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IRREGULARITY OF THE RATE OF DECREASE OF SEQUENCES OF POWERS IN THE VOLTERRA ALGEBRA

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1. Introduction. G. R. Allan and A. M. Sinclair proved in [**1**] that if a commutative radical Banach algebra $\mathscr{R}$ possesses bounded approximate identities then for every sequence (α_n) of real numbers such that $\lim_{n\to\infty} \alpha_n = 0$ there exists $b \in \mathscr{R}$ such that

$$\liminf_{n\to\infty} \frac{\|b^n\|^{1/n}}{\alpha_n} = +\infty.$$

In the other direction it is shown in [**6**] that if $\mathscr{R}$ is separable and if the nilpotents are dense in $\mathscr{R}$ then for every sequence (β_n) of positive reals there exists $b \in \mathscr{R}$ such that

$$[b\mathscr{R}]^- = \mathscr{R} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\|b^n\|^{1/n}}{\beta_n} = 0.$$

(This result was given in [**2**] for the Volterra algebra.)

We are concerned here with the irregularity of the rate of decrease of sequences of powers. It is known [**5**] that if a nonnilpotent element b of a commutative radical Banach algebra $\mathscr{R}$ satisfies $b \in [b\mathscr{R}]^-$ then there exists a nonnilpotent $c \in \mathscr{R}$ such that

$$\limsup_{n\to\infty} \frac{\|b^n\|^{1/n}}{\|c^n\|^{1/n}} = \limsup_{n\to\infty} \frac{\|c^n\|^{1/n}}{\|b^n\|^{1/n}} = +\infty.$$

We prove here that if $\mathscr{R}$ is a commutative separable radical Banach algebra with b.a.i in which the nilpotents are dense then for any sequences (α_n) and (β_n) of positive reals which converge to zero there exists $a \in \mathscr{R}$ such that $[a\mathscr{R}]^- = \mathscr{R}$ and

$$\limsup_{n\to\infty} \frac{\|a^n\|^{1/n}}{\alpha_n} = +\infty, \quad \liminf_{n\to\infty} \frac{\|a^n\|^{1/n}}{\beta_n} = 0.$$

This result does not extend to the weighted convolution algebra $L^1(R^+, e^{-t^2})$ because there exists a sequence (λ_n) of positive reals such that $\liminf \|b^n\|^{1/n}\lambda_n = +\infty$ for every nonzero element b of $L^1(R^+, e^{-t^2})$ (see [**2**] or [**6**]).

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2. Irregularity of the rate of decrease of sequences of powers.

THEOREM 1. *Let $\mathscr{R}$ be a commutative nonzero separable Banach algebra with bounded approximate identities. If the nilpotents are dense in $\mathscr{R}$ then for all sequences (α_n) and (β_n) of positive reals which converge to zero there exists $a \in \mathscr{R}$ such that $[a\mathscr{R}]^- = \mathscr{R}$ and*

$$\limsup_{n\to\infty} \frac{\|a^n\|^{1/n}}{\alpha_n} = +\infty, \quad \liminf_{n\to\infty} \frac{\|a^n\|^{1/n}}{\beta_n} = 0.$$

Proof. Put, for every $n \in N: \mu_n = (\beta_n H/n)^n$. Add a unit e to $\mathscr{R}$. By the Johnson-Varopoulos extension of Cohen's factorization theorem [**3**], [**7**], [**8**] there exists $x \in \mathscr{R}$ such that $[x\mathscr{R}]^- = \mathscr{R}$ and it follows from a result of [**1**] that there exists $b \in \mathscr{R}$ such that $x \in b\mathscr{R}$ and

$$\liminf_{n\to\infty} \frac{\|b^n\|^{1/n}}{\alpha_n} = +\infty.$$

So $[b\mathscr{R}]^- = \mathscr{R}$.

Define by induction a sequence (λ_n) of positive reals, two sequences (p_n) and (q_n) of positive integers and two sequences (f_n) and (g_n) of elements of $\mathscr{R}$ such that if we put

$$X_n = (\lambda_1 e + f_1)(2\lambda_1^{-1}e + g_1)\ldots (\lambda_{n-1}e + f_{n-1})(2\lambda_{n-1}^{-1}e + g_{n-1})(\lambda_n e + f_n)$$

$$Y_n = (\lambda_1 e + f_1)(2\lambda_1^{-1}e + g_1)\ldots(\lambda_n e + f_n)(2\lambda_n^{-1}e + g_n)$$

the following conditions are satisfied (we put for convenience $X_0 = Y_0 = e$).

(1) $\|bX_m^{-1}Y_{n-1} - bX_m^{-1}X_n\| < 2^{-n}$ $(0 \leqq m \leqq n-1, n \geqq 1)$

(2) $\|b^{p_m}Y_{n-1}^{p_m} - b^{p_m}X_n^{p_m}\| < 2^{-n}\mu_{p_m}$ $(1 \leqq m \leqq n-1, n \geqq 2)$

(3) $\|b^{q_m}Y_{n-1}^{q_m} - b^{q_m}X_n^{q_m}\| < 2^{-n}\|b^{q_m}\|$ $(1 \leqq m \leqq n-1, n \geqq 2)$

(4) $\|bX_m^{-1}X_n - bX_m^{-1}Y_n\| < 2^{-n}$ $(1 \leqq m \leqq n, n \geqq 1)$

(5) $\|b^{p_m}X_n^{p_m} - b^{p_m}Y_n^{p_m}\| < 2^{-n}\mu_{p_m}$ $(1 \leqq m \leqq n, n \geqq 1)$

(6) $\|b^{q_m}X_n^{q_m} - b^{q_m}Y_n^{q_m}\| < 2^{-n}\|b^{q_m}\|$ $(1 \leqq m \leqq n-1, n \geqq 2)$

(7) $\|b^{p_n}X^{p_n}\| < \mu_{p_n}$ $(n \geqq 1)$

(8) $\|b^{q_n}Y_n^{q_n}\| > \|b^{q_n}\|$ $(n \geqq 1)$.

There exists a sequence (e_k) of elements of $\mathscr{R}$ such that $\lim_{k\to\infty} xe_k = x$ for every $x \in \mathscr{R}$, and we may assume that e_k is nilpotent for every $k \in \mathbf{N}$. Taking $f_1 = e_k$ with k large enough we may arrange that $\|b - bf_1\| < \frac{1}{2}$. Let p_1 be a positive integer such that $f_1^{p_1} = 0$. Then

$$\lim_{\lambda\to 0} \|b - b(\lambda e + f_1)\| < \tfrac{1}{2} \quad \text{and} \quad \lim_{\lambda\to\infty} \|b^{p_1}(\lambda e + f_1)^{p_1}\| = 0.$$

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So taking $\lambda_1 > 0$ small enough we may arrange that f_1, p_1 and λ_1 satisfy the conditions (1) and (7). Then

$$\lim_{k\to\infty} bX_1[2\lambda_1^{-1}e + (1 - 2\lambda_1^{-1})e_k] = bX_1$$

so

$$\lim_{k\to\infty} b^m X_1^m U^m[2\lambda^{-1}e + (1 - 2\lambda_1^{-1})e_k]^m = b^m X_1^m U^m$$

for every $U \in \mathscr{R} \oplus \mathbf{C}e$ and for every $m \in \mathbf{N}$. So taking $g_1 = (2\lambda_1^{-1} - 1)e_k$ with k large enough we may arrange the conditions (4) and (5) to be satisfied. Then

$$\lim_{m\to\infty} \frac{\|b^m Y_1^m\|^{1/m}}{\|b^m\|^{1/m}} \geqq \lim \frac{1}{\|Y_1^{-m}\|^{1/m}} = 2$$

and there exists $q_1 \in \mathbf{N}$ such that

$$\|b^{q_1} Y_1^{q_1}\| > \|b^{q_1}\|.$$

Now suppose that we have constructed finite families $(\lambda_1, \ldots, \lambda_n)$, $(f_1, \ldots, f_n)$, $(g_1, \ldots, g_n)$, $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ satisfying the eight conditions. As $\lim_{k\to\infty} be_k = b$ we have

$$\lim_{k\to\infty} b^m U^m e_k^m = b^m U^m$$

for every $k \in \mathbf{N}$ and every $U \in \mathscr{R} \oplus \mathbf{C}e$. Taking $f_{p+1} = e_k$ with k large enough we may arrange that the following inequalities hold:

$$\|bX_m^{-1} Y_n - bX_m^{-1} Y_n f_{n+1}\| < 2^{-n-1} \ (0 \leqq m \leqq n)$$
$$\|b^{p_m} Y_n^{p_m} - b^{p_m} Y_n^{p_m} f_{n+1}^{p_m}\| < 2^{-n-1}\mu_{p_m} \ (1 \leqq m \leqq n)$$
$$\|b^m Y_n^m - b^m Y_n^m f_{n+1}^m\| < 2^{-n-1}\|b^{q_m}\| \ (1 \leqq m \leqq n).$$

Let $p_{n+1} > p_n$ be a positive integer such that $f_{n+1}^{p_{n+1}} = 0$. We have

$$\lim_{\lambda\to 0}\|(\lambda e + f_{n+1})^{p_{n+1}}\| = 0 \quad \text{and} \lim_{\lambda\to 0} x^m(\lambda e + f_{n+1})^m = x^m f_{n+1}^m$$

for every $m \in \mathbf{N}$ and every $x \in \mathscr{R}$. So taking $\lambda_{n+1} > 0$ small enough we may arrange the conditions (1), (2), (3) and (7) to be satisfied by λ_{n+1}, p_{n+1} and f_{n+1}.

Then

$$\lim_{k\to\infty} bX_{n+1}[2\lambda_{n+1}^{-1}e + (1 - 2\lambda_{n+1}^{-1}e_k)] = bX_{n+1}$$

so

$$\lim_{k\to\infty} b^m X_{n+1}^m U^m[2\lambda_{n+1}^{-1}e + (1 - 2\lambda_{n+1}^{-1}e_k)]^m = b^m X_{n+1}^m U^m$$

for every $U \in \mathscr{R} \oplus \mathbf{C}e$ and every $m \in \mathbf{N}$. So taking $g_{n+1} = (1 - 2\lambda_{n+1}^{-1})e_k$ with k large enough we can arrange the conditions (4), (5) and (6) to be satisfied. Then

$$\lim_{m\to\infty} \frac{\|b^m Y_{n+1}^m\|^{1/m}}{\|b^m\|^{1/m}} \geqq \lim_{m\to\infty} \frac{1}{\|Y_{n+1}^{-m}\|^{1/m}} = 2^n.$$

So we can choose $q_{n+1} > q_n$ satisfying (8).

We thus see that we can construct by induction sequences (λ_n), (f_n), (g_n), (p_n) and (q_n) satisfying the eight conditions. It follows from (1) and (4) that

$$\|bX_n - bX_{n+1}\| < 3.2^{-n-1}$$

for every $n \geqq 0$ and that

$$\lim_{n\to\infty} \|bX_n - bY_n\| = 0.$$

So the sequence (bX_n) is Cauchy. Denote by a its limit. Then $a = \lim_{n\to\infty} bY_n$. We have, for every $m \geqq 0$ and every $n \geqq m$,

$$\|bX_m^{-1}X_n - bX_m^{-1}X_{n+1}\| \leqq \|bX_m^{-1}X_n - bX_m^{-1}Y_n\| + \|bX_m^{-1}Y_n - bX_m^{-1}X_{n+1}\| < 3.2^{-n-1}.$$

So

$$\|aX_m^{-1} - b\| \leqq \sum_{n=m}^{\infty} \|bX_m^{-1}X_{n+1} - bX_m^{-1}X_n\| < 3.2^{-m}, \quad \text{and}$$

$$b = \lim_{m\to\infty} aX_m^{-1}.$$

So $b \in [a(\mathscr{R} + \mathbf{C}e)]^-$, $b\mathscr{R} \subseteq [a\mathscr{R}]^-$ and $[a\mathscr{R}]^- = \mathscr{R}$. It follows from (2) and (5) that we have, for every $m \geqq 1$ and every $n \geqq m$,

$$\|b^m X^m - b^m X_{n+1}{}^m\| < 3.2^{-n-1} \,.\, \mu_{p_m}.$$

So

$$\|a^{p_m}\| \leqq \|b^{p_m}\| + \sum_{m=n}^{\infty} \|b^{p_m} X_{n+1}{}^{p_m} - b^{p_m} \cdot X_n{}^{p_m}\|$$

$$< (1 + 3.2^{-n})\mu_{p_m} < 3\mu_{p_m}.$$

We obtain

$$\|a^{p_m}\|^{1/p_m} < 3^{1/p_m} \cdot \mu_{p_m}{}^{1/p_m} = 3^{1/p_m} \cdot \frac{\beta_{p_m}}{p_m}.$$

So

$$\lim_{m\to\infty} \frac{\|a^{p_m}\|^{1/p_m}}{\beta_{p_m}} = 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\|a^n\|^{1/n}}{\beta_n} = 0.$$

Also it follows from (3) and (6) that we have, for every $m \geqq 1$ and every $n \geqq m$:

$$\|b^{q_m} Y_n{}^{q_m} - b^{q_m} Y_{n+1}{}^{q_m}\| < 2^{-n}\|b^{q_m}\|.$$

So

$$\|a^{q_m}\| > \|b^{q_m} Y_m{}^{q_m}\| - \sum_{m=n}^{\infty} \|b^{q_m} \cdot Y_n{}^{q_m} - b^{q_m} \cdot Y_{n+1}{}^{q_m}\|$$

$$> \|b^{q_m}\|[1 - 2^{-m+1}].$$

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We obtain

$$\liminf_{m\to\infty} \frac{\|a^{q_m}\|^{1/q_m}}{\|b^{q_m}\|^{1/q_m}} \geqq 1,$$

$$\liminf_{m\to\infty} \frac{\|a^{q_m}\|^{1/q_m}}{\alpha_{q_m}} = +\infty.$$

So

$$\limsup_{n\to\infty} \frac{\|a^n\|^{1/n}}{\alpha_n} = +\infty.$$

This completes the proof of the theorem. The theorem applies in particular to the "Volterra algebra" $L_*^1(0, 1)$ discussed in [**4**], Section 7.

References

1. G. R. Allan and A. M. Sinclair, *Power factorization in Banach algebras with bounded approximate identity*, Studia Math. *56* (1976), 31–38.
2. W. G. Bade and H. G. Dales, *Norms and ideals in some radical Banach algebra s*, preprint.
3. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. *26* (1959), 199–206.
4. H. G. Dales, *Automatic continuity, a survey*, Bull. London Math. Soc. *10* (1978), 129–183.
5. J. Esterle, *Theorems of Gelfand Mazur type and continuity of epimorphisms from* $\mathscr{C}(\mathscr{K})$, J. Functional Analysis, to appear.
6. ——— *Rate of decrease of sequences of powers in commutative radical Banach algebras*, Pacific J. Math., to appear.
7. B. E. Johnson, *Continuity of centralizers on Banach algebras*, J. London Math. Soc. *41* (1966), 639–640.
8. N. Varopoulos, *Continuité des formes linéaires positives sur une algèbre de Banach avec involution*, C.R. Acad., Sci. Paris Ser. A*258* (1964), 1121.

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