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On the critical regularity of nilpotent groups acting on the interval: the metabelian case

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Abstract. Let G be a torsion-free, finitely generated, nilpotent and metabelian group. In this work, we show that G embeds into the group of orientation-preserving $C^{1+\alpha}$-diffeomorphisms of the compact interval for all $\alpha < 1/k$, where k is the torsion-free rank of G/A and A is a maximal abelian subgroup. We show that, in many situations, the corresponding $1/k$ is critical in the sense that there is no embedding of G with higher regularity. A particularly nice family where this happens is the family of $(2n+1)$-dimensional Heisenberg groups, for which we can show that the critical regularity is equal to $1 + 1/n$.

Key words: nilpotent groups, Heisenberg groups, diffeomorphism groups, Hölder continuity, critical regularity
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1. *Introduction*

Given an integer $n \geq 0$ and a real number $\alpha \in [0, 1)$, we denote by $\mathrm{Diff}_+^{n+\alpha}([0, 1])$ the group of all orientation-preserving C^n-diffeomorphisms of the closed interval $[0, 1]$ whose nth derivative is α-Hölder continuous ($C^{n+\alpha}$-diffeomorphisms for short). Observe that, with this notation, the group $\mathrm{Diff}_+^{0}([0, 1])$ is the group of all homeomorphisms of $[0, 1]$ isotopic to the identity. Finally, observe that the family of groups $\mathrm{Diff}_+^{n+\alpha}([0, 1])$ is totally ordered by inclusion because $\mathrm{Diff}_+^{n+\alpha}([0, 1]) \supseteq \mathrm{Diff}_+^{n'+\alpha'}([0, 1])$ if and only if $n < n'$, or $n = n'$ and $\alpha \leq \alpha'$.

We are interested in computing the critical regularity of an abstract group G acting on the interval $[0, 1]$. Recall that, given a group G, the *critical regularity* of G on $[0, 1]$ is, by definition,

$$\mathrm{Crit}_{[0,1]}(G) = \sup\{n + \alpha \mid n \geq 0, \alpha \in [0, 1) \text{ and } G \text{ embeds into } \mathrm{Diff}_+^{n+\alpha}([0, 1])\},$$

where we set $\mathrm{Crit}_{[0,1]}(G) = -\infty$ if G does not embed into $\mathrm{Diff}^0_+([0,1])$. The problem of computing the critical regularity of a group G is quite natural and turns out to be very interesting in the case when G is finitely generated (the reader may wish to consult [15] for an introduction). For example, we know from a theorem of Deroin, Kleptsyn and Navas [8] (see also [7]) that every countable subgroup of $\mathrm{Diff}^0_+([0,1])$ is conjugated to a group of bi-Lipschitz transformations, and hence $1 \leq \mathrm{Crit}_{[0,1]}(G)$ for every countable subgroup of $\mathrm{Diff}^0_+([0,1])$ (for uncountable subgroups of $\mathrm{Diff}^0_+([0,1])$ this is no longer true; see [5]). However, the celebrated stability theorem of Thurston [26] implies that every finitely generated subgroup of $\mathrm{Diff}^1_+([0,1])$ admits a surjective homomorphism onto the integers, and so not every group of homeomorphisms of the interval can be realized as a group of diffeomorphisms. (Concrete examples of finitely generated subgroups of $\mathrm{Diff}^0_+([0,1])$ having trivial abelianization can be found in [2, 24, 26]. However, Thurston's obstruction is not the only obstruction for C^1 smoothability, as there are also known examples of finitely generated and locally indicable groups having no *faithful* C^1 action on the interval; see [3, 5, 16, 19].) Further obstructions appear in higher regularity: for C^2 there is the important Kopell obstruction [17], and between C^1 and C^2 there is the generalized Kopell obstruction from [8]. In a related spirit, Kim and Koberda [14], and later Mann and Wolff [18], have shown that, for every $n \geq 1$ and every α in $[0, 1)$, there is a finitely generated group whose critical regularity on $[0, 1]$ is exactly $n + \alpha$.

In this work, we focus on actions on the interval of finitely generated and torsion-free nilpotent groups (basic definitions will be recalled in §2). Let G be one such group. It follows from the work of Mal'cev that G embeds into $\mathrm{Diff}^0_+([0,1])$ (see, for example, [25, §5.2] and [9, §1.2]), and we know from the work of Farb and Franks [10] that every action of G on $[0, 1]$ by homeomorphisms can be conjugated inside $\mathrm{Diff}^1_+([0,1])$ (see also the universal construction from Jorquera [12]). This was further refined by Parkhe [21] who showed that actually any C^0-action of G on $[0, 1]$ can be conjugated inside $\mathrm{Diff}^{1+\alpha}_+([0,1])$ as long as $\alpha < 1/\tau$, where τ is the degree of the polynomial growth of the nilpotent group G. On the other hand, Plante and Thurston [23] have shown that every nilpotent subgroup of $\mathrm{Diff}^2_+([0,1])$ must be abelian. So, if G is a torsion-free, finitely generated and nilpotent group that is non-abelian, then

$$1 + 1/\tau \leq \mathrm{Crit}_{[0,1]}(G) \leq 2.$$

The exact critical regularity of concrete nilpotent groups has been computed in only few cases and one important goal of this work is to provide new explicit computations of critical regularity for certain groups. Castro, Jorquera and Navas [6] build a family of nilpotent abelian-by-cyclic groups whose critical regularity is two. These examples can be made of arbitrarily large nilpotency degree, yet they are all metabelian (that is their commutator subgroup is abelian). Jorquera, Navas and the second author showed in [13] that the critical regularity of N_4 (the group of 4-by-4 upper triangular matrices with 1 on the diagonal) is $1 + 1/2$. We point out that, at the time of writing this article, N_4 is the only torsion-free nilpotent group whose critical regularity is known and turns out not to be an integer. Note that N_4 is also a metabelian group.

The main purpose of this article is to exhibit many other nilpotent groups whose critical regularity is strictly between one and two. Our main technical result is an improvement of Parkhe's lower bound for the critical regularity in the class of finitely generated, torsion-free nilpotent groups that are metabelian (see Remark 1.1). For the statement, recall that the torsion-free rank of an abelian group H is the dimension of the $\mathbb{Q}$-vector space $H \otimes \mathbb{Q}$. We denote this rank by $\mathrm{rank}(H)$.

THEOREM A. *Let G be a non-abelian, torsion-free, finitely generated nilpotent group that is metabelian, and let A be a maximal abelian subgroup containing $[G, G]$. If $k = \mathrm{rank}(G/A)$, then*

$$G \text{ embeds into } \mathrm{Diff}_+^{1+\alpha}([0, 1]) \quad \text{for all } \alpha < 1/k.$$

In particular, $1 + 1/k \le \mathrm{Crit}_{[0,1]}(G)$.

Remark 1.1. By the Bass–Guivarc'h formula [**1**, **11**], the degree of the polynomial growth of a nilpotent group G is $\tau = \sum_{i\geq 1} i\,\mathrm{rank}(\gamma_i/\gamma_{i+1})$, where $G = \gamma_1 \geqslant \gamma_2 \geqslant \cdots$ is the lower central series of G. In particular, for a nilpotent group G as in Theorem A with maximal abelian subgroup A, we have that $\mathrm{rank}(G/A) < \tau$. Hence, the lower bound for $\mathrm{Crit}_{[0,1]}(G)$ in Theorem A is (strictly) greater than Parkhe's lower bound.

The proof of Theorem A is given in §3. Taking inspiration from the abelian-by-cyclic action from [**6**, §4], in §3.1, we build, for a metabelian and finitely generated torsion-free nilpotent group G, a family of actions of G on the interval $[0, 1]$ by orientation-preserving homeomorphisms. This is done by first building actions of G on $\mathbb{Z}^{k+1}$ that preserve a lexicographic order and then 'projecting' them into the interval. In §3.2, we use the Pixton–Tsuboi technique [**22**, **27**] to show that these actions can be smoothed to actions by $C^{1+\alpha}$-diffeomorphisms for any $\alpha < 1/k$. This section closely follows the work in [**6**], the main difference being that we do not have explicit polynomials in the construction of the actions, but only bounds on them (see Proposition 3.1). Although these actions may not be faithful, in §3.3, we explain how to glue some of these actions to obtain an embedding of G into $\mathrm{Diff}_+^{1+\alpha}([0, 1])$ for any $\alpha < 1/k$.

In some situations, even the lower bound in Theorem A is not sharp in the sense that there are groups for which the theorem applies yet their critical regularity is strictly greater than the predicted lower bound. This is related to the possibility of splitting the group as a product of two groups that each allow an embedding with higher regularity. We provide an easy example of this phenomenon in §4.3. However, in many cases, we can ensure that the inequality in Theorem A is indeed optimal and, in §§4.1 and 4.2, we provide two families of examples where we can obtain upper bounds for the regularity and hence compute the critical regularity.

The first family of examples are the $(2n + 1)$-dimensional discrete Heisenberg groups, which we denote by $\mathscr{H}_n$. Recall that, by definition,

$$\mathscr{H}_n := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^{\,t} & I_n & \vec{y}^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x}, \vec{y} \in \mathbb{Z}^n \quad \text{and} \quad c \in \mathbb{Z} \right\},$$

where I_n is the identity matrix of size n and $\vec{0}^{\,t}$, $\vec{y}^{\,t}$ are the transposes of $\vec{0}$, $\vec{y}$, respectively. It is easy to see that these groups are nilpotent of degree two and hence they are metabelian. Moreover, a maximal abelian subgroup A of $\mathscr{H}_n$ is given by the set of matrices whose

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corresponding vector $\vec{x} = 0$. In particular, $\mathscr{H}_n/A$ has torsion-free rank equal to n. For this family, we show in §4.1 that there is no embedding of $\mathscr{H}_n$ into $\mathrm{Diff}_+^{1+\alpha}([0, 1])$ for $\alpha > 1/n$. In particular, we obtain the following theorem.

THEOREM B. *Let $\mathscr{H}_n$ be the $(2n + 1)$-dimensional discrete Heisenberg group. Then*

$$\mathrm{Crit}_{[0,1]}(\mathscr{H}_n) = 1 + \frac{1}{n}.$$

Finally, in §4.2, we produce examples of metabelian and torsion-free nilpotent groups for which we can compute the critical regularity but whose nilpotency degree can be chosen to be arbitrarily large. More precisely, we show the following theorem.

THEOREM C. *For any integers k and d with $d > k$, there is a nilpotent group G and a maximal abelian subgroup A containing $[G, G]$ such that d is the nilpotency degree of G, k is the torsion-free rank of G/A and*

$$\mathrm{Crit}_{[0,1]}(G) = 1 + \frac{1}{k}.$$

In both cases, the key to obtaining an upper bound for the regularity is to use the internal algebraic structure of the groups to be able to apply the generalized Kopell lemma from [8].

Remark 1.2. We know from the results of Kim and Koberda [14] that, for any real number $\alpha \geq 1$, there is a finitely generated group whose critical regularity is exactly α. However, in all known cases where the critical regularity of a torsion-free nilpotent group has been computed, it is of the form $1 + 1/n$ for some integer n. See Theorems B, C and [6, 13]. So, we wonder whether this is always the case for torsion-free and finitely generated nilpotent groups (not necessarily metabelian).

2. *Preliminaries on nilpotent groups and invariant orders*

Given a group G and two elements $f, g \in G$, we let $[f, g] = fgf^{-1}g^{-1}$ denote the commutator of f and g. Further, if G is finitely generated and S is a finite generating set, an element of the form $[s_1, s_2]$ with $s_1, s_2 \in S$ is called a simple commutator of weight two. Inductively, a *simple commutator of weight n* is defined as an element of the form

$$[s_1, \ldots, s_n] := [s_1, [s_2, \ldots, s_n]], \quad s_1, \ldots, s_n \in S.$$

Note that, given n, there exists only a finite number of simple commutators of weight n.

Let H and K be subgroups of G. $[K, H]$ denotes the subgroup of G generated by commutators $[g, h]$ with $g \in K$ and $h \in H$. The subgroup $[G, G]$ is called the *commutator subgroup* and we say that G is *metabelian* if $[G, G]$ is abelian.

Remember that the *lower central series* of G is

$$G = \gamma_0 \geqslant \gamma_1 \geqslant \gamma_2 \geqslant \cdots,$$

where $\gamma_1 = [G, G]$ and $\gamma_i = [G, \gamma_{i-1}]$; and the *upper central series* of G is

$$\{e\} = \zeta_0 \leqslant \zeta_1 \leqslant \zeta_2 \leqslant \cdots,$$

where $\zeta_i/\zeta_{i-1} = Z(G/\zeta_{i-1})$, and $Z(G)$ denotes the center of G.

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We recall some classic results about nilpotent groups. See [25] for an in-depth exposition of them. The group G is *nilpotent* of degree n if $\zeta_n = G$ but $\zeta_{n-1} \neq G$. In this case, it also happens that $\gamma_n = \{e\}$ but $\gamma_{n-1} \neq \{e\}$. Therefore, in a finitely generated nilpotent group of degree n, we only have a finite number of simple commutators (for a fixed generating set). This is because all simple commutators of weight n are trivial.

It is a result of Mal'cev that, if G is a torsion-free nilpotent group, then the factors ζ_i/ζ_{i-1} are also torsion free for all $i \in \{1, \ldots, n\}$ (see [25, Proposition 5.2.19]). Recall also that finitely generated nilpotent groups are polycyclic, and hence every subgroup of a finitely generated nilpotent group is finitely generated as well (see [25, Proposition 5.4.12]). In addition, nilpotent groups also satisfy that their non-trivial normal subgroups always intersect non-trivially the center of the group (see [25, Proposition 5.2.1]). An immediate consequence of this is the following useful result.

PROPOSITION 2.1. *Let G be a nilpotent group, let H be a group and let $\varphi : G \to H$ be a group homomorphism. Then φ is injective if and only if $\varphi\mid_{Z(G)}$ (the restriction of φ to $Z(G)$) is injective.*

For $g \in G$, $\mathrm{Centr}(g) = \{h \in G \mid gh = hg\}$ denotes the centralizer of g. The following proposition, although elementary, will be very important for building actions of G on $\mathbb{Z}^{k+1}$ in §3.1.

PROPOSITION 2.2. *Let G be a torsion-free and finitely generated nilpotent group that is metabelian.*

- *Given $g \in G$ and $0 \neq m \in \mathbb{Z}$, we have that* $\mathrm{Centr}(g^m) = \mathrm{Centr}(g)$.
- *Let $A \leqslant G$ be a maximal abelian subgroup. If A is normal in G, then G/A is torsion free.*

Proof. Assume that $f, g \in G$ and $m \in \mathbb{Z}$ are such that $[f, g^m] = e$. Define $H := \langle f, g\rangle$, the subgroup generated by f and g. Since $[g, g^m] = [f, g^m] = e$, we have that $g^m \in Z(H)$, and, since $H/Z(H)$ is torsion free (see [25, Proposition 5.2.19]), we have that $g \in Z(H)$. Therefore, $\mathrm{Centr}(g^m) \subseteq \mathrm{Centr}(g)$ (the other inclusion is obvious).

The second point follows from the first. Let A be a maximal abelian subgroup that is also normal and assume that G/A is not torsion free. Suppose $g \in G$ is such that $g \notin A$ but $g^m \in A$ for some $m \neq 0$. Then, since A is abelian, we have that $A \subseteq \mathrm{Centr}(g^m) = \mathrm{Centr}(g)$. In particular, $\langle A, g\rangle$, the group generated by A and g, is an abelian subgroup larger than A, which contradicts our assumption. □

2.1. *On the action of G/A on A.* Let G be a torsion-free and finitely generated nilpotent group of degree n that is also metabelian. Let A be a maximal abelian subgroup containing $[G, G]$ (in particular, it is normal). In view of Proposition 2.2, we have that G is an extension of $\mathbb{Z}^k$ by $\mathbb{Z}^d$,

$$1 \longrightarrow \mathbb{Z}^d \longrightarrow G \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where $A \simeq \mathbb{Z}^d$ and $G/A \simeq \mathbb{Z}^k$. In this section, we study the natural action of G/A on A coming from the conjugacy action of G on A.

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Let $\{g_1, \ldots, g_d\}$ and $\{f_1A, \ldots, f_kA\}$ be generating sets of A and G/A, respectively. Since A is normal, the subgroup of G generated by $f_1, \ldots, f_k$ acts on A by automorphisms yielding a homomorphism

$$\langle f_1, \ldots, f_k\rangle \longrightarrow \mathrm{Aut}(\mathbb{Z}^d).$$

Therefore, the action of each $f \in \langle f_1, \ldots, f_k\rangle$ is given by a matrix $A_f \in GL_d(\mathbb{Z})$, which depends on the set $\{g_1, \ldots, g_d\}$. We call A_f the *conjugacy matrix* of f. In the special case of the generators $f_1, \ldots, f_k$, we denote the conjugacy matrix of f_i simply by A_i.

In the next lemma, we will see that we can always choose a generating set of A such that the conjugacy matrices of the elements $f_1, \ldots, f_k$ belong to $U_d(\mathbb{Z})$, the group of upper triangular matrices with 1 in the diagonal. This is due to Mal'cev in the case where the matrix coefficients belong to a field. We write a direct proof in our special case. For the proof, we say that a generating set of a group is *minimal* if it has least possible cardinality.

LEMMA 2.3. *Let $A \leqslant G$ be a maximal abelian subgroup satisfying that $[G, G] \subseteq A$. Suppose that $\mathbb{Z}^d \simeq A$ and $\mathbb{Z}^k \simeq G/A = \langle f_1A, \ldots, f_kA\rangle$. Then there exists a generating set $\{g_1, \ldots, g_d\}$ of A such that the conjugacy matrices of the elements $f_1, \ldots, f_k$ belong to $U_d(\mathbb{Z})$. In particular, the nilpotency degree of G is bounded by $d + 1$.*

Proof. Since G is nilpotent of degree n, the upper central series

$$\{e\} = \zeta_0 \leqslant \zeta_1 \leqslant \cdots \leqslant \zeta_n = G,$$

is finite. Remember that all the factors ζ_i/ζ_{i-1} are torsion free. Combining this with the fact that G/A is also torsion free (see Proposition 2.2), we have, for $g \in G$, that

$$g^j \in \zeta_i \cap A \Rightarrow g \in \zeta_i \cap A \quad \text{for all } i \in \{0, \ldots, n\},\ j \in \mathbb{Z}. \tag{2.1}$$

Define $\Gamma_i := \zeta_i \cap A$ and let m be the smallest element in $\{1, \ldots, n\}$ such that $\Gamma_m = A$. This yields the filtration

$$\{e\} = \Gamma_0 \leqslant \Gamma_1 \leqslant \cdots \leqslant \Gamma_m = A,$$

such that

$$[G, \Gamma_i] \subseteq \Gamma_{i-1}, \tag{2.2}$$

and, by (2.1), it also has the property that each factor Γ_{i+1}/Γ_i is torsion-free abelian.

Note that, if $\Gamma_{m-1} \simeq \mathbb{Z}^{n_{m-1}}$ and $\Gamma_m/\Gamma_{m-1} \simeq \mathbb{Z}^{n_m}$, then, since $\Gamma_m = A \simeq \mathbb{Z}^d$ is abelian, we have that $d = n_{m-1} + n_m$. Therefore, if $\{g_1, \ldots, g_{n_{m-1}}\}$ and $\{g_{n_{m-1}+1}\Gamma_{m-1}, \ldots, g_{n_{m-1}+n_m}\Gamma_{m-1}\}$ are minimal generating sets of Γ_{m-1} and Γ_m/Γ_{m-1}, respectively, then $\{g_1, \ldots, g_{n_{m-1}}, g_{n_{m-1}+1}, \ldots, g_d\}$ is a minimal generating set of $\Gamma_m = A$.

Recursively, we obtain a minimal generating set $\{g_1, \ldots, g_d\}$ of A which, by (2.2), has the property that, for $g_s \in \{g_1, \ldots, g_d\} \cap \Gamma_i$, it holds that $[f_j, g_s] \in \Gamma_{i-1} \subseteq \langle g_1, \ldots, g_{s-1}\rangle$ for all $j \in \{1, \ldots, k\}$. In other words, the conjugacy matrices of each $f \in \{f_1, \ldots, f_k\}$ belong to $U_d(\mathbb{Z})$.

The fact that G has nilpotency degree bounded by $d + 1$ follows from the fact that $U_d(\mathbb{Z})$ has nilpotency degree $d + 1$. □

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2.2. *Invariant orders and their dynamical versions.* We close these preliminaries with the concepts of order and dynamical realization. A group G is left-orderable if it admits a total order relationship, say, $\preceq$, which is invariant under multiplication from the left: that is, if $f \preceq g$, then $hf \preceq hg$ for all $h \in G$. An important family of left-orderable groups is that of finitely generated and torsion-free abelian groups. Indeed, we will repetitively use the *lexicographic order* of $\mathbb{Z}^n$ defined by

$$(i_1, \ldots, i_n) \prec (i'_1, \ldots, i'_n) \Leftrightarrow \text{there exists } k \in \{1, \ldots, n\} \quad \text{such that}$$
$$i_k < i'_k \quad \text{and} \quad i_s = i'_s \quad \text{for } s < k. \tag{2.3}$$

What is important for this work is that a countable group is left-orderable if and only if it embeds into $\mathrm{Diff}^0_+(\mathbb{R})$ (see [20, §2] or [9, §1.1.3] for details). Since $\mathrm{Diff}^0_+(\mathbb{R})$ is isomorphic to $\mathrm{Diff}^0_+([0, 1])$, left-orderability of a countable group is equivalent to being isomorphic to a subgroup of $\mathrm{Diff}^0_+([0, 1])$. More generally, given a group G that acts on a countable and totally ordered set $(\Omega, \preceq)$ by order-preserving bijections, say, $\omega \mapsto g(\omega)$, for $g \in G$ and $\omega \in \Omega$, then there is a *dynamical realization* of this action. This means that there is an order-preserving map $i : (\Omega, \preceq) \to ([0, 1], \leq)$ and a homomorphism $\psi : G \to \mathrm{Diff}^0_+([0, 1])$ satisfying that $\psi(g)(i(\omega)) = i(g(\omega))$ for every $\omega \in \Omega$ and every $g \in G$. See [4, Lemma 2.40] for a proof. Clearly, ψ is an embedding whenever the G action on Ω is faithful.

3. *Proof of Theorem A*

Throughout this section, G will denote a non-abelian, torsion-free, finitely generated nilpotent group that is metabelian, and A will denote a maximal abelian subgroup containing $[G, G]$ (in particular, it is normal). Recall that then G is an extension of $G/A \simeq \mathbb{Z}^k$ by $A \simeq \mathbb{Z}^d$ (see §2.1). In particular, the nilpotency degree of G is bounded by $d + 1$ (see Lemma 2.3).

3.1. *An action of G on a totally ordered set.*

PROPOSITION 3.1. *Fix a generating set $\{g_1, \ldots, g_d, f_1, \ldots, f_k\}$ of G such that $\{g_1, \ldots, g_d\}$ is a generating set of A given by Lemma 2.3 and $\langle f_1 A, \ldots, f_k A\rangle = G/A \simeq \mathbb{Z}^k$. Then, for a fixed $s \in \{1, \ldots, d\}$, there is an action of G on $\mathbb{Z}^{k+1}$ that satisfies the following.*

(1) *For all $m \in \{1, \ldots, d\}$ and all $t \in \{1, \ldots, k\}$, there exist functions $\ell_t, r_m : \mathbb{Z}^k \to \mathbb{Z}$, such that*

$$f_t \cdot (i_1, .., i_t, .., i_k, j) = (i_1, .., i_t + 1, .., i_k, j + \ell_t(i_1, \ldots, i_k)),$$
$$g_m \cdot (i_1, \ldots, i_k, j) = (i_1, \ldots, i_k, j + r_m(i_1, \ldots, i_k)).$$

In particular, the action of G on $\mathbb{Z}^{k+1}$ preserves the lexicographic order. In addition, $r_s \equiv 1$ and $r_1 = r_2 = \cdots = r_{s-1} \equiv 0$.

(2) *There exists a positive constant M such that, for all $t \in \{1, \ldots, k\}$, $m \in \{1, \ldots, d\}$ and $(i_1, \ldots, i_k) \neq (0, \ldots, 0)$,*

$$|\ell_t(i_1, \ldots, i_k)| \leq M(|i_1| + \cdots + |i_k|)^d, \quad |r_m(i_1, \ldots, i_k)| \leq M(|i_1| + \cdots + |i_k|)^d.$$

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Proof. We start by showing item 1. To this end, fix $s \in \{1, \ldots, d\}$ and consider the subgroup $H_s = \langle \{g_1, \ldots, g_d\} \setminus \{g_s\} \rangle$. Since the sets $\{f_1^{i_1} \cdots f_k^{i_k} A : i_1, \ldots, i_k \in \mathbb{Z}\}$ and $\{g_s^j H_s : j \in \mathbb{Z}\}$ are partitions of G and A, respectively, the coset space can be described by the *normal forms*

$$G/H_s = \{f_1^{i_1} \cdots f_k^{i_k} g_s^j H_s \ : \ i_1, \ldots, i_k, j \in \mathbb{Z}\}. \tag{3.1}$$

Hence, we can identify G/H_s with $\mathbb{Z}^{k+1}$ (as sets) by identifying $f_1^{i_1} \cdots f_k^{i_k} g_s^j H_s$ with $(i_1, \ldots, i_k, j)$. In particular, the left-multiplication action of G on G/H_s provides an action of G on $\mathbb{Z}^{k+1}$. This is the action that we want to consider.

Now, by Lemma 2.3, we have that, for all $i, j \in \{1, \ldots, k\}$ and $l \in \{1, \ldots, d\}$, it holds that

$$f_i f_j \in f_j f_i \langle g_1, \ldots, g_d \rangle \quad \text{and} \quad g_l f_j \in f_j g_l \langle g_1, \ldots, g_{l-1} \rangle.$$

Therefore, for $t \in \{1, \ldots, k\}$, the action of f_t is addition by 1 on the t coordinate and the action on the $k+1$ coordinate depends on previous k coordinates, and hence the function ℓ_t. The function r_m, for $m \in \{1, \ldots, d\}$, can be found analogously. Finally, as the maps ℓ_t and r_m depend only on the coordinates $(i_1, \ldots, i_k)$, the reader can easily verify that the G action on $\mathbb{Z}^{k+1}$ preserves the lexicographic order.

Now we check item 2. Let $t \in \{1, \ldots, k\}$. Recall that the action of f_t on $\mathbb{Z}^{k+1}$ is nothing but the left-multiplication action of f_t on G/H_s. Hence, in order to compute the image of $f_1^{i_1} \cdots f_t^{i_t} \cdots f_k^{i_k} g_s^j H_s$ under f_t, we need to multiply and find the representative in normal form (3.1). To do this, observe that $f_t f_j = [f_t, f_j] f_j f_t$. Hence, bringing f_t to the tth position generates at most $|i_1| + \cdots + |i_k|$ simple commutators of weight two, which we now need to move to the rightmost place (i.e. after the $f_k^{i_k}$ but before g_s^j). Since G is metabelian, the commutators commute with each other. So, moving them all to the rightmost place generates at most $(|i_1| + \cdots + |i_k|)^2$ simple commutators of weight three. Analogously, moving them all to the rightmost place, we have at most $(|i_1| + \cdots + |i_k|)^3$ simple commutators of weight four, and so on. Since G has nilpotency degree bounded by $d+1$, all simple commutators of this weight are trivial (see Lemma 2.3). Therefore, repeating the previous argument $d+1$ times gives

$$f_t.(f_1^{i_1} \cdots f_t^{i_t} \cdots f_k^{i_k} g_s^j H_s) = f_1^{i_1} \cdots f_t^{i_t+1} \cdots f_k^{i_k} g g_s^j H_s,$$

where $g \in A$ is the product of at most

$$\sum_{i=1}^{d} (|i_1| + \cdots + |i_k|)^i \leq d(|i_1| + \cdots + |i_k|)^d$$

simple commutators. Now note that

$$g.g_s^j H_s = g_s^{\ell_t(i_1,\ldots,i_k)} g_s^j H_s,$$

since $\ell_t(i_1, \ldots, i_k)$ agrees with the exponent of g_s in the expression of g over the generators $g_1, \ldots, g_d$. Therefore, letting $\mathcal{S} \subseteq A$ be the set of all simple commutators of G (which is finite), and defining

$$\lambda := \max\{|m_s| : \text{there exists } m_1, \ldots, m_d \text{ for which } (g_1^{m_1} \cdots g_s^{m_s} \cdots g_d^{m_d}) \in \mathcal{S}\},$$

we see that $\ell_t(i_1, \dots, i_k)$ is bounded by λ times the number of simple commutators that were used to write g. Hence,

$$|\ell_t(i_1, \dots, i_k)| \leq \lambda d(|i_1| + \cdots + |i_k|)^d.$$

Analogous computations give the inequality for the functions r_m. $\square$

Remark 3.2. Note that the action built in Proposition 3.1 is not necessarily faithful. However, it is such that the elements $g_1, \dots, g_{s-1}$ act trivially and $g_s(i_1, \dots, i_k, j) = (i_1, \dots, i_k, j+1)$. This will be used in §3.3 to build a faithful action.

3.2. *Action by diffeomorphisms of* $[0, 1]$. For a fixed $s \in \{1, \dots, d\}$, Proposition 3.1 builds an action of G on $\mathbb{Z}^{k+1}$ that preserves the lexicographic order. Hence, we can consider the dynamical realization of this action (see the beginning of §2.2) to get a G-action by orientation-preserving homeomorphisms of $[0, 1]$.

Now, since the group is nilpotent and we have good control from the polynomials appearing in Proposition 3.1, we will see that this action can actually be smoothed to an action by diffeomorphisms of $[0, 1]$. For this, we need the following result from Pixton and Tsuboi [**22**, **27**]. See the proof of Proposition 1.2 in [**27**] for details.

LEMMA 3.3. *There exists a family of C^∞-diffeomorphisms $\varphi_{I',I}^{J',J} : I \to J$, ranging over all bounded intervals I, I', J, J' of $\mathbb{R}$, where I' (respectively, J') is adjacent to I (respectively, J) by the left, such that:*

(1) *for all I, I', J, J', K, K' as above,*
$$\varphi_{J',J}^{K',K} \circ \varphi_{I',I}^{J',J} = \varphi_{I',I}^{K',K};$$

(2) *for all I, I', J, J',*
$$D\varphi_{I',I}^{J',J}(x_-) = \frac{|J'|}{|I'|} \quad \text{and} \quad D\varphi_{I',I}^{J',J}(x_+) = \frac{|J|}{|I|},$$
where x_- (respectively, x_+) is the left (respectively, right) endpoint of I;

(3) *there is a constant M such that, for all I, I', J, J' as above, and all $x \in I$,*
$$D\log(D\varphi_{I',I}^{J',J})(x) \leq \frac{M}{|I|}\left|\frac{|I||J'|}{|J||I'|} - 1\right|,$$
provided that $\max\{|I|, |I'|, |J|, |J'|\} \leq 2\min\{|I|, |I'|, |J|, |J'|\}$; *and*

(4) *given $I, I', J, J', K, K', L, L'$, as above, then*
$$|\log(D\varphi_{I',I}^{K',K})(x) - \log(D\varphi_{J',J}^{L',L})(y)| \leq \left|\log\frac{|K||J|}{|I||L|}\right| + \left|\log\frac{|K'||I|}{|I'||K|}\right| + \left|\log\frac{|L'||J|}{|J'||L|}\right|,$$
for all $x \in I$, $y \in J$.

Now, let $\{I_{i_1,\dots,i_k,j} : (i_1, \dots, i_k, j) \in \mathbb{Z}^{k+1}\}$ be a family of intervals whose disjoint union is dense in $[0, 1]$ and that are disposed preserving the lexicographic order of $\mathbb{Z}^{k+1}$. We identify the generators $g_1, \dots, g_d, f_1, \dots, f_k$ from Lemma 2.3 with elements in $\mathrm{Diff}_+^0([0, 1])$ as follows: f_t and g_s will be homeomorphisms of $[0, 1]$ whose restriction

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to $I_{i_1,\ldots,i_k,j}$ coincides, respectively, with

$$\varphi_{I_{i_1,\ldots,i_k,j-1},\, I_{i_1,\ldots,i_k,j}}^{I_{i_1,\ldots,i_t+1,\ldots,i_k,j+\ell_t(i_1,\ldots,i_k)-1},\, I_{i_1,\ldots,i_t+1,\ldots,i_k,j+\ell_t(i_1,\ldots,i_k)}} \quad \text{and} \quad \varphi_{I_{i_1,\ldots,i_k,j-1},\, I_{i_1,\ldots,i_k,j}}^{I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)-1},\, I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)}},$$

for $t \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, d\}$. Thus, by (1) in Lemma 3.3, we have a group homomorphism $G \to \mathrm{Diff}_+^0([0, 1])$. The main technical step for proving Theorem A is the following proposition.

PROPOSITION 3.4. *Given $\alpha < 1/k$, there is a choice of lengths of the intervals $|I_{i_1,\ldots,i_k,j}|$ such that the homeomorphisms $f_1, \ldots, f_k, g_1, \ldots, g_d$ are diffeomorphisms of class $C^{1+\alpha}$.*

The rest of §3.2 is devoted to the proof of Proposition 3.4. We assume that $k \geq 2$ since, after Condition (3) in Proposition 3.1, we can use the estimates from [6, §4] to ensure that, when $k = 1$, the action is by $C^{1+\alpha}$ diffeomorphisms for any $\alpha < 1$.

So let $k \geq 2$ and consider $\alpha < 1/k$. Choose positive real numbers $p_1, \ldots, p_k, r$ such that, for all $n \in \{1, \ldots, k\}$:

(I) $\alpha + r \leq 2$;
(II) $d(r - 1) \leq (1 - \alpha)$;
(III) $2dr \leq p_n$;
(IV) $2d \leq p_n(1 - \alpha)$;
(V) $1/p_1 + \cdots + 1/p_k + 1/r < 1$; and
(VI) $\alpha \leq 1/p_n + 1/r$ and $\alpha \leq r/p_n(r - 1)$.

For example, one can take $p_1 = \cdots = p_k = 3d/\alpha$ and $r = 3d/(3d - 1)$.

Now define the lengths of the intervals $I_{i_1,\ldots,i_k,j}$ as

$$|I_{i_1,\ldots,i_k,j}| = \frac{1}{|i_1|^{p_1} + \cdots + |i_k|^{p_k} + |j|^r + 1}.$$

From condition (V,) it follows that $\sum |I_{i_1,\ldots,i_k,j}| < \infty$, and hence this family of intervals can be disposed on a finite interval respecting the lexicographic order. After renormalization, we can assume that this interval is [0, 1].

Following [13], we say that two real-valued functions f and g satisfy $f \prec g$ if there is a constant $M > 0$ such that $|f(x)| \leqslant Mg(x)$ for all x. We also write $f \asymp g$ if $f \prec g$ and $g \prec f$.

Let θ be a non-negative C^2 real-valued function satisfying $\theta(\xi) = |\xi|^r$ for $|\xi| \geqslant 1$ and $\theta(0) = 0$. Consider the auxiliary functions (C^2 with respect to ξ):

- $\psi(i_1, \ldots, i_k, \xi) := 1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k} + \theta(\xi)$; and
- $\Psi_{i_1,\ldots,i_k}(\xi) := \log(\psi(i_1, \ldots, i_k, \xi))$.

LEMMA 3.5. *Let $S = 1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k}$. Given $C > 0$, there exists a positive constant M such that the inequality*

$$\frac{1}{M}\psi(i_1, \ldots, i_k, j) \leq \psi(i_1, \ldots, i_k, \xi) \leq M\psi(i_1, \ldots, i_k, j),$$

holds for any ξ satisfying $|\xi - j| \leq C(S^{1/r} + (|i_1| + \cdots + |i_k|)^d)$.

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We remark that, in the situation of Lemma 3.5, we will still use the notation $\psi(i_1,\ldots,i_k,j) \asymp \psi(i_1,\ldots,i_k,\xi)$. Even if this is a slight abuse of notation, it is justified by comparing the functions ψ_j and ψ_ξ defined by $\psi_\xi(i_1,\ldots,i_k) = \psi(i_1,\ldots,i_k,\xi)$ and $\psi_j(i_1,\ldots,i_k) = \psi(i_1,\ldots,i_k,j)$ whenever the inequality $|\xi - j| \leq C(S^{1/r} + (|i_1| + \cdots + |i_k|)^d)$ holds.

Proof of Lemma 3.5. By symmetry, it is enough to show that $\psi(i_1,\ldots,i_k,\xi)/\psi(i_1,\ldots,i_k,j)$ is bounded above. For this we note that

$$\begin{aligned}\frac{\psi(i_1,\ldots,i_k,\xi)}{\psi(i_1,\ldots,i_k,j)} &\prec \frac{1+|i_1|^{p_1}+\cdots+|i_k|^{p_k}+|j|^r+|\xi-j|^r}{\psi(i_1,\ldots,i_k,j)}\\ &\prec 1+\frac{S+(|i_1|+\cdots+|i_k|)^{dr}}{\psi(i_1,\ldots,i_k,j)}\\ &\prec 2+\frac{(|i_1|+\cdots+|i_k|)^{dr}}{\psi(i_1,\ldots,i_k,j)},\end{aligned}$$

where we repeatedly use the inequality $|x+y|^a \prec |x|^a + |y|^a$, which holds for any $a > 0$. Now, notice that the last expression is bounded. Indeed, since $(|i_1| + \cdots + |i_k|)^{dr} \prec |i_1|^{dr} + \cdots + |i_k|^{dr}$, it is enough to observe that, for each $n \in \{1,\ldots,k\}$,

$$|i_n|^{dr} \leq (\psi(i_1,\ldots,i_k,j))^{dr/p_n} \leq \psi(i_1,\ldots,i_k,j),$$

which holds due to condition (III). □

3.2.1. *The maps g_s are $C^{1+\alpha}$-diffeomorphisms.* We start the proof of Proposition 3.4 by showing that the maps g_s, for $s \in \{1,\ldots,d\}$, are of class $C^{1+\alpha}$. That is, we want to show that g_s is a C^1-diffeomorphism and that there is a constant $C > 0$ such that

$$\frac{|\log Dg_s(x) - \log Dg_s(y)|}{|x-y|^\alpha} \leq C \quad \text{for all different } x, y \in [0,1].$$

To check this, it is enough to find a uniform C, as above, for points x, y in $\bigcup_j I_{i_1,\ldots,i_k,j}$ (independent of $i_1,\ldots,i_k$). Indeed, after condition (2) in Lemma 3.3 and the definition of g_s, it follows that g_s has derivative 1 at the end points of the intervals $\bigcup_j I_{i_1,\ldots,i_k,j}$. Hence the conditions from [**20**, Lemma 4.1.22] are satisfied, and therefore we obtain that the g_s are of class $C^{1+\alpha}$.

Case 1: The points x, y belong to the same $I := I_{i_1,\ldots,i_k,j}$. Condition (3) in Lemma 3.3 provides a Lipschitz constant for $\log(Dg_s)$. So it is enough to bound

$$\frac{1}{|I|^\alpha}\left|\frac{|I||J'|}{|J||I'|} - 1\right|,$$

where $I' = I_{i_1,\ldots,i_k,j-1}$, $J = I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)}$ and $J' = I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)-1}$.

We will, in fact, bound the following (*a posteriori*) asymptotically equivalent expression

$$\frac{1}{|I|^\alpha} \log \frac{|I||J'|}{|J||I'|}.$$

For this, notice that $\log |I||J'|/|J||I'|$ is equal to

$$\Psi_{i_1,\ldots,i_k}(j + r_s(i_1,\ldots,i_k)) - \Psi_{i_1,\ldots,i_k}(j + r_s(i_1,\ldots,i_k) - 1) - (\Psi_{i_1,\ldots,i_k}(j) - \Psi_{i_1,\ldots,i_k}(j-1)).$$

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So, applying the mean value theorem first to the function $x \mapsto \Psi_{i_1,\ldots,i_k}(j+1+x) - \Psi_{i_1,\ldots,i_k}(j+x)$ and then to the function $x \mapsto D\Psi_{i_1,\ldots,i_k}(x)$ gives

$$\left|\log \frac{|I||J'|}{|J||I'|}\right| = |r_s(i_1,\ldots,i_k)||D^2(\Psi_{i_1,\ldots,i_k})(\xi)|, \tag{3.2}$$

where ξ is a point in the convex hull of $\{j-1, j, j-1+r_s, j+r_s\}$. We find an upper bound for $|D^2(\Psi_{i_1,\ldots,i_k})(\xi)|$. Since $D\theta$ and $D^2\theta$ are bounded on $[-1,1]$, and

$$D^2(\Psi_{i_1,\ldots,i_k})(\xi) = \frac{D^2\theta(\xi)}{\psi(i_1,\ldots,i_k,\xi)} - \frac{(D\theta(\xi))^2}{(\psi(i_1,\ldots,i_k,\xi))^2},$$

we have that

$$D^2(\Psi_{i_1,\ldots,i_k})(\xi) \prec \frac{1}{\psi(i_1,\ldots,i_k,\xi)}$$

for all $\xi \in [-1,1]$. On the other hand, for $\xi \notin [-1,1]$, we have that $\theta(\xi) = |\xi|^r$. So, since $|\xi|^{r-2} < 1$ and $|\xi|^r/\psi(i_1,\ldots,i_k,\xi) < 1$, it follows that

$$D^2(\Psi_{i_1,\ldots,i_k})(\xi) \prec \frac{|\xi|^{r-2}}{\psi(i_1,\ldots,i_k,\xi)} \prec \frac{1}{\psi(i_1,\ldots,i_k,\xi)}. \tag{3.3}$$

Now, going back to equation (3.2) and using (3) of Proposition 3.1,

$$\log \frac{|I||J'|}{|J||I'|} \prec \frac{|i_1|^d + \cdots + |i_k|^d}{\psi(i_1,\ldots,i_k,\xi)}.$$

Note that, for all $n \in \{1,\ldots,k\}$, condition (IV) yields

$$|i_n|^d \leq (\psi(i_1,\ldots,i_k,\xi))^{d/p_n} \leq (\psi(i_1,\ldots,i_k,\xi))^{(1-\alpha)}.$$

Finally, thanks to the fact that ξ belongs to the convex hull of $\{j-1, j, j-1+r_s, j+r_s\}$, we use the bounds of r_s from Proposition 3.1 to apply Lemma 3.5 and conclude that

$$\frac{1}{|I|^\alpha}\log \frac{|I||J'|}{|J||I'|} \prec \frac{(\psi(i_1,\ldots,i_k,\xi))^{-\alpha}}{|I|^\alpha} \prec \frac{(\psi(i_1,\ldots,i_k,j))^{-\alpha}}{|I|^\alpha} = 1,$$

as desired.

Case 2: The point x belongs to $I_{i_1,\ldots,i_k,j}$ and y belongs to $I_{i_1,\ldots,i_k,j'}$. We assume, without loss of generality, that $j < j'$. Condition (4) of Lemma 3.3 tells us that $|\log Dg_s(x) - \log Dg_s(y)|$ is bounded above by

$$\left|\log \frac{|I_{i_1,\ldots,i_k,j+r_s}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j}||I_{i_1,\ldots,i_k,j'+r_s}|}\right| + \left|\log \frac{|I_{i_1,\ldots,i_k,j+r_s-1}||I_{i_1,\ldots,i_k,j}|}{|I_{i_1,\ldots,i_k,j-1}||I_{i_1,\ldots,i_k,j+r_s}|}\right| + \left|\log \frac{|I_{i_1,\ldots,i_k,j'+r_s-1}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j'-1}||I_{i_1,\ldots,i_k,j'+r_s}|}\right|.$$

The estimates in Case 1 allow us to control the last two terms (divided by $|x-y|^\alpha$), and thus we only need to bound the first term. So we look for a uniform bound for

$$\frac{1}{|x-y|^\alpha}\left|\log \frac{|I||J'|}{|I'||J|}\right|, \tag{3.4}$$

where $I = I_{i_1,\ldots,i_k,j}$, $I' = I_{i_1,\ldots,i_k,j'}$, $J = I_{i_1,\ldots,i_k,j+r_s}$ and $J' = I_{i_1,\ldots,i_k,j'+r_s}$. Assume that j, j' are positive (the case where both are negative follows by symmetry, and if they have different sign, it suffices to consider an intermediate comparison with the term corresponding to $j'' = 0$). Assume, further, that $j' - j \geq 2$ (the case where $j' - j = 1$ follows from Case 1, passing through the point that separates the intervals and using the triangle inequality). Again, applying the mean value theorem first to the function $x \mapsto \Psi_{i_1,\ldots,i_k}(j + 1 + x) - \Psi_{i_1,\ldots,i_k}(j + x)$ and then to the function $x \mapsto D\Psi_{i_1,\ldots,i_k}(x)$ gives

$$\left|\log \frac{|I||J'|}{|I'||J|}\right| = |j - j'| \cdot |r_s(i_1, \ldots, i_k)| \cdot |D^2(\Psi_{i_1,\ldots,i_k})(\xi)|, \tag{3.5}$$

for a certain ξ in the convex hull of $\{j, j', j + r_s, j' + r_s\}$.

We start by bounding $|x - y|^{-\alpha}$. For this, note that, by Case 1 and the triangle inequality, we can (and will) assume that x is the left endpoint of I and y is the right endpoint of I'. This yields

$$\frac{1}{|x - y|^\alpha} = \left(\frac{1}{\sum_{\ell=j}^{j'} |I_{i_1,\ldots,i_k,\ell}|}\right)^\alpha \leq \left(\frac{1}{|j - j'||I_{i_1,\ldots,i_k,j'}|}\right)^\alpha,$$

where the last inequality holds because $|I_{i_1,\ldots,i_k,j'}| < |I_{i_1,\ldots,i_k,\ell}|$ for $\ell < j'$. Note that if, in addition, $|j' - j| \leq C(S^{1/r} + (|i_1| + \cdots + |i_k|)^d)$, for some $C > 0$, we can use Lemma 3.5 to compare $|I|$ with $|I'|$, and we eventually obtain the inequality

$$\frac{1}{|x - y|^\alpha} \prec \left(\frac{1}{|j - j'||I_{i_1,\ldots,i_k,j}|}\right)^\alpha. \tag{3.6}$$

We now exhibit a bound for (3.4). We consider three separate cases. Let M be the constant in Proposition 3.1.

(i) The integers j, j' belong to $[0, 2M(|i_1| + \cdots + |i_k|)^d]$. Since $\xi \in \operatorname{conv}\{j, j', j + r_s, j' + r_s\}$, it follows from (3.3) and Lemma 3.5 that

$$|D^2(\Psi_{i_1,\ldots,i_k})(\xi)| \prec \frac{1}{\psi(i_1, \ldots, i_k, \xi)} \asymp \frac{1}{\psi(i_1, \ldots, i_k, j)}.$$

Furthermore, we have that

$$|j - j'||r_s(i_1, \ldots, i_k)| \prec (|i_1| + \cdots + |i_k|)^{2d} \prec (\psi(i_1, \ldots, i_k, j))^{1-\alpha},$$

where the last inequality holds by condition (IV). If we combine this with (3.5), (3.6), we conclude that

$$\frac{1}{|x - y|^\alpha}\left|\log \frac{|I||J'|}{|I'||J|}\right| \prec \frac{1}{|I|^\alpha} \frac{(\psi(i_1, \ldots, i_k, j))^{1-\alpha}}{\psi(i_1, \ldots, i_k, j)} = 1.$$

(ii) The integers j, j' belong to (the constant k^d is just to ensure that the interval is non-empty) $[2M(|i_1| + \cdots + |i_k|)^d, 2Mk^d S^{1/r}]$. Similarly to i), the reader can check that we are in the hypotheses of Lemma 3.5 and that $|\xi| \geq M(|i_1| + \cdots +$

$|i_k|)^d$. Therefore, by (3.3), (3.5) and (3.6), we get

$$\frac{1}{|x-y|^\alpha}\left|\log\frac{|I||J'|}{|I'||J|}\right| \prec \left(\frac{1}{|j-j'||I_{i_1,\ldots,i_k,j}|}\right)^\alpha |j'-j| \frac{(|i_1|+\cdots+|i_k|)^d|\xi|^{r-2}}{\psi(i_1,\ldots,i_k,\xi)}$$
$$\prec |j'-j|^{1-\alpha}\frac{(|i_1|+\cdots+|i_k|)^{d(r-1)}}{\psi(i_1,\ldots,i_k,j)^{1-\alpha}}.$$

To prove that this last expression is bounded, it is enough to show that

$$|j'-j|^{1-\alpha}(|i_1|+\cdots+|i_k|)^{d(r-1)} \prec \psi(i_1,\ldots,i_k,j)^{1-\alpha}.$$

Since $j'-j \le 2Mk^dS^{1/r}$, it follows that

$$|j'-j|^{1-\alpha}(|i_1|+\cdots+|i_k|)^{d(r-1)} \prec (1+|i_1|^{p_1}+\cdots+|i_k|^{p_k})^{(1-\alpha)/r}$$
$$\times (|i_1|+\cdots+|i_k|)^{d(r-1)},$$

so it suffices to prove that, given $n, m \in \{1,\ldots,k\}$,

$$|i_n|^{p_n(1-\alpha)/r}|i_m|^{d(r-1)} \prec (\psi(i_1,\ldots,i_k,j))^{1-\alpha}. \tag{3.7}$$

However, note that

$$|i_n|^{p_n(1-\alpha)/r}|i_m|^{d(r-1)} \le (\psi(i_1,\ldots,i_k,j))^{(1-\alpha)/r+d(r-1)/p_m},$$

and that conditions (II) and (V) guarantee that $(1-\alpha)/r + d(r-1)/p_m \le (1-\alpha)$, which implies (3.7).

(iii) Finally suppose that the integers j, j' belong to $[2Mk^dS^{1/r}, \infty)$.

If $j' \le 2j$, then

$$\frac{\psi(i_1,\ldots,i_k,j')}{\psi(i_1,\ldots,i_k,j)} \prec 1+\frac{|j-j'|^r}{\psi(i_1,\ldots,i_k,j)} \prec 1+\frac{|j|^r}{\psi(i_1,\ldots,i_k,j)} \le 2. \tag{3.8}$$

In particular, the intervals $|I'|$ and $|I|$ have comparable size and hence we conclude that (3.6) still holds. Also note that $j' \le 2j$ implies that $|\xi - j| \le |j| + M(|i_1| + \cdots + |i_k|)^d$. Then, proceeding as in ii), we have that

$$\frac{1}{|x-y|^\alpha}\left|\log\frac{|I||J'|}{|I'||J|}\right| \prec \frac{|j|^{1-\alpha}(|i_1|+\cdots+|i_k|)^{d(r-1)}}{\psi(i_1,\ldots,i_k,j)^{1-\alpha}}.$$

The reader can check, again as in ii), that this last expression is bounded.

For the case $j' > 2j$, we have

$$|x-y| = \sum_{\ell=j}^{j'}|I_{i_1,\ldots,i_k,\ell}| = \sum_{\ell=j}^{j'}\frac{1}{|i_1|^{p_1}+\cdots+|i_k|^{p_k}+|\ell|^r}$$
$$\succ \sum_{\ell=j}^{j'}\frac{1}{|\ell|^r} \succ \int_{\ell=j}^{j'}\frac{1}{x^r}\,dx \succ \frac{1}{|j|^{r-1}},$$

where the last inequality holds because $j' > 2j$. On the other hand, applying the mean value theorem, it follows that

$$\log \frac{|I||J'|}{|I'||J|} = |r_s(i_1, \ldots, i_k)||D(\Psi_{i_1,\ldots,i_k})(\xi) - D(\Psi_{i_1,\ldots,i_k})(\tilde{\xi})|,$$

where $\xi \in \mathrm{conv}\{j, j + r_s\}$ and $\tilde{\xi} \in \mathrm{conv}\{j', j' + r_s\}$. Therefore, observing that the function $\xi \mapsto D(\Psi_{i_1,\ldots,i_k})(\xi) = r|\xi|^{r-1}/\psi(i_1, \ldots, i_k, \xi)$ is decreasing,

$$\frac{1}{|x-y|^\alpha}\left|\log \frac{|I||J'|}{|I'||J|}\right| \prec (|i_1| + \cdots + |i_k|)^d \frac{|j|^{(\alpha+1)(r-1)}}{\psi(i_1, \ldots, i_k, j)}.$$

Now we want to see that this last expression is bounded, in other words, that the inequality $|j|^{(\alpha+1)(r-1)}(|i_1| + \cdots + |i_k|)^d \prec \psi(i_1, \ldots, i_k, j)$ holds. For this, arguing as in (3.7), it is enough to check that, for all $n \in \{1, \ldots, k\}$, the inequality

$$\frac{(\alpha+1)(r-1)}{r} + \frac{d}{p_n} \leq 1$$

holds. To see this, note that, from IV) it follows that $d/p_n \leq (1-\alpha)/2$. Finally notice that $(\alpha + 1)(r - 1)/r + (1 - \alpha)/2 \leq 1 \Leftrightarrow r \leq 2$, which is ensured by condition (I).

3.2.2. *The maps f_t are $C^{1+\alpha}$-diffeomorphisms.* In the same way as for the maps g_s, we want to see that there is a constant $C > 0$ such that

$$\frac{|\log Df_t(x) - \log Df_t(y)|}{|x-y|^\alpha} \leqslant C \quad \text{for all different } x, y \in [0, 1].$$

To simplify notation, we only work with $t = 1$, as the other cases are analogous. As for the case of the maps g_s, we only have two cases to analyze.

Case 1: The points x, y belongs to the same interval $I_{i_1,\ldots,i_k,j}$. By Lemma 3.3, it is enough to show that the expression

$$\frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha} \log \frac{|I_{i_1,\ldots,i_k,j}||I_{i_1+1,i_2,\ldots,i_k,j+\ell_1-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}||I_{i_1,\ldots,i_k,j-1}|}$$

is uniformly bounded. To see this, simply note that the above expression is equal to

$$\frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha} \log \frac{|I_{i_1,\ldots,i_k,j}||I_{i_1+1,i_2,\ldots,i_k,j-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j}||I_{i_1,\ldots,i_k,j-1}|} + \frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha} \log \frac{|I_{i_1+1,i_2,\ldots,i_k,j}||I_{i_1+1,i_2,\ldots,i_k,j+\ell_1-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}||I_{i_1+1,i_2,\ldots,i_k,j-1}|}.$$

By condition (VI), we know from [6, §3.3] that the first term is uniformly bounded. The second term is bounded as well since it is the same as we bounded when dealing with g_s (changing i_1 to $i_1 + 1$).

Case 2: The point $x \in I = I_{i_1,\ldots,i_k,j}$ and $y \in J = I_{i_1,\ldots,i_k,j'}$, with $j < j'$. Here we can use (4) from Lemma 3.3 to bound $|\log Df_1(x) - \log Df_1(y)|$ by

$$\left|\log \frac{|I_{i_1+1,\ldots,i_k,j+\ell_1}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j}||I_{i_1+1,\ldots,i_k,j'+\ell_1}|}\right| + \left|\log \frac{|I_{i_1+1,\ldots,i_k,j+\ell_1-1}||I_{i_1,\ldots,i_k,j}|}{|I_{i_1,\ldots,i_k,j-1}||I_{i_1+1,\ldots,i_k,j+\ell_1}|}\right| + \left|\log \frac{|I_{i_1+1,\ldots,i_k,j'+\ell_1-1}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j'-1}||I_{i_1+1,\ldots,i_k,j'+\ell_1}|}\right|,$$

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and then work in the same way as for the functions g_s. For example, we express the term

$$\frac{1}{|x-y|^\alpha} \log \frac{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j}||I_{i_1+1,i_2,\ldots,i_k,j'+\ell_1}|}$$

as

$$\frac{1}{|x-y|^\alpha} \log \frac{|I_{i_1+1,i_2,\ldots,i_k,j}||I_{i_1,\ldots,i_k,j'}|}{|I_{i_1+1,i_2,\ldots,i_k,j'}||I_{i_1,\ldots,i_k,j}|} + \frac{1}{|x-y|^\alpha} \log \frac{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}||I_{i_1+1,i_2,\ldots,i_k,j'}|}{|I_{i_1+1,i_2,\ldots,i_k,j}||I_{i_1+1,i_2,\ldots,i_k,j'+\ell_1}|}.$$

The first term is bounded by [6, §3.3] and the second is also bounded by the same argument as used for the functions g_s.

3.3. *Faithful actions.* Given $s \in \{1, \ldots, d\}$ and a compact interval I_s, we have seen in Proposition 3.4 how to produce an action

$$\phi_s : G \to \mathrm{Diff}_+^{1+\alpha}(I_s).$$

Recall that the action from Proposition 3.4 is a smoothing of the dynamical realization of the action given in Proposition 3.1. In particular, the subgroup $\langle g_1, \ldots, g_{s-1}\rangle$ acts trivially, whereas $\langle g_s\rangle$ acts faithfully.

To obtain a faithful action of G, we do the following. Consider compact intervals $I_1, \ldots, I_d$ such that, for all $s \in \{1, \ldots, d-1\}$, I_{s+1} is contiguous to I_s by the right. Then define on $I := I_1 \cup \cdots \cup I_d$ the action $\phi : G \to \mathrm{Diff}_+^{1+\alpha}(I)$ as

$$\phi\,|_{I_s} = \phi_s.$$

We claim that ϕ is injective. Indeed, since $Z(G) \leqslant A = \langle g_1, \ldots, g_d\rangle$, by Proposition 2.1 we only need to check that $\phi\,|_A$ is injective. Let $g \in A$ be an element that acts trivially on I. Then, there exist $j_1, \ldots, j_d \in \mathbb{Z}$ such that $g = g_1^{j_1} \cdots g_d^{j_d}$. Now, since $\phi(g) = \mathrm{id}$, it follows that

$$\phi_s(g) = \mathrm{id} \quad \text{for all } s \in \{1, \ldots, d\}.$$

This yields that $j_d = \cdots = j_1 = 0$ and hence g is the trivial element. This finishes the proof of Theorem A.

4. *Examples*

In this section, we give examples of nilpotent groups for which we can compute the critical regularity. In each case, we use Theorem A to obtain a lower bound for the critical regularity, and we argue that, in our examples, this is also an upper bound for the regularity.

We begin by recalling that if G is a finitely generated nilpotent group of homeomorphisms of $(0, 1)$ that has no global fixed points, then there is a well-defined group homomorphism $\rho : G \to \mathbb{R}$, which is usually called the *translation number* of the action. This map characterizes the elements of G that have fixed points, in the sense that $\rho(g) = 0$ if and only if g has a fixed point in $(0, 1)$. Further, the action of G on the interval has *no crossings*. By this, we mean that if an element $f \in G$ fixes an open subinterval I of $(0, 1)$ and satisfies that $f(x) \neq x$ for all x in I, then, for any other $g \in G$, we have that $g(I) = I$ or $g(I) \cap I = \emptyset$. See [20, §2.2.5] for details. With this, it is easy to prove the following result that we will repeatedly use.

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LEMMA 4.1. *Let $G \leqslant \mathrm{Diff}_+^0(0,1)$ be a nilpotent group and let $c \in G$ be a non-trivial element such that $c = [a,b]$ for some elements $a, b \in G$. If c fixes an open interval I and has no fixed point inside, then either a or b moves I (disjointly).*

Proof. Looking for a contradiction, assume that a and b fix I. Then we have the translation number homomorphism for the group $\langle a, b, c\rangle \leqslant \mathrm{Diff}_+^0(I)$. Since c is a commutator, it is in the kernel of this homomorphism. Hence, we conclude that c has a fixed point inside I, which is contrary to our assumptions. □

To obtain upper bounds for the regularity of our groups we will use a result from Deroin, Kleptsyn and Navas [**8**]. We use the version from [**6**, Proposition 2.1].

THEOREM 4.2. *Let $f_1, \ldots, f_k$ be C^1-diffeomorphisms of the interval $[0,1]$ that commute with a C^1-diffeomorphism g. Assume that g fixes a subinterval I of $[0,1]$ and that its restriction to I is non-trivial. Moreover, assume that, for a certain $0 < \alpha < 1$ and a sequence of indexes $i_j \in \{1, \ldots, k\}$, the sum*

$$\sum_{j\geq 0} |f_{i_j} \cdots f_{i_1}(I)|^{\alpha} < \infty.$$

Then, $f_1, \ldots, f_k$ cannot be all of class $C^{1+\alpha}$.

The following lemma is useful to get into the hypotheses of Theorem 4.2. Although it is stated in a slightly different way, the reader can check that the proof is exactly the same as that of [**8**, Lemma 3.3].

LEMMA 4.3. *Let $f_1, \ldots, f_k$ be C^1-diffeomorphisms of $[0,1]$ and let I be a subinterval of $[0,1]$ such that $\mathbb{Z}^k \simeq \langle f_1, \ldots, f_k\rangle/\mathrm{Stab}(I)$, where $\mathrm{Stab}(I)$ is the stabilizer of I (which is assumed to be a normal subgroup). Then, if $\alpha > 1/k$, there exists a sequence $(f_{i_j})_{j\in\mathbb{N}}$ of elements in $\{f_1, \ldots, f_k\}$ such that*

$$\sum_{j\geq 0} |f_{i_j} \cdots f_{i_1}(I)|^{\alpha} < \infty.$$

4.1. *Heisenberg groups.* For a natural number $n \geq 1$, the discrete $(2n+1)$-dimensional *Heisenberg group*, is defined as the set of matrices

$$\mathscr{H}_n := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^t & I_n & \vec{y}^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x}, \vec{y} \in \mathbb{Z}^n, c \in \mathbb{Z} \text{ and } I_n \text{ is the identity matrix of size } n \right\},$$

with the usual matrix product. Note that the center of $\mathscr{H}_n$ coincides with the commutator subgroup and is generated by the matrix

$$\mathbf{C} := \begin{pmatrix} 1 & \vec{0} & 1 \\ \vec{0}^t & I_n & \vec{0}^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix}.$$

We want to prove Theorem B, but before this, it will be useful for us to bound the rank of maximal abelian subgroups. Assume that there exists a maximal abelian subgroup of $\mathscr{H}_n$ of rank m. Then we can choose elements

$$\mathbf{A}_i := \begin{pmatrix} 1 & \vec{a}_i & c_i \\ \vec{0}^t & I_n & \vec{b}_i^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix} \in \mathscr{H}_n \quad \text{for } i \in \{1, \ldots, m-1\},$$

such that $\langle \mathbf{A}_1, \ldots \mathbf{A}_{m-1}, \mathbf{C}\rangle \simeq \mathbb{Z}^m$. Note that the commutativity of these matrices is equivalent to the equations

$$\vec{a}_i \cdot \vec{b}_j = \vec{a}_j \cdot \vec{b}_i \quad \text{for all } i, j \in \{1, \ldots, m-1\}. \tag{4.1}$$

Note also that if $\{\mathbf{A}_1, \ldots, \mathbf{A}_{m-1}, \mathbf{C}\}$ generates a free abelian group of rank m, then the set of vectors $\mathscr{B} := \{(\vec{b}_i, \vec{a}_i) \in \mathbb{Z}^n \times \mathbb{Z}^n : 1 \le i \le m-1\}$ generates a free abelian group of rank $m-1$. Indeed, if we have a dependency relationship, say, $r(\vec{b}_1, \vec{a}_1) \in \langle(\vec{b}_2, \vec{a}_2), \ldots, (\vec{b}_{m-1}, \vec{a}_{m-1})\rangle$ for some $0 \neq r \in \mathbb{Z}$, then $\mathbf{A}_1^r \in \langle \mathbf{A}_2, \ldots, \mathbf{A}_{m-1}, \mathbf{C}\rangle$, which contradicts that the abelian group has rank m.

Having said this, we claim that $m \le n+1$. To see the latter, note that, by equations (4.1), any vector of the form $(\vec{a}_i, -\vec{b}_i)$, with $1 \le i \le m-1$, is perpendicular to $\langle \mathscr{B}\rangle$. Hence, we have two orthogonal subgroups of $\mathbb{Z}^n \times \mathbb{Z}^n$ that both have rank $m-1$, and thus $m-1 \le n$, which proves our claim.

Realization. Consider the abelian subgroup

$$A := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^{\,t} & I_n & \vec{0}^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x} \in \mathbb{Z}^n \quad \text{and} \quad c \in \mathbb{Z} \right\}.$$

Notice that A has rank equal to $n+1$, which is the largest we can expect. Since the rank of $\mathscr{H}_n/A$ is n, we have that Theorem A provides an injective group homomorphism

$$\mathscr{H}_n \hookrightarrow \mathrm{Diff}_+^{1+\alpha}([0,1]) \quad \text{for } \alpha < 1/n.$$

Bounding the regularity. Now we consider a faithful action $\phi : \mathscr{H}_n \hookrightarrow \mathrm{Diff}_+^1([0,1])$. Abusing notation, we can think that $\mathscr{H}_n \leqslant \mathrm{Diff}_+^1([0,1])$.

Since the commutator subgroup of $\mathscr{H}_n$ is generated by $\mathbf{C}$, we deduce from Lemma 4.1 that $\mathbf{C}$ has fixed points inside $(0,1)$. Therefore, we can find an interval $I \subsetneq [0,1]$ such that $\mathbf{C}(I) = I$ and $\mathbf{C}(x) \neq x$ for all x in the interior of I. Let $\mathrm{Stab}(I)$ be the stabilizer of I. It is easy to see that this is an abelian subgroup. Indeed, if we take $\mathbf{A}, \mathbf{B} \in \mathrm{Stab}(I)$ and assume that they do not commute, then there must exist $m \in \mathbb{Z}$ such that $[\mathbf{A}, \mathbf{B}] = \mathbf{C}^m$. Since $\mathbf{C}$ has no fixed points inside I, Lemma 4.1 tells us that either $\mathbf{A}$ or $\mathbf{B}$ moves I, which is a contradiction. Note that Stab(I) is a normal subgroup as it contains the commutator subgroup. Further, we know that there is a natural number k and elements $\mathbf{B}_1, \ldots, \mathbf{B}_k \in \mathscr{H}_n$ such that

$$\mathbb{Z}^k \simeq \frac{\mathscr{H}_n}{\mathrm{Stab}(I)} = \frac{\langle \mathbf{B}_1, \ldots, \mathbf{B}_k\rangle}{\mathrm{Stab}(I)}.$$

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So, given $\alpha > 1/k$, we can find by Lemma 4.3 a sequence $(\mathbf{B}_{i_j})_{j\in\mathbb{N}}$ of elements in $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$ such that

$$\sum_{j\geq 0} |\mathbf{B}_{i_j} \cdots \mathbf{B}_{i_1}(I)|^{\alpha} < \infty,$$

and hence Theorem 4.2 yields that ϕ is not an action by $C^{1+\alpha}$-diffeomorphisms.

Now, since the rank of $\mathrm{Stab}(I)$ is bounded above by $n-1 = \mathrm{rank}(A)$,

$$k = \mathrm{rank}\left(\frac{\mathscr{H}_n}{\mathrm{Stab}(I)}\right) \geq \mathrm{rank}\left(\frac{\mathscr{H}_n}{A}\right) = n,$$

which implies that the regularity of the action ϕ is bounded above by $1 + 1/n$. So we conclude that

$$\mathrm{Crit}_{[0,1]}(\mathscr{H}_n) = 1 + \frac{1}{n}.$$

Remark 4.4. If G is a finitely generated, torsion-free nilpotent group whose center is cyclic and satisfies $[G, G] \leqslant Z(G)$, then the proof of Theorem B yields that the lower bound for $\mathrm{Crit}_{[0,1]}(G)$ given by Theorem A is the critical one.

4.2. *Examples with large nilpotency degree.* Theorem B gives us the critical regularity for the Heisenberg groups, which are groups having nilpotency degree two. In this section, we provide more examples of nilpotent groups for which we can compute the critical regularity, but whose nilpotency degree can be arbitrarily large. As for the Heisenberg groups, in these examples, we show that the lower bound provided by Theorem A is also an upper bound.

Fix $d, k \in \mathbb{N}$, assume that $d \geq k$ and consider a matrix $(m_{i,s}) \in M_k(\mathbb{Z})$ with non-zero determinant and positive entries. We let G be the group generated by the set

$$\{g_0\} \cup \{g_{i,j} : (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, d\}\} \cup \{f_1, \ldots, f_k\},$$

subject to the relationships:

- $[g_0, g_{i,j}] = [g_0, f_i] = [f_s, f_i] = [g_{i,j}, g_{l,m}] = e$, for all $s, i, l \in \{1, \ldots, k\}$, $j, m \in \{1, \ldots, d\}$;
- $[f_s, g_{i,j}] = g_{i,j-1}^{m_{i,s}}$ for all $s, i \in \{1, \ldots, k\}$ and $j \in \{2, \ldots, d\}$; and
- $[f_s, g_{i,1}] = g_0^{m_{i,s}}$ for all $s, i \in \{1, \ldots, k\}$.

Note that, from the identities $[ab, c] = a[b, c]a^{-1}[a, c]$ and $[a, bc] = [a, b]b[a, c]b^{-1}$, we immediately have the following additional relationships:

- $[f_s^{-1}, g_{i,j}] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle g_{i,j-1}^{-m_{i,s}}$ for all $s, i \in \{1, \ldots, k\}$, $j \in \{2, \ldots, d\}$; and
- $[f_s^{-1}, g_{i,1}] = g_0^{-m_{i,s}}$ for all $s, i \in \{1, \ldots, k\}$.

It is easy to see that G is a nilpotent group of degree $d+1$, and that $A = \langle \{g_0\} \cup \{g_{i,j} : (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, d\}\}\rangle$ is a maximal abelian subgroup containing the commutator of G (see Lemma 4.5 below). Moreover, k is the torsion-free rank of G/A. Therefore, in view of Theorem A, we know that G embeds in $\mathrm{Diff}_+^{1+\alpha}([0, 1])$ for $\alpha < 1/k$.

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To show that $1 + 1/k$ is actually an upper bound for the regularity, we need the following elementary lemma.

LEMMA 4.5. *For all* $(i, j) \in \{1, \ldots, k\} \times \{2, \ldots, d\}$ *and* $n_1, \ldots, n_k \in \mathbb{Z}$*:*

(1) $[f_1^{n_1} \cdots f_k^{n_k}, g_{i,j}] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle g_{i,j-1}^{\lambda_i}$*; and*

(2) $[f_1^{n_1} \cdots f_k^{n_k}, g_{i,1}] = g_0^{\lambda_i}$,

where $\lambda_i = \sum_{s=1}^k n_s m_{i,s}$*. In particular, the subgroup A is a maximal abelian subgroup.*

Proof. We show (1) by induction on $n = \sum_{s=1}^k |n_s|$.

Note that, when $n = 1$, we have the result by the relationships of G. So, consider an arbitrary natural number $n = \sum_{j=1}^k |n_j|$ and assume that $n_k < 0$ (the other case is similar). For all $i \in \{1, \ldots, k\}$ and $j \in \{2, \ldots, d\}$,

$$[f_1^{n_1} \cdots f_k^{n_k}, g_{i,j}] = [f_1^{n_1} \cdots f_k^{n_k+1}, [f_k^{-1}, g_{i,j}]]\, [f_k^{-1}, g_{i,j}]\, [f_1^{n_1} \cdots f_k^{n_k+1}, g_{i,j}],$$

and since $[f_k^{-1}, g_{i,j}]$ belongs to $\langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle g_{i,j-1}^{-m_{i,k}}$, it follows that $[f_1^{n_1} \cdots f_k^{n_k+1}, [f_k^{-1}, g_{i,j}]] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle$. Also, by induction,

$$[f_1^{n_1} \cdots f_k^{n_k+1}, g_{i,j}] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle g_{i,j-1}^{(\sum_{s=1}^{k-1} n_s m_{i,s} + (n_k+1) m_{i,k})}.$$

Plugging these into the previous equation yields assertion (1). The proof of assertion (2) is analogous. □

Remark 4.6. The most useful part of Lemma 4.5 is the explicit expression for the integers λ_i appearing. These will be used in the proof of Theorem C.

Proof of Theorem C. Suppose that G embeds into $\mathrm{Diff}_+^{1+\alpha}([0, 1])$ for some $\alpha > 1/k$. Let x_0 be a point in $(0, 1)$ such that $g_0(x_0) \neq x_0$ and define the intervals

$$I_0 := \Big(\inf_n g_0^n(x_0), \sup_n g_0^n(x_0)\Big) \quad \text{and} \quad I_{i,j} := \Big(\inf_n g_{i,j}^n(x_0), \sup_n g_{i,j}^n(x_0)\Big).$$

Case 1: $f(I_0) \cap I_0 = \emptyset$ for all $f \in \langle f_1, \ldots, f_k\rangle \simeq \mathbb{Z}^k$. In this case, I_0 is a wandering interval for the dynamics of $\langle f_1, \ldots, f_k\rangle$. A contradiction is provided by Lemma 4.3 followed by Theorem 4.2, because the central element g_0 acts non-trivially on I_0.

Case 2: There is a non-trivial element $f \in \langle f_1, \ldots, f_k\rangle$ such that $f(I_0) = I_0$. We put $f = f_1^{n_1} \cdots f_k^{n_k}$. Given $i \in \{1, \ldots, k\}$, by Lemma 4.5,

$$[f, g_{i,1}] = g_0^{\lambda_i} \quad \text{and} \quad [f, g_{i,j}] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2}\rangle g_{i,j-1}^{\lambda_i} \quad \text{for all } j \in \{2, \ldots, d\}, \tag{4.2}$$

where $\lambda_i = \sum_{j=1}^k n_j m_{i,j}$. Since the vectors $(m_{i,1}, \ldots, m_{i,k})$ are linearly independent in $\mathbb{R}^k$, we can choose i to obtain $\lambda_i \neq 0$. Then, the relationships (4.2) and Lemma 4.1 imply that $g_{i,1}(I_0) \cap I_0 = \emptyset$. Since the action has no crossings, the element f also fixes the intervals $I_{i,j}$ and hence the same argument also yields that $g_{i,j}(I_{i,j-1}) \cap I_{i,j-1} = \emptyset$ for all $j > 2$. Therefore, I_0 is a wandering interval for the action of $\langle g_{i,1}, \ldots, g_{i,k}\rangle \simeq \mathbb{Z}^k$. So, a contradiction is reached using Lemma 4.3 and Theorem 4.2, as before. □

4.3. *An example with even higher regularity.* It is easy to see that, in some situations, the regularity given by Theorem A is not critical. In the examples that we know of, this is related to the fact that the group can be split as a direct product of groups, each of which allows an embedding with better regularity. Take, for example, the groups of [**6**, §4]. These are given by the presentation

$$G_d := \langle f, g_1, \ldots, g_d : [g_i, g_j] = \mathrm{id}, [f, g_1] = \mathrm{id}, [f, g_i] = g_{i-1} \text{ for all } j \geq 1, i > 1\rangle.$$

Note that G_d is isomorphic to a non-trivial semidirect product of the form $\mathbb{Z}^d \rtimes \mathbb{Z}$. Now define the group $G := G_d \times G_d$. On one hand, it is easy to see that

$$G \simeq \mathbb{Z}^{2d} \rtimes \mathbb{Z}^2,$$

and $\mathbb{Z}^{2d} \times \{0\}$ is a maximal abelian subgroup of G. Therefore, if we apply Theorem A, we obtain an embedding of G into $\mathrm{Diff}^{1+\alpha}_+([0, 1])$ for all $\alpha < 1/2$. However, on the other hand, the critical regularity of G is two. Indeed, we can apply Theorem A to each factor of G to obtain an embedding of the factor into $\mathrm{Diff}^{1+\alpha}_+([0, 1])$ for all $\alpha < 1$. If we put these two actions together acting on disjoint intervals (as we did in §3.3), we end up with an embedding of G into $\mathrm{Diff}^{1+\alpha}_+([0, 1])$ for all $\alpha < 1$.

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