil only eight are independent, and every cubic which contains these must contain the ninth also. It follows from this that nine independent points are necessary and sufficient to determine a cubic. The general proposition, of which that proved above is a particular case, runs: all curves of the

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p^{th} degree which contain $pn - \frac{1}{2}(n-1)(n-2)$ points of a curve of the n^{th} degree, shall contain in addition $\frac{1}{2}(n-1)(n-2)$ points dependent on these. When $p = n$ we get the proposition: of the n^2 ground-points of a pencil of the n^{th} degree only $n^2 - \frac{1}{2}(n-1)(n-2)$, that is, $\frac{1}{2}(n^2 + 3n - 2)$ are independent. Hence a curve of the n^{th} degree is determined by $\frac{1}{2}n(n+3)$ independent points.

Multiple Points.

A li e through an m^{fold} point on a curve can meet the curve at $n - m$ additional points at most.

We shall look first at double points. Let K^n have a double point at D. Suppose K^n generated by

$$d_1 d_2 d_3, \ldots \barwedge K_1^{n-1}, K_2^{n-1}, K_3^{n-1} \ldots$$

d_1 is to meet K^n, and, therefore, K_1^{n-1} at only $n-2$ points outside of D: $\therefore$ K_1^{n-1} must pass through D. Similiarly for K_2^{n-1}, &c. So, in order that D shall be a double point, K^n must be capable of being generated by a ray-pencil and a curve-pencil both passing through D.

Let $c_1 c_2 c_3 \ldots$ be the tangents to the K^{n-1}'s at D; then tangents to K^n at D will be the double rays of $c_1 c_2 c_3 \ldots \barwedge d_1 d_2 d_3 \ldots$ Let these be t and t^1; if they are real and distinct, D is a *node* on K^n; if real and coincident, D is a *cusp*; if imaginary, D is a *conjugate point.*

In order to be a triple point of K^n, D must be a double point of K_1^{n-1}, K_2^{n-1}, &c.; and, generally, in order to be an m^{fold} point of K^n, it must be an $\overline{m-1}^{fold}$ point of the $\overline{n-1}^{st}$ degree pencil.

Again, multiple points may, as we saw, be discussed in connection with the other mode of generating curves: in order that D should be an m^{fold} point, the intersections of any pencil $p_1 p_2 p_3 \ldots$ with K^n must determine an $(n, n-m)$ involution at D.

Polar Curves.

If from a point P a line l be drawn to meet a curve K^n, then the locus of the double elements of the involution $(P)^n$, $l(K^n)$, as l revolves round P, is called the "first polar" of P with respect to K^n. That is, we take the involution of the n^{th} degree determined by two groups, of which one is the n^{fold} element P, and the other the intersections of l with K^n; we find all the groups which take the form $X^2 X_1 X_2 \ldots X_{n-2}$, and the locus of X as l revolves is our required polar curve. There are in general $n-1$ such double points in such an involution, and the curve generated is of the $\overline{n-1}^{st}$ degree. We shall denote it by P^{n-1}. When l touches K^n at a point Q, then $l(K^n)$ itself takes the form $Q^2Q_1Q_2 \dots Q^n$. Hence Q lies on the polar; or, the first polar of P passes through all the points of contact of tangents from P to K^n.

The polar of P with respect to P^{n-1}, is called the second polar with respect to K^n, and is denoted by P^{n-2}; finally, P^1 is called the polar line of P. If P lies on K^n it will also lie on P^{n-1}, P^{n-2}, &c., and P^1 will be the tangent at P to K^n and its successive polars.

The polars of the points of a range with respect to a curve will constitute a curve-pencil projective to the range.

The polars of a point with respect to the curves of a pencil will constitute a projective curve-pencil.

If through P we take n lines, $a_1a_2a_3 \dots a_n$, and let these along with K^n determine an n^{th} degree pencil, P shall have the same polars with respect to all the curves of the pencil. For, through P take a line l; it will cut the pencil $a_1a_2 \dots a_n$, K^n, $K_1{}^n$, $K_2{}^n$ ... in an involution $(P)^n$, $l(K^n)$, $l(K_1{}^n)$, $l(K_2{}^n)$ .. Accordingly, all the involutions $(P)^n$, $l(K^n)$; $(P)^n$, $l(K_1{}^n)$; $(P)^n$, $l(K_2{}^n)$, &c., are one and the ame involution, and hence the polars, which are defined by the aids of these involutions, are also identical. Q.E.D.

If the first polar of P contains a point Q, the polar line of Q shall contain P. For, take through P n lines $a_1a_2 \dots a_n$. In the pencil $a_1a_2 \dots a_n$, K^n take the particular curve $K_0{}^n$ which contains Q. Then the polar line $Q_0{}^1$ will be the tangent at Q to $K_0{}^n$. Again, since all polars of P are identical, P^{n-1} is the first polar of $K_0{}^n$; and, therefore, its intersections with $K_0{}^n$ give points of contact of tangents from P. But Q lies in both P^{n-1} and $K_0{}^n$. $\therefore$ PQ is the tangent to $K_0{}^n$ at Q, that is, $Q_0{}^1$ is PQ. Again, the polar of Q, with respect to $a_1a_2 \dots a_n$, is easily seen to be a curve made up of double rays of the ray-involution $(PQ)^n$, $a_1a_2 \dots a^n$; that is, it consists of $n-1$ rays through P. Similarly for successive polars, until the polar line of Q, with respect to $a_1a_2 \dots a_n$, is seen to be a line through P. Now we have proved that the polar line of Q, with respect to *two* curves of the n^{th} degree pencil, is a line through P; therefore this is true for all curves of the pencil, that is, Q^1 contains P. Q.E.D.

The extended form of this proposition is: if P^{n-m} hold Q, Q^m shall hold P.

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Plücker's First Equation.

A curve and its first polar meet in general in $n(n-1)$ points; hence $n(n-1)$ tangents can in general be drawn from a given point to the curve. But in each double point of K^n two of the points of intersection of K^n and P^{n-1} coincide; and, moreover, this point is no longer a point of contact in the usual sense; hence, for every double point of K^n two of the possible tangents from P disappear. Similarly, for every cusp three points of contact disappear. Hence we get Plücker's first equation for a curve which does not contain singularities of a higher order:—$k = n(n-1) - 2d - 3s$ (k being the "class" of the curve, d the number of its nodes, and s of its cusps).

Dual Treatment of Curves as Envelopes.

Again, we might prove that a curve K^n could equally be developed as an envelope by means of ranges and point-involutions, and we could then dualize all the results obtained. For example, in Plücker's equation n becomes k, d becomes t (the number of double tangents), and s becomes w (the number of points of inflection), and we get $n = k(k-1) - 2t - 3w$.

The above may serve as illustrations of the application of Dr Kötter's methods to the theory of curves.

Eighth Meeting, June 14*th*, 1889.

GEORGE A. GIBSON, Esq., M.A., President, in the Chair.

Sur une propriété projective des sections coniques.

Par M. PAUL AUBERT.

Théorème.—*On considère tous les cercles σ passant par deux points fixes, dont l'un c est sur la circonférence d'un cercle donné s, et l'autre d sur une droite donnée l. Chacun des cercles σ rencontre la droite l en un second point d', et la circonférence s en un second point c': La droite c'd' passe par un point fixe i de la circonférence s, quel que soit le cercle σ considéré.*

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