NOTE ON A PAPER OF TSUZUKU

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In [**2**], Tosiro Tsuzuku gave a proof of the following:

THEOREM. *Let G be a doubly transitive permutation group of degree n, let K be any commutative ring with unit element and let ρ be the natural representation of G by $n \times n$ permutation matrices with elements* 0, 1 *in K. Then ρ is decomposable as a matrix representation over K if and only if n is an invertible element of K.*

For G the symmetric group this result follows from Theorems (2.1) and (4.12) of [**1**]. The proof given by Tsuzuku is unsatisfactory, although it is perfectly valid when K is a field. The purpose of this note is to give a correct proof of the general case.

Let M be the representation module realizing the representation ρ, that is, M is the free left K-module generated by the permuted letters $e_1, \ldots, e_n$. Suppose that n is invertible in K. Then for arbitrary elements k_i of K we have

$$\sum_{i=1}^{n} k_i e_i = n^{-1}k \,.\, e + \sum_{i=2}^{n} (k_i - n^{-1}k)(e_i - e_1),$$

where $k = k_1 + \ldots + k_n$, $e = e_1 + \ldots + e_n$. This shows that M is the direct sum of the representation submodules Ke and $\sum_{i=2}^{n} K(e_i - e_1)$.

Conversely, suppose that M is the direct sum of two representation submodules: $M = M' + M''$. By definition of representation modules, M' and M'' are K-free (this point was missed by Tsuzuku), and every $m \in M$ has a unique decomposition in the form $m = m' + m''$, where $m' \in M'$, $m'' \in M''$. Clearly $(gm)' = gm'$, $(gm)'' = gm''$ for $g \in G$, $m \in M$. Let $e_i' = \sum_{j=1}^{n} \kappa(e_i, e_j)e_j$, where $\kappa(e_i, e_j) \in K$. If now g is any permutation belonging to G, then

$$(ge_i)' = ge_i' = \sum_{j=1}^{n} \kappa(e_i, e_j)ge_j.$$

This shows that, for all $g \in G$,

$$\kappa(e_i, e_j) = \kappa(ge_i, ge_j) \quad (i, j = 1, \ldots, n).$$

Put $\kappa(e_1, e_1) = \lambda$, $\kappa(e_1, e_2) = \mu$. Since G is doubly transitive, we have $\kappa(e_i, e_i) = \lambda$ (for all i) and $\kappa(e_i, e_j) = \mu$ for $i \neq j$. Hence, again writing $e = e_1 + \ldots + e_n$, we get

$$e_i' = \mu e + \sigma e_i, \quad \text{where} \quad \sigma = \lambda - \mu.$$

Hence $e' = (n\mu + \sigma)e$ and therefore

$$(e_i')' = \mu(n\mu + 2\sigma)e + \sigma^2 e_i.$$

But $(e_i')' = e_i'$; consequently

$$\sigma^2 = \sigma \quad \text{and} \quad \mu\chi = 0, \quad \text{where} \quad \chi = n\mu + 2\sigma - 1.$$

From this and the fact that $(m')''=0$, we find that

$$0=(\chi e_i')''=(\chi\sigma e_i)''=\chi\sigma\,.\,e_i'' \quad (i=1,\ldots,n).$$

This implies that $\chi\sigma M''=0$ and, as M'' is K-free, we deduce that $\chi\sigma=0$, and consequently $\chi e_i'=0$ $(i=1,\ldots,n)$. Again this shows that $\chi M'=0$; whence $\chi=0$. We now have

$$(n\mu)^2=(1-2\sigma)^2=1-4\sigma+4\sigma^2=1,$$

which proves that n is an invertible element of K. This completes the proof.

REFERENCES

1. H. K. Farahat, On the natural representation of the symmetric groups, *Proc. Glasgow Math. Assoc.* **5** (1962), 121–136.

2. T. Tsuzuku, On decompositions of the permutation representation of a permutation group, *Nagoya Math. J.* **22** (1963), 79–82.

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