GASEOUS MODELS

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ABSTRACT

We can understand physics of self-gravitating system in terms of gaseous models in so far that their global natures and effects of self-gravity are concerned. Here summarized what are known in idealized gaseous models. They include gravothermal collapse/expansion in linear and non-linear regimes, and post-collapse evolution with gravothermal oscillation. Also discussed are their relations with discrete system and with treatment in statistical mechanics.

1. ROLE OF GASEOUS MODEL

Self-gravitating system shows some peculiar behaviors when it is seen from common sense of normal thermodynamics and statistical mechanics. They come originally from the infinite range of the gravitational force. If the gravitational energy is included, the internal energy is not extensive, i.e., not proportional to the mass of the system any more, but proportional to the square of the mass. It could bring about some phenomena out of common sense, because the common sense in thermodynamics tacitly assumes extensiveness of the internal energy. (Comments and warnings are, of course, given in standard text books. However, in an example of surface tension, the surface energy is proportional to $M^{2/3}$. It depends more weakly on mass than the extensive quantities, and in this sense it lies on the opposite side of the gravothermodynamics.)

The most prominent phenomenon is gravothermal instability or catastrophe, i.e., the instability of an isothermal system contained in an adiabatic wall, which was first pointed out by Antonov (1962) and later analyzed somewhat more closely by Lyndel-Bell and Wood (1968). They showed that the system is unstable if the density contrast between the center and a point just inside the adiabatic wall exceeds the critical value of 709, and they discussed that the instability develops into the collapse of the core.

J. Goodman and P. Hut (eds.), Dynamics of Star Clusters, 207–218.

In their approach only a single value of temperature was considered throghout the system. However, the gravothermal instability or collapse of the core is, in itself, the developement of spatial structure of the system. Therefore, a theory was desirable which treats the specific entropy and the temperature as functions of spatial coordinate.

Such work was done by Hachisu and Sugimoto (1978) by means of linearized stability theory. For the spatial structure and its evolution we have a wealth of knowledge from the theory of stellar structure and evolution. The gravothermal catastrophe is the same phenomenon as the gravitational contraction of the star except for the outer boundary conditions. Computer code for calculating steller evolution can easily be applied for calculating the gravothermal catastrophe of finite amplitude. Such computations were done by Hachisu et al. (1978) and also by Lynden-Bell and Eggleton (1980).

The aim of Hachisu and Sugimoto (1978) and of Hachisu et al. (1978) was to construct physics of the self-gravitating system which will be described in the next section. The aim of Lynden-Bell and Eggleton (1980) was somewhat different. In addition to correct Hachisu et al's (1978) expression for the heat conductivity, they found the existence of similarity solution and showed that the central density reaches infinity in a finite time. They seemed to be more interested in finding functional form to compare with astronomical observation than in exploring physics of self-gravitating system.

When we talk about gaseous models we should discuss how they simulate or represent real stellar systems which are almost collision-free systems. In gaseous models we can utilize many concepts which matured in the theory of stellar evolution, and we can establish physical concepts in well defined forms as will be discussed in the next section. In particle systems, on the other hand, physics has not been posed as well defined but talked about rather vaguely. It is true in particular when the final state of the system is talked about. Though, physics of gaseous model is an approximation to the particle systems, it can play a role of a guide to explore physics of the particle systems. Then we can ask what are same or different and what are limitations of the gaseous model. Here we should be reminded that physics of ideal gas was constructed before physics of interacting gas and that the concepts from the ideal gas proved a powerful tool to investigate the interacting gas.

2. CONCEPTS ESTABLISHED IN GASEOUS MODELS

For the sake of definitness we shall confine ourselves to physics of a gas system which is enclosed within a spherical adiabatic wall and which is described by equation of states for ideal gas. Once physics is established for such case, it is relatively easy to extend it to introduce isothermal wall, other equations of states, multi-components of gas particles etc., though they will not be discussed in the present paper. Here we shall summarize the results only as lemmas (L), and corollaries (C) in weak sense. Most of them are discussed in HS (Hachisu and Sugimoto 1978), HNNS (Hachisu, Nakada, Nomoto and Sugimoto 1978) and BS (Bettwieser and Sugimoto 1984; see also Sugimoto and Bettwieser 1983).

2-1. Linear regime (HS)

It is assumed that the gas is isothermal and in hydrostatic equilibrium with the density contrast,

$$D \equiv \rho_c/\rho_b, \tag{1}$$

where ρ_c and ρ_b are the densities at the center and just inside the wall. For a redistribution of heat or specific entropy hydrostatic equilibrium is recovered and we consider change in thermodynamical quantities after this hydrostatic readjustment. We shall not talk the hydrostatic readjustmet explicitly. Therefore, every quantity of response is an "apparent" quantity in the sense that the effect of gravitational interaction has been transferred into it.

If we describe changes in temperature and pressure distributions as

$$\delta \ln T(q) = \int_0^1 F(q,q')\delta\sigma(q')dq' \quad , \tag{2}$$

$$\delta \ln P(q) = \int_0^1 G(q,q')\delta\sigma(q')\, dq' \quad , \tag{3}$$

where $\underline{q}$ is the mass fraction coordinate and $\delta\sigma$ is non-dimensional specific entropy. The inverse of the tensor specific heat F(q,q') is related to the response function of the pressure G(q,q') by

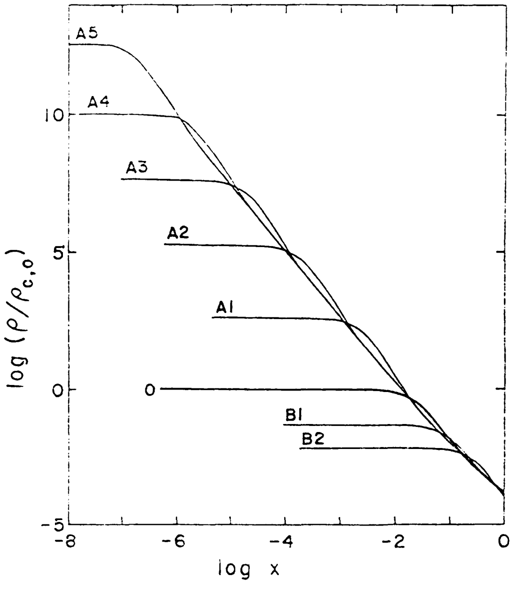

Fig. 1. Gravothermal collapse (sequence A) and expansion (B). The initial state is indicated with Stage 0. The density distribution is plotted against the radial coordinate $\underline{x}$. The outer adiabatic wall corresponds to log x = 0. Stage B2 which is the final state of the gravothermal expansion is now in a state of thermal system. Taken from HNNS.

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$$F(q,q') = (2/5)[G(q,q') + \delta(q-q')]. \quad (4)$$

Here the delta function in the right hand side describes local specific heat and the function G describes the gravothermal effect.

L1: $G(q,q')$ is negative in the central region and positive in the outer region. If its effect overrides the effect of the delta-function, the system is regarded to have "negative specific heat" as a whole.

L2: The system is thermally unstable when $D > D_{cr} = 709$. This is "gravothermal instability".
C2-1: When $D < D_{cr}$, we call it a thermal system to which the usual concepts of thermodynamics can be applied. In fact the gravo-thermodynamics tends to the usual thermodynamics in the limit of $D \to 1$.
C2-2: When $D > D_{cr}$, we call it a gravothermal system and its behavior can be different from one imagined in the usual thermodynamics.
C2-3: As is expected from the formalism of linearized theory, the gravothermal instability proceeds in both ways (Fig. 1), i.e., to contraction as well as expansion of the core according as the initial perturbation.

The entropy of the system is defined as

$$S = M\int_0^1 s(q)dq, \quad (5)$$

where M is the total mass of the system and s is the specific entropy.

L3: The equilibrium isothermal state lies at a local maximum of entropy for $D < D_{cr}$ and at a local minimum of entropy for $D > D_{cr}$.
C3-1: Both in the cases of contraction and expansion the gravothermal instability proceeds in the direction to increase S as is described by

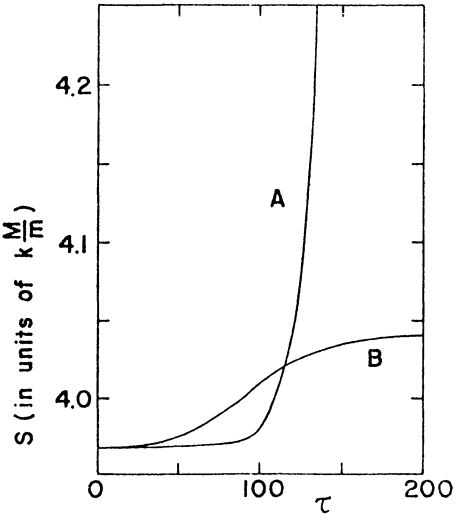

Fig. 2. Total entropy of the system is plotted against the non-dimensional time for the gravothermal contraction (A) and expansion (B) of Figure 1. In the gravothermal contraction the total entropy increases indefinitely within the non-general relativistic regime. In the gravothermal expansion it levels off to a value which corresponds to a local maximum for the system having the same energy, and the system tends to a stable thermal system. Taken from HNNS.

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the second law of thermodynamics (Fig. 2). It is an irreversible process; entropy is produced as a result of heat transport.

2-2. Non-linear regime (HNNS)

When the amplitude becomes finite, the gravothermal instability is regarded to be in non-linear regime.

L4: As the instablity proceeds, the entropy of the system increases by heat conduction as an irreversible process, which is described by

$$\frac{ds}{dt} = \int_0^1 L_r \frac{\partial}{\partial q} \left(\frac{1}{T}\right) dq, \qquad (6)$$

where L_r is the heat flux through a spherical shell at the radial distance r.

L5: For gravothermal expansion S increases but levels off to the value that corresponds to the state of stable equilibrium for the thermal system with the same energy (Fig. 2).

L6: For gravothermal contraction S increases indefinitely (Fig. 2) within a finite time.

L7: For the gravothermal contraction the central region contracts while the outer shells expand somewhat. The system results in a core-halo structure.

L8: The mass of the contracting core decreases in time (HNNS; Lynden-Bell and Eggleton 1980).
C8-1: If the total mass of the system is not large enough as compared with the mass of a gas-particle, the mass of the core could become as small as the mass of the gas-particle.
C8-2: If it is large enough, the radius of the core could become smaller than the gravitational radius of the core. The result is the collapse of the core into a black hole.

Similar things can be discussed also in the context of cosmology where the thermodynamics of the black hole and the radiation field plays a role. Sugimoto et al. (1981) discussed that the development of spatial structure or, non-uniformity if we state more exactly, is formed in the universe consistently with the second law of thermodynamics. Concerning the initial state of the universe Sugimoto et al. (1981) and later but independently Frautschi (1982) showed that at the Planck time its state was at the global maximum of entropy if we consider only the causally connected region or, in other words, the region within a particle horizon. As the universe expands, such region expands relative to a co-moving volume and the state falls out of the change in the state with the maximum entropy. In other words, the state of thermal equilibrium, i.e., the state with the maximum entropy, changes as the boundary condition of the system changes by the expansion of the universe, and the real system is left behind and now lies out of thermal equilibrium. Therefore, the entropy of the comoving volume can increase thereafter, and it is associated with generation of non-uniformites through the gravothermal catastrophe.

L9: The core collapse is described by similarity solution of almost isothermal core surrounded by an envelope of the density distribution $\rho \sim r^{-\alpha}$ with $\alpha = 2.2$. Its central density becomes infinitely high in finite time (Lynden-Bell and Eggleton 1980).

2-3. Post-collapse evolution (BS)

The gravothermal collapse is a result of the change in the mean field. When the central density becomes high enough, binaries of gas-particles (or stars) will be formed. This can be regarded as the development of two-body correlations. The binaries play a role to input energies into the mean field. To investigate the post-collapse evolution a gaseous model with energy source is considered. Since it will be discussed by Bettwieser in this Symposium, here we will summarize only their results.

Energy input by binaries is considered in analogy with the nuclear energy input in a main-sequence star. If the outermost shells of our system are regarded as the place where the entropy generated in the inner core can be dumped away, both the inner core of our system and the main-sequence star are similar dissipative open systems. The main-sequence star lies in a steady state.

L10: The core of our system makes gravothermal oscillation repeating gravothermal contraction and expansion.

L11: If the amount of the energy input into the mean field is neglected, it tends to a limit cycle encircling the collapsed singular isothermal solution. This is another state of the dissipative open system.

The difference between the main-sequence star and our system comes from the difference in the functional form of heat conductivity. In the main-sequence star the time scale of heat transport is shorter in the outer shells, while in our system it is shorter in the inner core. Therefore, the outer shells respond quickly to a change of the inner core in the case of the main-sequence star but they do not in our system. This is also the reason why the size of the core is relatively large in the interior of the star but is small in our system.

L12: Because of a net energy input to the mean field from the binaries, the energy of the mean field increases secularly. Finally, it becomes so high that the system can not be regarded to be the gravothermal system any more but a thermal system.
C12-1: When the system becomes a thermal system, the gravothermal oscillation does not occur any more, and the system makes an expansion on input of energy until an equilibrium thermal state is reached.
C12-2: If the rate of the energy input is too high (BS), the secular

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change dominates over the limit cycle so that the core expands simply to a thermal state.

L13: The expansion phase of the gravothermal oscillation is driven by the gravothermal instability. Therefore, it requires no energy input after an initial small amount of expansion has taken place. Further expansion takes place with almost the same time scale as the contraction.

During the main phase of the gravothermal expansion the so-called gravitational energy release $\varepsilon_g \equiv -T\, ds/dt$ just balances the conduction loss $\varepsilon_{cond} \equiv dL_r/Mdq$, i.e., $\varepsilon_g = \varepsilon_{cond}$, in the core region. If one used inappropriately long time step Δt in numerical computation, ε_g could not be computed correctly but the conduction loss could balance the energy generation ε_{bin} from binaries, i.e., $\varepsilon_{cond} = \varepsilon_{bin}$, within a small but finite allowance of the assumed accuracy. Then one would obtain a slow general expansion. Heggie (1984 and in this Symposium) obtained such general expansion in his numerical computation with long time steps. This seems to be an interpretation of his results, but some more studies will be necessary in order to conclude that the gravothermal oscillation should take place in our idealized system.

Inagaki and Lynden-Bell (1983) showed that there exists a self-similar solution also for the post-collapse expansion. However, their solution determines the required strength and time change of the energy generation which has nothing to do with physical mechanism. In actual, however, the energy generation is determined by binaries, i.e., by local process which is independent from global mechanism controlling expansion of the system. There is no guarantee at all that the energy generation is tuned so as to match such prescribed requirement. If the energy generation is slightly deficient, for instance, the expansion is slowed down and deviates from the similarity solution, because the configuration and temperature distribution of the similarity solution correspond to those of a gravothermally contracting system.

3. RELATION WITH PARTICLE SYSTEM OR GRAVITATIONAL MANY-BODY PROBLEM

There are several important time scales in evolution of the many body system. The shortest is the time scale of violent relaxation t_{viol} which is of the order of the crossing time. The second is the time scale of heat transport t_{heat} which is of the order of the two-body close encounter. The third is the time scale of the growth of two-body correlations or the formation of binaries t_{bin} which is of the order of two-body tidal capture or three body collision. The fourth is the time scale of the growth of many-body correlations t_{corr}. Because many-body correlation is formed successively through collision of particles with a lower order correlation, its time scale t_{corr} is longer than but is of the order of t_{bin}. Finally, many-body correlations will prevail the whole system.

Let us consider a system consisting of N particles (stars) which are enclosed with a specularly reflecting wall. For such a system the orders

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of magnitude of these time scales are related by $t_{heat} \sim (N/\ln N)t_{viol}$, and $t_{bin} \sim n^{-2}$ where _n_ is the particle density. In many numerical computations these time scales are not always well separated so that different physical phenomena appear in mixture. If we choose appropriate set of initial parameters, i.e., relatively large value of _N_ and low density _n_, we can separate and visualize phenomena of different time scales.

Because the particle system is not the main topics of the present talk, I will show only one example. It was calculated recently by Bettwieser in contact with myself for a system with N = 100 by means of a computer code NBODY3 which was kindly supplied by Professor S. Aarseth. The computation covers through $200t_{viol}$. Figure 3 shows time evolution of logarithm of the density in the innermost region in which 10 particles are contained. Initially the particles were distributed uniformly within the radius R_0 and with a constant velocity dispersion. The total energy in units of $GM^2/4R_0$ is E = -581 upto t = $47t_{viol}$ and E = -170 after t = 50 t_{viol}. We see that the fluctuation is a factor of about $10^{1/2}$, that the gravothermal catastrophe takes place, and that the formation of hard binaries excites the gravothermal oscillation.

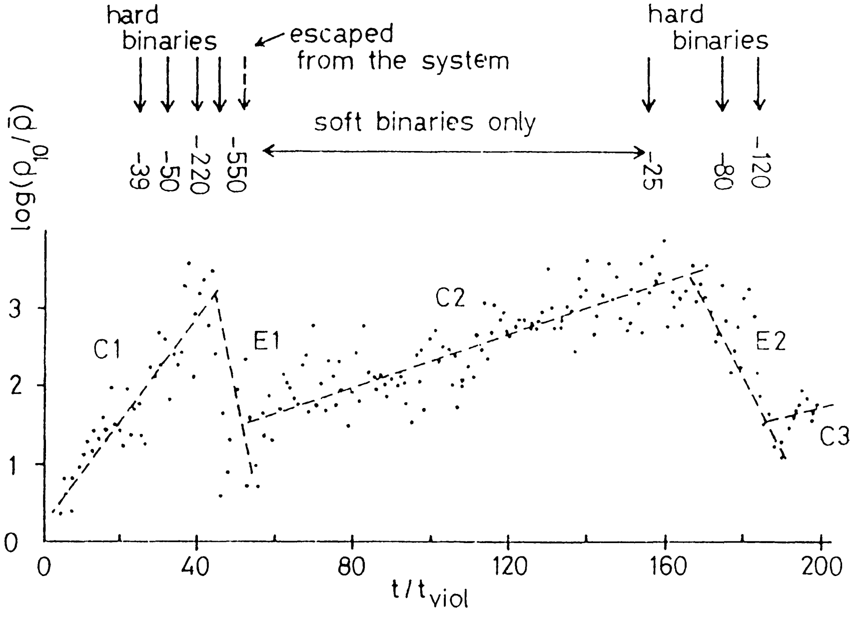

Fig. 3. Time evolution of the gravitational 100 body system. Phases of contraction (C) and expansion (E) are clearly seen. For each hard binary the time of its formation or escape is indicated by an arrow to which its energy is also attached. See the text for more details.

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Consulting with such numerical computation and the lemmas given in the preceding section, we may conjecture a picture for the evolution of self-gravitating particle system as follows.

Phase 1: Violent relaxation. It takes place in the time scale t_{viol}. It may be described by collisionless Boltzmann equation in μ-space which includes only the mean fild. An equilibrium state will be reached for $t \to t_\infty$.

Phase 2: Gravothermal catastrophe. The infinite time cited in Phase 1 is not a real infinity but the time $t_{viol} \ll t_\infty \ll t_{heat}$. Therefore, two-body collisions proceed, and the equilibrium state in the sense of Phase 1 will be shaffled. Then a new equilibrium state will be settled which has a higher entropy. If the condition for the gravothermal system is met, this is the real progress of the gravothermal instability. This phase can still be described in μ-space.

Phase 3: Gravothermal oscillation. When the time becomes of the order of $t_{bin} (\gg t_{heat})$, binaries will be formed. Now the distribution function can not be described only by multiplication of a single particle distribution functions, but at least the two-particle distribution function in binary space is necessary. It can be divided into the part of independent particles and the part of two-body correlations, i.e.,

$$f(1,2) = f(1)f(2) + g(1,2). \tag{7}$$

Correspondingly, the energy can also be divided into the energy E_{mf} which is associated to the mean field and translations of single particles and multiplets, and the energy of correlations E_{corr} which corresponds to the binding energy of the multiplets, i.e.,

$$E = E_{mf} + E_{corr} . \tag{8}$$

If the particle numbur N is large enough, there could be a situation where E_{corr} is still low compared with E_{mf} in their absolute values. Then the gravothermal oscillation can take place.

Phase 4: Prevalence of Correlations. If the system is contained in a specularly reflecting wall, E is kept constant. As the correlations devlop, E_{corr} becomes negative and large in its absolute value. Correspondingly E_{mf} becomes negative and small in its absolute value in the first place, and then can become even positive. This is a reaction to the mean field from the development of the correlations. Then the system becomes to be regarded as thermal system, if we see only the mean field, i.e., if the multiplets are considered to be constituents of the system and if E_{corr} is regarded as sub-constituent energy (such as binding energy of molecule). Then the more probable state becomes one for the thermal system, which is an isothermal state with weak density contrast. The gradual change to such state is the secular change during the gravothermal oscillation.

Phase 5: Final state. If we consider only the two-body correlation, the final state will be a uniform distribution of binaries because E could becomes much greater than E.

We have to notice that there is an important difference between the gaseous model and the usual statistical treatment of the problem. In the gaseous model we discuss the evolution of the system, while in the statistical mechanics we discuss only the final state in many cases. Therefore, the final state of Phase 5, which is obtaind from H-theorem in binary space for instance, does not necessarily imply that there is no evolution through Phases 2-3 of the gravothermal catastrophe and oscillation. In order to understand physics correctly for a system with possible strong correlation, we have to discuss their evolution by means of non-equilibrium statistical mechanics or, at least, by introducing different levels of infinite times t_∞ as done in the present section.

Nevertheless, we may ask what is the real final state as inferred from the statistical mechanics. After the formation of binaries the formation of more body correlations will proceed. Finally N' body correlation will develop where N' is of the order of N. Then we shall ask what is the spatial distribution of such N'-multiplets. However, it can not be answered by the statistical mechanics, because the number of N'-multiplets is too small to be treated in the statistical mechanics.

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DISCUSSION

SHAPIRO: One must always be wary of "hydrodynamical" models of intrinsically collisionless or Fokker-Planck (secularly collisional) systems. Of course, one can always take *moments* of the collisional Boltzmann equation and get equations which, under suitable approximations, "look like" gas-dynamical equations. This is reliable in many instances and is exactly what R. Larson (1970) did in his pioneering work on the Fokker-Planck equation over a decade ago. How exactly do your equations compare with his? What is the analog of your conduction equation in his equations? Should one not be skeptical of the gas-dynamic approach to the extent that it differs from Larson's formalism?

SUGIMOTO: Our equations can be obtained from the moment equations when the third moment is truncated by means of the heat conduction equation. Larson (1970) took account of the higher moments upto the fourth moment. Therefore in his equations non-local heat flow and deviation from Maxwellian distribution function were taken into account. Here, we can ask what are described by each moment. As far as the mean field is concerned our equation describes the essential point. In other words we see that the gravothermal contraction is described very well by our model. Higher moments will introduce only quantitative difference because the deviation from the Maxwellian distribution makes mainly a local effect which can be transferred to a correction to the equation of state. Concerning the heat conduction, Bettwieser and Sugimoto (M.N.R.A.S., submitted) have recently computed a 1000-body problem, and calculated moments from numerical results and compared it with gas-dynamical concepts. They have found that the gas-dynamical concepts can be applied fairly well. Development of correlations are described by higher moments or in many-body space. Thus, it was not taken into account even in Larson's (1970) work. However, we can take account of its main effects phenomenologically by means of the energy generation rate by binaries, for instance. Anyhow, the gaseous model of an idealized system is appropriate to explore the fundamental behavior of self-gravitating systems. When this behaviour is understood exactly in idealized systems, there will be no difficulties to include astronomical details though they may be very complex. Thus, we can and have to follow two different approaches to the problem.

LARSON: I have some reservations about whether "gravothermal expansion" can occur in real star clusters. I can imagine that it could occur in a conducting gas sphere with an adiabatic boundary, where the presence of the boundary allows heat to be trapped in the outer part of the system; this trapped heat might then in some circumstances be conducted back into the core and drive expansion. Real clusters, however, do not have an adiabatic boundary but are surrounded by an infinite heat sink, namely the rest of the universe, so that heat can not accumulate in the outer part of the cluster. All of the observations that I know of show velocity dispersions decreasing monotonically outward, and this is also true for all of the models I know of for systems not surrounded by an adiabatic boundary. Thus, in real systems the sense of heat flow should always be outward.

SUGIMOTO: In the system the timescale of heat transport is much longer (even 10^6 times though depending on the central density) in outer shells. In this sense the relatively outer shells can be regarded as the "adiabatic wall" to a good approximation. This is an essential difference of our system from the stellar structure (In the stellar interior heat is transported by radiation and its timescale is much shorter in outer shells.) See my answer to Grindlay's question as for the relation with observations.

GRINDLAY: My question follows naturally from Dr. Larson's remark as well as from the viewgraph you now have on the screen. The question is: in a real globular cluster (of high mass), what is the maximum difference in "temperature" or velocity dispersion between the inner and outer core for a cluster in the expansion phase? As I mentioned in my talk yesterday, I suspect the X-ray globular NGC 6712 may be in such a post-collapse expansion phase and we are beginning a major program to measure V_{disp} vs. R. What is the maximum *increase* in V_{disp} v. radius R we might expect to measure if indeed the cluster is in an expansion phase?

SUGIMOTO: As can be seen in one of my figures (Figure 2 of Sugimoto and Bettwieser 1983 M.N.R.A.S. 204, 19P), the maximum possible temperature difference between its peak and the central value is $\log(T_{peak}/T_{center}) \simeq 0.05$. The peak lies at the edge of the core. The value given above is one at the stage when the mass fraction of the core is equal to 1×10^{-3}. Both in earlier and later stages of expansion, i.e., for smaller and larger core mass, respectively, the temperature difference is smaller.

KING: In a real cluster, where the density drops to zero at the boundary, what quantity corresponds to the magic density contrast of your physical idealization?

COHN (after Sugimoto referred the question to him): The best observational parameter to look at in order to determine whether the gravothermal instability is likely to have occurred is your W_0 parameter or equivalently the concentration parameter c. Clusters are expected to become unstable for $W_0 \geq 8.5$ that is $c \geq 2.0$. Indeed the clusters with possible cusps satisfy this criterion.

KING: That is very interesting. In the linear-sequence sense to which Spitzer referred yesterday, $W_0 = 8.5$ is very close to the turning point.