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SOME RESULTS ON THE COUNTABLE COMPACTNESS AND PSEUDOCOMPACTNESS OF HYPERSPACES

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1. Introduction. Let X be a Hausdorff space. Let 2^X denote the set of all non-empty closed subsets of X. For a subset A of X, we set $2^A = \{F \in 2^X : F \subseteq A\}$. Recall that the finite topology on 2^X is that topology having as a sub-basis the family $\{2^G : G \text{ is open in } X\} \cup \{2^X - 2^F : F \text{ is closed in } X\}$. When endowed with this topology, 2^X is referred to as the *hyperspace of* X. For the fundamental properties of hyperspaces, we refer the reader to [**6**; **7**]. Following [**6**], we adopt the following notation: If $A_0, A_1, \ldots, A_n$ are subsets of X, we set $B(A_0; A_1, \ldots, A_n) = 2^{A_0} \cap \bigcap_{i=1}^n (2^X - 2^{X-A_i}) = \{F \in 2^X : F \subseteq A_0$ and $F \cap A_i \neq \emptyset$ for all $i = 1, 2, \ldots, n\}$. Using this notation, we see that the sets of the form $B(G_0; G_1, \ldots, G_n)$, where $G_0, G_1, \ldots, G_n$ are open in X and $\bigcup_{i=1}^n G_i \subseteq G_0$, form a basis for the finite topology on 2^X.

We are concerned here with the countable compactness and pseudocompactness of 2^X. At this point, let us recall several concepts related to countable compactness and pseudocompactness, which will be useful in our discussion.

Let X be a topological space. Let $(S_n : n \in N)$ be a sequence of subsets of X. A point $p \in X$ is a *limit point of the sequence* $(S_n : n \in N)$, if, for each neighborhood W of p, $\{n \in N : W \cap S_n \neq \emptyset\}$ is infinite. Let $\mathscr{D}$ be a free ultrafilter on the set (discrete space) N of positive integers. We say that p is *a* $\mathscr{D}$*-limit point of the sequence* $(S_n : n \in N)$ if, for every neighborhood W of p, $\{n : W \cap S_n \neq \emptyset\} \in \mathscr{D}$.

X is *countably compact* if every sequence of points in X has a limit point.

Let $\mathscr{D}$ be a free ultrafilter on N. Following [**1**], we say that X is $\mathscr{D}$*-compact* if every sequence of points in X has a $\mathscr{D}$-limit point in X.

X is *pseudocompact* if every continuous real-valued function on X is bounded.

X is $\mathscr{G}$*-pseudocompact* if every sequence of non-empty open subsets of X has a limit point in X.

Let $\mathscr{D}$ be a free ultrafilter on N. We say that X is $\mathscr{D}$*-pseudocompact*, if every sequence of non-empty open subsets of X has a $\mathscr{D}$-limit point.

Finally, X is $\aleph_0$*-bounded* if every countable subset of X is contained in a compact subset of X.

The following facts relating these concepts are elementary and easy to verify. A $\mathscr{D}$-compact space is countably compact. A $\mathscr{G}$-pseudocompact space is

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pseudocompact. A completely regular space X is pseudocompact if, and only if' it is $\mathscr{G}$-pseudocompact. A $\mathscr{D}$-pseudocompact space is $\mathscr{G}$-pseudocompact. Every $\mathscr{D}$-compact space is $\mathscr{D}$-pseudocompact.

We shall have occasion to use the following two theorems.

THEOREM 1.1. (Bernstein) *$\mathscr{D}$-compactness is closed-hereditary, and productive. A completely regular space X is $\aleph_0$-bounded if, and only if, it is $\mathscr{D}$-compact for every free ultrafilter $\mathscr{D}$ on N.* (See [**1**].)

THEOREM 1.2. (Ginsburg-Saks) *Let X be a Hausdorff space. Then the following are equivalent:*
(i) *Every power of X is countably compact;*
(ii) *X^{2^c} is countably compact;*
(iii) *$X^{|X|^{\aleph_0}}$ is countably compact;*
(iv) *X is $\mathscr{D}$-compact for some free ultrafilter $\mathscr{D}$ on N.* (See [**4**].)

For a discussion of the above results and related ideas, we refer the reader to [**1**; **4**].

As mentioned above, this paper is concerned with the countable compactness and pseudocompactness of hyperspaces. We will establish the following results. X is $\mathscr{D}$-compact if, and only if, 2^X is $\mathscr{D}$-compact. X is $\mathscr{D}$-pseudocompact if, and only if, 2^X is $\mathscr{D}$-pseudocompact. If all powers of X are countably compact, then 2^X is countably compact. If 2^X is countably compact, then all finite powers of X are countably compact. If X is completely regular and 2^X is pseudocompact, then all finite powers of X are pseudocompact. We give an example of a completely regular space X, all of whose finite powers are countably compact, such that 2^X is not pseudocompact.

2. Some theorems on the countable compactness and pseudocompactness of 2^X. Our first result compares the $\mathscr{D}$-compactness of 2^X with that of X.

THEOREM 2.1. *Let X be a Hausdorff space, and let $\mathscr{D}$ be a free ultrafilter on N. Then X is $\mathscr{D}$-compact if, and only if, 2^X is $\mathscr{D}$-compact.*

Proof. If X is Hausdorff, then the singletons in 2^X form a closed subset homeomorphic to X. (See 2.4 of [7].) Since $\mathscr{D}$-compactness is closed hereditary by 1.1, if 2^X is $\mathscr{D}$-compact, so is X.

For the converse, suppose X is $\mathscr{D}$-compact. We show 2^X is $\mathscr{D}$-compact. Thus, let $(F_n : n \in N)$ be a sequence in 2^X. Let $L = \{p \in X : p$ is a $\mathscr{D}$-limit point of the sequence $(F_n : n \in N)\}$. Clearly L is a non-empty, closed subset of X. That is, $L \in 2^X$. We claim that L is a $\mathscr{D}$-limit point of the sequence $(F_n : n \in N)$ in 2^X. To see this, let $\mathscr{W} = B(G_0; G_1, \ldots, G_T)$ be a basic neighborhood of L in 2^X. We must show that $\{n : F_n \in \mathscr{W}\} \in \mathscr{D}$. Now let $N_0 = \{n \in N : F_n \subseteq G_0\}$, and for $i \in \{1, 2, \ldots, T\}$ let $N_i = \{n \in N : F_n \cap G_i \neq \emptyset\}$. Clearly $\{n \in N : F_n \in \mathscr{W}\} = \bigcap_{i=0}^{T} N_i$. Thus, to show that $\{n \in N : F_n \in \mathscr{W}\} \in \mathscr{D}$, we need prove that $N_i \in \mathscr{D}$ for each $i \in \{0, 1, \ldots, T\}$. Now, since $L \in \mathscr{W}$, we have $L \cap G_i \neq \emptyset$. Let $p \in L \cap G_i$. Then p is a $\mathscr{D}$-limit point of the sequence $(F_n : n \in N)$ and G_i is a neighborhood of p, so $\{n : G_i \cap F_n \neq \emptyset\} = N_i \in \mathscr{D}$. Thus $N_i \in \mathscr{D}$ for $i = 1, 2, \ldots, T$. Finally, we show $N_0 \in \mathscr{D}$. For the sake of contradiction, assume $N_0 \notin \mathscr{D}$. Then $N - N_0 \in \mathscr{D}$. For each $n \in N - N_0$, choose a point $x_n \in F_n - G_0$. For each $n \in N_0$, choose a point x_n arbitrarily from F_n. The sequence $(x_n : n \in N)$ so obtained has a $\mathscr{D}$-limit point a, by the $\mathscr{D}$-compactness of X. Clearly a is a $\mathscr{D}$-limit point of the sequence $(F_n : n \in N)$, and so $a \in L$. But $L \in \mathscr{W}$, so that $L \subseteq G_0$. Therefore $a \in G_0$. Since a is a $\mathscr{D}$-limit point of the sequence $(x_n : n \in N)$, we have $\{n : x_n \in G_0\} \in \mathscr{D}$. But this last set is disjoint from $N - N_0$, which also lies in $\mathscr{D}$. This is a contradiction. Therefore $N_0 \in \mathscr{D}$, and L is a $\mathscr{D}$-limit point of the sequence $(F_n : n \in N)$ in 2^X. Thus 2^X is $\mathscr{D}$-compact.

From 2.1 we can obtain, as a corollary, the following theorem due to J. Keesling [**5**].

COROLLARY 2.2. *Let X be a normal space. Then X is $\aleph_0$-bounded if, and only if, 2^X is $\aleph_0$-bounded.*

Proof. If X is normal, then, by 4.9.5 of [**7**], 2^X is completely regular. By 1.1, 2^X is $\aleph_0$-bounded if, and only if, it is $\mathscr{D}$-compact for every free ultrafilter $\mathscr{D}$ on N. By 2.1, this happens exactly when X is $\mathscr{D}$-compact for all free ultrafilters $\mathscr{D}$ on N, which by 1.1, is equivalent to X being $\aleph_0$-bounded.

Theorem 2.1 also allows us to establish the following relation between the countable compactness of 2^X and that of powers of X.

COROLLARY 2.3. *Let X be a Hausdorff space. If all powers of X are countably compact, then 2^X is countably compact. If 2^X is countably compact, then all finite powers of X are countably compact.*

Proof. If all powers of X are countably compact, then, by 1.2, there is a free ultrafilter $\mathscr{D}$ on N such that X is $\mathscr{D}$-compact. By 2.1, 2^X is also $\mathscr{D}$-compact, and so, in particular, is countably compact.

Suppose 2^X is countably compact. For each $n \in N$, let $\mathscr{F}_n(X) = \{F \in 2^X : |F| \leqq n\}$. By 2.4 of [**7**], $\mathscr{F}_n(X)$ is a closed subspace of 2^X for each $n \in N$. For each n, define the map $S_n : X^n \to \mathscr{F}_n(X)$ by $S_n(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$. Then, for each n, S_n is a continuous, closed, finite-to-one map from X^n onto $\mathscr{F}_n(X)$ [**3**]. As countable compactness is closed hereditary and preserved under perfect pre-images, the countable compactness of 2^X implies that of X^n for each $n \in N$.

We next turn to pseudocompactness. The next result is an analogy to 2.1.

THEOREM 2.4. *Let $\mathscr{D}$ be a free ultrafilter on N. Then X is $\mathscr{D}$-pseudocompact if, and only if, 2^X is $\mathscr{D}$-pseudocompact.*

Proof. Suppose 2^X is $\mathscr{D}$-pseudocompact. We show that X is $\mathscr{D}$-pseudocompact. Thus, let $(G_n : n \in N)$ be a sequence of non-empty open subsets of X. Then $(2^{G_n} : n \in N)$ is a sequence of non-empty open subsets of 2^X. As 2^X is $\mathscr{D}$-pseudocompact, this sequence has a $\mathscr{D}$-limit point $F \in 2^X$. Choose any point $p \in F$. We show that p is a $\mathscr{D}$-limit point in X, of the sequence $(G_n : n \in N)$. For, let W be any neighborhood of p in X. Then, since $F \cap W \neq \emptyset$, $2^X - 2^{X-W}$ is a neighborhood of F in 2^X. Since F is a $\mathscr{D}$-limit point of the sequence $(2^{G_n} : n \in N)$, $\{n : 2^{G_n} \cap (2^X - 2^{X-W}) \neq \emptyset\} \in \mathscr{D}$. But $2^{G_n} \cap (2^X - 2^{X-W}) \neq \emptyset$ if, and only if, $G_n \cap W \neq \emptyset$. Thus $\{n : G_n \cap W \neq \emptyset\} \in \mathscr{D}$, and so p is a $\mathscr{D}$-limit point of the sequence $(G_n : n \in N)$. Therefore X is $\mathscr{D}$-pseudocompact.

Conversely, suppose X is $\mathscr{D}$-pseudocompact. Since the sets $B(G_0; C_1, \ldots, G_T)$, with $G_0, G_1, \ldots, G_T$ open in X and $\bigcup_{i=1}^{T} G_i \subseteq G_0$, form a basis for the topology on 2^X, to show that 2^X is $\mathscr{D}$-pseudocompact we need only show that open sequences of these sets have $\mathscr{D}$-limit points. Thus, suppose we are given a sequence $(\mathscr{G}_n : n \in N)$ of non-empty basic open sets $\mathscr{G}_n$ in 2^X. Write $\mathscr{G}_n$ as $B(G_{0,n}; G_{1,n}, \ldots, G_{T_n,n})$, with $G_{i,n}$ open in X and $\bigcup_{i=1}^{T_n} G_{i,n} \subseteq G_{0,n}$. Let $L = \{p \in X : p$ is a $\mathscr{D}$-limit point of the sequence $(G_{0,n} : n \in N)\}$. Then L is a non-empty, closed subset of X. That is, $L \in 2^X$. We claim that L is a $\mathscr{D}$-limit point in 2^X of the sequence $(\mathscr{G}_n : n \in N)$. Now the sets of the form 2^G and $B(X; G)$ form a sub-basis for 2^X. Since filters are closed under finite intersection, to show that L is a $\mathscr{D}$-limit point of $(\mathscr{G}_n : n \in N)$, it is enough to establish the following two statements.

(i) If G is open in X and $L \in 2^G$, then $\{n \in N : 2^G \cap \mathscr{G}_n \neq \emptyset\} \in \mathscr{D}$.

(ii) If G is open in X and $L \in B(X; G)$, then $\{n \in N : B(X; G) \cap \mathscr{G}_n \neq \emptyset\} \in \mathscr{D}$.

Let us first establish (i). Note that $2^G \cap \mathscr{G}_n \neq \emptyset$ if, and only if, $G \cap G_{i,n} \neq \emptyset$ for all $i = 1, 2, \ldots, T_n$. Let $S = \{n \in N : 2^G \cap \mathscr{G}_n \neq \emptyset\}$ and let $T = N - S$. For the sake of contradiction, suppose $S \notin \mathscr{D}$. Then $T \in \mathscr{D}$. For each $n \in T$, find an integer $i_n \in \{1, 2, \ldots, T_n\}$ such that $G \cap G_{i_n,n} = \emptyset$. Define a sequence $(H_n : n \in N)$ of non-empty open subsets of X as follows. For $n \in T$, $H_n = G_{i_n,n}$, and for $n \in S$, $H_n = G_{1,n}$. Now, since X is $\mathscr{D}$-pseudocompact, the sequence $(H_n : n \in N)$ has a $\mathscr{D}$-limit point $a \in X$. Clearly $a \in L$. Since $L \in 2^G$, we have $L \subseteq G$, and so $a \in G$. Since a is a $\mathscr{D}$-limit point of $(H_n : n \in N)$, $\{n \in N : G \cap H_n \neq \emptyset\} \in \mathscr{D}$. But this latter set is disjoint from T, and $T \in \mathscr{D}$. This is a contradiction. Therefore $S \in \mathscr{D}$, establishing (i). To establish (ii), suppose G is open in X and $L \in B(X; G)$. Observe that $B(X; G) \cap \mathscr{G}_n \neq \emptyset$ if, and only if, $G \cap G_{0,n} \neq \emptyset$. Let $M = \{n \in N : B(X; G) \cap \mathscr{G}_n \neq \emptyset\} = \{n \in N : G \cap G_{0,n} \neq \emptyset\}$. Now, since $L \in B(X; G)$, $L \cap G \neq \emptyset$. Let $p \in L \cap G$. Then p is a $\mathscr{D}$-limit point of the sequence $(G_{0,n} : n \in N)$, and G is a neighborhood of p. Therefore $\{n \in N : G \cap G_{0,n} \neq \emptyset\} \in \mathscr{D}$. That is, $M \in \mathscr{D}$, establishing (ii).

We have thus shown that L is a $\mathscr{D}$-limit point of $(\mathscr{G}_n : n \in N)$. Therefore 2^X is $\mathscr{D}$-pseudocompact, as desired.

Even having established 2.4, we cannot conclude that the pseudocompactness of all powers of X implies the pseudocompactness of 2^X, at least not by an argument analogous to the one used in 2.3. The problem here is that $\mathscr{D}$-pseudocompactness is not a necessary condition for pseudocompact powers (see **4**). We can, however, establish a pseudocompact counterpart to the second assertion in 2.3.

THEOREM 2.5. *Let X be regular. If 2^X is $\mathscr{G}$-pseudocompact, then all finite powers of X are $\mathscr{G}$-pseudocompact.*

Proof. Assume 2^X is $\mathscr{G}$-pseudocompact. Firstly, X is $\mathscr{G}$-pseudocompact. For, if $(G_n : n \in N)$ is a sequence of non-empty open subsets of X, the sequence $(2^{G_n} : N)$ has a limit point L in 2^X. Choosing any point $p \in L$, it is easy to see that p is a limit point of $(G_n : n \in N)$. Thus every sequence of non-empty open subsets of X has a limit point in X. That is, X is $\mathscr{G}$-pseudocompact.

Next, we show that $X \times X$ is $\mathscr{G}$-pseudocompact, for which it suffices to show that every sequence $(U_n \times V_n : n \in N)$ where U_n, V_n are non-empty open subsets of X, has a limit point in X. We will assume not, and we will derive a contradiction. So assume $(U_n \times V_n : n \in N)$ has no limit point in $X \times X$. Now X is $\mathscr{G}$-pseudocompact, as has already been established, so the sequence $(U_n : n \in N)$ has a limit point $p \in X$. Since $(U_n \times V_n : n \in N)$ has no limit point in $X \times X$, in particular, (p, p) is not a limit point of $(U_n \times V_n : n \in N)$. Therefore, there is a neighborhood W of p in X such that $\{n \in N : (W \times W) \cap (U_n \times V_n) \neq \emptyset\}$ is finite. Let $S = \{n \in N : (W \times W) \cap (U_n \times V_n) \neq \emptyset\}$. By regularity, find a neighborhood W_1 of p in X such that $\mathrm{cl}_X W_1 \subseteq W$. Let $T = \{n \in N : W_1 \cap U_n \neq \emptyset\}$. Since p is a limit point of $(U_n : n \in N)$, T is infinite. Let $N_1 = T - S$. Then N_1 is infinite. Consider the sequence $((W_1 \cap U_n) \times V_n : n \in N_1)$. Being a refinement of a subsequence of $(U_n \times V_n : n \in N)$, the sequence $((W_1 \cap U_n) \times V_n : n \in N_1)$ also has no limit point in $X \times X$. Let $A = \mathrm{cl}_X W_1$, and let $B = \mathrm{cl}_X (\bigcup_{n \in N_1} V_n)$. Then A and B are disjoint regular-closed subsets of X, and $\bigcup_{n \in N_1} [(W_1 \cap U_n) \times V_n] \subseteq A \times B$. Now, since A and B are disjoint closed sets, $A \cup B$ is homeomorphic to $A + B$, the free union of A and B. By 5a., page 166 of [**6**], 2^{A+B} is homeomorphic to $2^A \times 2^B$. Now $\mathscr{G}$-pseudocompactness is evidently inherited by regular-closed subsets. As 2^X is $\mathscr{G}$-pseudocompact, so is $2^{A \cup B}$, and so, by the above remarks, is $2^A \times 2^B$. It follows easily that $A \times B$ is $\mathscr{G}$-pseudocompact. But $((W_1 \cap U_n) \times V_n : n \in N_1)$ has no limit point in $X \times X$, which is a contradiction. Thus $X \times X$ is $\mathscr{G}$-pseudocompact.

One can now prove by induction on n that X^n is $\mathscr{G}$-pseudocompact for all $n \in N$. The essential idea in going from X^n to X^{n+1} is the same as going from X^2 to X^3, but the details are more cumbersome. Accordingly, we will show how to deduce the $\mathscr{G}$-pseudocompactness of X^3 from that of X^2 (and that of 2^X, of course), and leave the induction as a straightforward extension of this step.

Thus, from the $\mathscr{G}$-pseudocompactness of 2^X and $X \times X$, we are to deduce the $\mathscr{G}$-pseudocompactness of $X \times X \times X$. We assume that $X \times X \times X$ is not $\mathscr{G}$-pseudocompact, and we will reach a contradiction. So, let $(A_n \times B_n \times C_n : n \in N)$ be an open sequence in X^3 which has no limit point. Now X^2 is

$\mathscr{G}$-pseudocompact, so the sequence $(A_n \times B_n : n \in N)$ has the limit point (a, b) in $X \times X$. Neither (a, b, a) nor (a, b, b) is a limit point of $(A_n \times B_n \times C_n : n \in N)$ in X^3. Thus we can find neighborhoods G and H of a and b respectively, such that the two sets

$$M_1 = \{n \in N : (G \times H \times G) \cap (A_n \times B_n \times C_n) \neq \emptyset\} \text{ and}$$
$$M_2 = \{n \in N : (G \times H \times H) \cap (A_n \times B_n \times C_n) \neq \emptyset\} \text{ are finite.}$$

Find neighborhoods G_1 and H_1 of a and b respectively such that $\mathrm{cl}_X\, G_1 \subseteq G$ and $\mathrm{cl}_X\, H_1 \subseteq H$. Let

$$M_3 = \{n \in N : (G_1 \times H_1) \cap (A_n \times B_n) \neq \emptyset\}.$$

Since (a, b) is a limit point of $(A_n \times B_n : n \in N)$, M_3 is infinite. Now let $N_1 = M_3 - (M_1 \cup M_2)$. Then N_1 is infinite. Let $A_n' = G_1 \cap A_n$, and let $B_n' = H_1 \cap B_n$. The sequence $(A_n' \times B_n' \times C_n : n \in N_1)$, being a refinement of a subsequence of $(A_n \times B_n \times C_n : n \in N)$, also has no limit point in X^3. But X^2 is $\mathscr{G}$-pseudocompact, so the sequence $(B_n' \times C_n : n \in N_1)$ has a cluster point (c, d) in X^2. Neither (c, c, d) nor (d, c, d) is a cluster point of $(A_n' \times B_n' \times C_n : n \in N_1)$. So we may find neighborhoods U and V of c and d respectively, such that the two sets

$$L_1 = \{n \in N_1 : (U \times U \times V) \cap (A_n' \times B_n' \times C_n) \neq \emptyset\} \text{ and}$$
$$L_2 = \{n \in N_1 : (V \times U \times V) \cap (A_n' \times B_n' \times C_n) \neq \emptyset\} \text{ are finite.}$$

Find neighborhoods U_1 and V_1 of c and d respectively, such that $\mathrm{cl}_X\, U_1 \subseteq U$ and $\mathrm{cl}_X\, V_1 \subseteq V$. Now, let

$$L_3 = \{n \in N_1 : (U_1 \times V_1) \cap (B_n' \times C_n) \neq \emptyset\}.$$

Since (c, d) is a limit point of $(B_n' \times C_n : n \in N_1)$, the set $N_2 = L_3 - (L_1 \cup L_2)$ is infinite. For $n \in N_2$, set $A_n'' = A_n'$, $B_n'' = U_1 \cap B_n'$, $C_n'' = V_1 \cap C_n$. The sequence $(A_n'' \times B_n'' \times C_n'' : n \in N_2)$ has no limit point in X^3. Let $A = \mathrm{cl}_X\, (\bigcup_{n \in N_2} A_n'')$, $B = \mathrm{cl}_X\, (\bigcup_{n \in N_2} B_n'')$, $C = \mathrm{cl}_X\, (\bigcup_{n \in N_2} C_n'')$. Then A, B, C are pairwise disjoint regular-closed subsets of X, and $\bigcup_{n \in N_2} (A_n'' \times B_n'' \times C_n'') \subseteq A \times B \times C$. By the same argument used earlier, $2^A \times 2^B \times 2^C$ is homeomorphic to $2^{A \cup B \cup C}$, which, as a regular-closed subspace of 2^X, inherits $\mathscr{G}$-pseudocompactness. Thus $A \times B \times C$ is $\mathscr{G}$-pseudocompact, which contradicts the fact that $(A_n'' \times B_n'' \times C_n'' : n \in N_2)$ has no limit point. This contradiction proves that X^3 is $\mathscr{G}$-pseudocompact.

As was mentioned in the introduction, a completely regular space X is $\mathscr{G}$-pseudocompact if, and only if, it is pseudocompact. Although 2^X is completely regular only when X is normal, these concepts remain equivalent for 2^X when X is completely regular, as we now show.

PROPOSITION 2.6. *Let X be completely regular. Then 2^X is $\mathscr{G}$-pseudocompact if, and only if, it is pseudocompact.*

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Proof. $\mathscr{G}$-pseudocompactness always implies pseudocompactness. We need only show that if 2^X is not $\mathscr{G}$-pseudocompact, then 2^X is not pseudocompact. If 2^X is not $\mathscr{G}$-pseudocompact, there is a sequence $\mathscr{G}_n = B(G_{0,n}; G_{1,n}, \ldots, G_{T_n,n})$ of non-empty basic open subsets of 2^X, which has no limit point in 2^X. For each n and each $i \in \{1, 2, \ldots, T_n\}$, choose a point $p_{n,i} \in G_{i,n}$. Let $F_n = \{p_{n,i} : i = 1, 2, \ldots, T_n\}$. Now $F_n \subseteq G_{0,n}$, so, by complete regularity, we can find, for each n, a continuous, real-valued function f_n on X such that $f_n(x) = 1$ for each $x \in F_n$, and $f_n(x) = 0$ for each $x \in X - G_{0,n}$, and such that $0 \leqq f_n \leqq 1$. Given n and $i \in \{1, 2, \ldots, T_n\}$, by complete regularity, we can find a continuous, real-valued function $g_{n,i}$ on X such that $0 \leqq g_{n,i} \leqq 1$, $g_{n,i}(p_{n,i}) = 1$, and $g_{n,i}(x) = 0$ for each $x \in X - G_{i,n}$. Now, for each n, define f_n^- on 2^X by $f_n^-(F) = \inf_{x \in F} f_n(x)$. For each n and each $i \in \{1, 2, \ldots, T_n\}$, define $g_{n,i}^+$ on 2^X by $g_{n,i}^+(F) = \sup_{x \in F} g_{n,i}(x)$. By 4.7 of [**7**], the functions f_n^- and $g_{n,i}^+$ are all continuous, real-valued functions on 2^X. Now, for each n, let $G_n = f_n^- \cdot g_{n,1}^+ \cdot \ldots g_{n,T_n}^+$. Then G_n is continuous and $G_n(F_n) = 1$, and $G_n(F) = 0$ for each $F \in 2^X - \mathscr{G}_n$. Since the sequence $(\mathscr{G}_n : n \in N)$ has no limit point, the function $\sum_{n \in N} nG_n$ is continuous on 2^X, and is clearly unbounded. Thus 2^X is not pseudocompact.

COROLLARY 2.7. *Let X be completely regular. If 2^X is pseudocompact, then all finite powers of X are pseudocompact.*

Proof. This follows immediately from 2.6 and 2.7.

3. An example. In [**2**], Z. Frolik constructs, for each positive integer n, a space X such that X^n is countably compact but X^{n+1} is not pseudocompact. In [**5**], J. Keesling shows that the hyperspaces of these spaces are not pseudocompact. This conclusion also follows from 2.7. Also in [**2**], Frolik constructs a space Y, all of whose finite powers are countably compact, such that $Y^{\aleph_0}$ is not pseudocompact. We will see below that 2^Y is not pseudocompact, thus providing a counterexample to the converse of 2.7 and to the converse of the last statement in 2.3.

3.1. *Example. A completely regular space Y, all of whose finite powers are countably compact, such that 2^Y is not pseudocompact.*

Frolik constructs a sequence X_i, for $i \in N$, of subspaces of $\beta N - N$, such that $\prod_{K \in N} N \cup X_K$ is not pseudocompact, while every finite subproduct is countably compact. In his example, $\bigcap_{i \in N} X_i = \emptyset$. The desired space Y is the free union of the spaces $N \cup X_i$, together with a point at infinity, whose neighborhoods are complements of finitely many of the spaces $N \cup X_i$. To avoid ambiguity, let us replace $N \cup X_i$ by $Y_i = (N \cup X_i) \times \{i\}$. The space Y is then $(\bigcup_{i \in N} Y_i) \cup \{\infty\}$, with the topology described above. We will show that 2^Y is not pseudocompact. We will in fact produce an open-closed subspace of 2^Y homeomorphic to N. For each n, we let

$$F_n = \{(n, 1), (n, 2), \ldots, (n, n)\}.$$

Since each point of each copy of N is isolated in Y, it follows that, for every n, F_n is an isolated point of 2^Y. Thus $D = \{F_n : n \in N\}$ is a discrete, open subspace of 2^Y, and our proof will be complete if we show D is closed in 2^Y. Let $A \in 2^Y$. We show that A is not a cluster point of D.

Case 1. $A \cap [\bigcup_{k \in N} N \times \{k\}] \neq \emptyset$. In this case, let $(n, k) \in A$. Now (n, k) is isolated in Y, so $B(Y; \{(n, k)\})$ is a neighborhood of A in 2^Y. At most one F_i is in $B(Y; \{(n, k)\})$. Therefore A is not a cluster point of D.

Case 2. There is an integer i such that $A \cap Y_i = \emptyset$. In this case, 2^{Y-Y_i} is a neighborhood of A in 2^Y meeting D in a finite set. Thus A is not a cluster point of D.

Case 3. For some integer i, $|A \cap Y_i| > 1$. In this case, A meets two disjoint open subsets G_1 and G_2 of Y_i. Since each F_n contains at most one element from each Y_i, $B(Y; G_1, G_2)$ is a neighborhood of A in 2^Y that is disjoint from D. So again, A is not a cluster point of D.

Case 4. In light of the first three cases, we may now assume that $A = \{(x_n, n) : n \in N\} \cup \{\infty\}$, where, for each n, $x_n \in X_n$. Now, since $\bigcap_{n \in N} X_n = \emptyset$, we can find integers n and m such that $x_n \neq x_m$. Find disjoint open sets U and V in βN such that $x_n \in U$ and $x_m \in V$. Now set $U_1 = [U \cap (N \cup X_n)] \times \{n\}$, and $V_1 = [V \cap (N \cap X_m)] \times \{m\}$. Then U_1 and V_1 are open in Y, and $(x_n, n) \in U_1$, $(x_m, m) \in V_1$. Thus $B(Y; U_1, V_1)$ is a neighborhood of A in 2^Y. Since $B(Y; U_1, V_1)$ is clearly disjoint from D, A is not a cluster point of D.

Cases 1 to 4 combine to show that D is closed in 2^Y, completing the proof.

3.2. *Remark*. In light of the results of 2.3 and 2.7, and Example 3.1, it is natural to ask whether there is any relation between the pseudocompactness (countable compactness) of $X^{\aleph_0}$ and that of 2^X. It would also be interesting to characterize those spaces X whose hyperspaces are countably compact (pseudocompact). The author has been unable to resolve these questions, and leaves them open to the reader. Natural examples of $\mathscr{D}$-compact and $\mathscr{D}$-pseudocompact spaces can be found in [**4**]. These spaces provide non-trivial examples of pseudocompact and countably compact hyperspaces.

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