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EMBEDDING THEOREMS IN GROUP C^*-ALGEBRAS†

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ABSTRACT. Let G be a locally compact group and H an open subgroup of G. The embeddings of group C^*-algebras associated with H into the group C^*-algebras associated with G are studied. Three conditions for the embeddings given in terms of C^*-norms of the group algebras, group representations and positive definite functions are shown to be equivalent. As corollary, we prove that the full C^*-algebra of H can be embedded into the full C^*-algebra of G in a natural way as well as the case for the reduced group C^*-algebras. We also show that the embeddings hold for their duals and double duals.

§0. **Introduction.** Group C^*-algebras provide some of the most interesting and important examples in the theory of operator algebras. In studying these algebras one naturally asks what are the connections between algebras associated with a group and algebras associated with its subgroups. The most popular choices for group C^*-algebras have been the full group C^*-algebras and the reduced group C^*-algebras. For locally compact abelian group G, these two group C^*-algebras associated with G can be identified [10; 1.17] and realized as $C_0(\hat{G})$, the algebra of the complex-continuous functions vanishing at infinity on the dual group $\hat{G}$ of G. If H is an open subgroup of G, then $C_0(\hat{H})$ can be embedded into $C_0(\hat{G})$ in a canonical way since $\hat{H}$ is a quotient group of G (cf. [14; Theorem 54, p. 274]). Furthermore, since the dual of $C_0(\hat{G})$ is $M^1(\hat{G})$, the Banach involutive algebra (under convolution) of bounded complex measures on $\hat{G}$, the embedding of the dual of $C_0(\hat{H})$ into the dual of $C_0(\hat{G})$ exists. In general, the full C^*-algebra is different from the reduced one. Eymard showed in [10] that the natural embedding holds in general for the reduced group C^*-algebras. This paper deals with questions concerning the embeddings of the general group C^*-algebras [see §1 for definition].

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Three conditions for the existence of the embedding in terms of C^*-algebra norms, group representations and continuous positive-definite functions are shown to be equivalent as in Theorem 2.5. As corollaries, we obtain that the embeddings hold for both the reduced and the full group C^*-algebras. Moreover, the embeddings exist for their duals and double duals too. In the last section, we give an example to show that embeddings of group C^*-algebras and their duals are independent. Also, we prove that if a locally compact group G contains a nontrivial amenable normal open subgroup, then the reduced C^*-algebra $C^*_\rho(G)$ is not simple (cf. [16; Proposition 1.6]).

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§1. **Notations and preliminaries.** We shall adopt the notations used by Eymard [10]. Throughout this paper G will denote a locally compact group with identity e. We choose, once and for all, a left Haar measure λ on G; and we denote by Δ the modular function of G. For $1 \leq p \leq \infty$, we define $L^p(G)$ as the usual L^p-space relative to λ. We identify $L^1(G)$ with the closed two sided ideal of $M^1(G)$ consisting of measures absolutely continuous with respect to λ by $f \to f\,d\lambda$. We designate the class of all continuous unitary representations of G by $\Sigma(G)$, as shown in [8, Propositions 13.3.1 and 13.3.4] we may identify $\Sigma(G)$ with the class of all non-degenerate representations of $L^1(G)$. For $\Gamma \subset \Sigma(G)$ we renorm $L^1(G)$ by defining $\|f\|_\Gamma = \sup_{\pi \in \Gamma} \|\pi(f)\|_{\mathscr{L}(\mathscr{H}_\pi)}$ where $\mathscr{L}(\mathscr{H}_\pi)$ denotes the algebra of bounded operators on $\mathscr{H}_\pi$, the Hilbert space corresponding to π.

Definition 1.1. The group C^*-algebra $C^*_\Gamma(G)$ is the completion of $L^1(G)/I_\Gamma$ under the norm $\|\cdot\|_\Gamma$ where

$$I_\Gamma = \{f \in L^1(G) \mid \pi(f) = 0, \pi \in \Gamma\}.$$

If $\Gamma = \Sigma(G)$, then we have $I_\Gamma = \{0\}$ and we shall use $C^*(G)$ to denote the completion of $L^1(G)$ under the norm $\|\cdot\|_{\Sigma(G)}$; $C^*(G)$ is called the full group C^*-algebra of G, and we are able to identify $\Sigma(G)$ with the class of all non-degenerate representations of $C^*(G)$ (cf. [8; 13.9.3]). If $\Gamma = \{\rho\}$ where ρ is the left regular representation of G in $L^2(G)$, then again $I_\rho = \{0\}$ and $C^*_\rho(G)$ is the C^*-algebra generated by $L^1(G)$ (or rather $\rho(L^1(G))$) in $\mathscr{L}(L^2(G))$. We define $P_\Gamma(G)$ and $B_\Gamma(G)$ as in [10, 1.21 and 2.2] for $\Gamma \subset \Sigma(G)$. We note that $P_\Gamma(G)$ can be identified as the set of all positive linear functionals of $C^*_\Gamma(G)$, and $B_\Gamma(G)$ can be regarded as the dual of $C^*_\Gamma(G)$. We denote by $W^*_\Gamma(G)$ the enveloping von-Neumann algebra of $C^*_\Gamma(G)$, and we may identify $W^*_\Gamma(G)$ as the second dual of $C^*_\Gamma(G)$ (cf. [8; 12.1.4]). We recall that $P(G)$ is the set of all continuous positive definite functions on G and $B(G)$ consists of all finite complex-linear combinations of functions in $P(G)$. As in [10; 2.16], $B(G)$ with the dual norm of $C^*(G)$ is a commutative Banach algebra; and it contains an ideal, $A(G)$ as defined in [10; 3.5], called the Fourier algebra of G. We denote by VN(G) the von-Neumann algebra generated by $L^1(G)$ in $\mathcal{L}(L^2(G))$, and note that $A(G)$ may be regarded as the predual of VN(G). We remark that $W^*(G)$, the enveloping von-Neumann algebra of $C^*(G)$, corresponds to the "big group algebra" defined by Ernest in [9].

If $K \subset G$, and if $f: G \to X$, $f \mid K$ is the restriction of f to K. If $\mathcal{B}$ is a family of functions of G to X, then $\mathcal{B} \mid K = \{f \mid K: f \in \mathcal{B}\}$. If $f: K \to \mathbb{C}$, then $f^\sigma: G \to \mathbb{C}$ is the extension of f such that f is identical zero outside K and $f^\sigma \mid K = f$, where $\mathbb{C}$ is the set of complex numbers.

Finally, for $\Lambda, \Gamma \subset \Sigma$ we shall write $\Lambda \cong \Gamma$ if Λ is weakly equivalent to Γ (cf. [12; chapter I,5]).

§2. **Embedding theorems for group C^*-algebras.** Let H be a locally compact subgroup of G with $\lambda(H) > 0$. Then $\lambda \mid H$ is a left Haar measure of H, and we shall simply write λ for $\lambda \mid H$. In this section, we are concerned about the extensions of the natural embedding $f \to f^\sigma$ from $L^1(H)$ into $L^1(G)$ to their corresponding group C^*-algebras. First, we note that $\lambda(H) > 0$ if and only if H is an open subgroup of G (cf. [13; Vol. I, Corollary 20.17]). Therefore, from now on H will denote an open subgroup of G for the extensions of the embeddings. The following proposition is immediate since $f \to f^\sigma$ is a $*$-homomorphism from $L^1(H)$ to $L^1(G)$.

PROPOSITION 2.1. *Let $\Gamma \subset \Sigma(H)$ and $\Lambda \subset \Sigma(G)$ be such that $\|f\|_\Gamma = \|f^\sigma\|_\Lambda$ for all $f \in L^1(H)$. Then the embedding $f \to f^\sigma$ from $L^1(H)$ into $L^1(G)$ induces an isometric $*$-homomorphism from $C^*_\Gamma(H)$ into $C^*_\Lambda(G)$.*

DEFINITION 2.2. We shall write $C^*_\Gamma(H) \le C^*_\Lambda(G)$ if $C^*_\Gamma(H)$ can be embedded in $C^*_\Lambda(G)$ as in Proposition 2.1.

THEOREM 2.3 (cf. [10; 2.31]). *Let us denote by ρ_1 the left regular representation of H. Then we have $C^*_{\rho_1}(H) \le C^*_\rho(G)$.*

Proof. We shall show that $\|f\|_{\rho_1} = \|f^\sigma\|_\rho$ for all $f \in L^1(H)$. Let $\{Hx \mid x \in I\}$ be the collection of right cosets of H in G. Then $L^2(G) = \sum_{x \in I} L^2(Hx)$, the direct sum of $L^2(Hx)$'s. Since

$$(\rho(f^\sigma))g(y) = f^\sigma * g(y) = \int_H f(z)g(z^{-1}y)\,dz,$$

$L^2(Hx)$ is invariant under $\rho(f^\sigma)$. For $g \in L^2(Hx)$, define $g' \in L^2(H)$ by $g'(y) = \Delta(x)^{-1/2} g(yx^{-1})$. Then $g \to g'$ is an isomorphism of $L^2(Hx)$ onto $L^2(H)$ and for every $y \in Hx$, $(f^\sigma * g)'(y) = \Delta(x)^{-1/2} \int_H f(z)g(z^{-1}yx^{-1})\,dz = f^\sigma * g'(z)$. Hence $\rho(f^\sigma) \mid L^2(Hx)$ is unitarily equivalent to $\rho(f^\sigma) \mid L^2(H) = \rho_1(f)$. It follows that $\|f\|_{\rho_1} = \|\rho_1(f)\|_{\mathcal{L}(L^2(H))} = \|\rho(f^\sigma)\|_{\mathcal{L}(L^2(G))} = \|f^\sigma\|_\rho$. Therefore, by Proposition 2.1, $C^*_{\rho_1}(H) \le C^*_\rho(G)$.

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THEOREM 2.4. *If $C^*_\Gamma(H) \le C^*_\Lambda(G)$ and α is the embedding, then ${}^t\alpha$, the transpose of α, is a norm decreasing homomorphism of $B_\Lambda(G)$ onto $B_\Gamma(H)$ such that ${}^t\alpha(u) = u \mid H$ for $u \in B_\Lambda(G)$. The bitranspose ${}^{tt}\alpha$ is an isomeric normal $*$-homomorphism of $W^*_\Gamma(H)$ into $W^*_\Lambda(G)$ which extends α.*

Proof. The transpose ${}^t\alpha$ maps $B_\Lambda(G)$ onto $B_\Gamma(H)$ since α is injective; and for $u \in B_\Lambda(G)$, $f \in L^1(H)$, we have

$$({}^t\alpha(u))(f) = \langle u, \alpha \circ f \rangle = \langle u, f^\sigma \rangle = \int_G f^\sigma(x) u(x)\, dx$$

$$= \int_H f(x)(u \mid H)(x)\, dx = \langle u \mid H, f \rangle.$$

Therefore, ${}^t\alpha(u) = u \mid H$. $\|{}^t\alpha(u)\| \le \|u\|$ is clear since α embeds the unit ball of $C^*_\Gamma(H)$ into the unit ball of $C^*_\Lambda(G)$. ${}^{tt}\alpha$ is a normal $*$-homomorphism of $W^*_\Gamma(H)$ into $W^*_\Lambda(G)$ which extends α by [8; 12.5.11]. Since ${}^{tt}\alpha$ is injective, it is an isometry.

THEOREM 2.5. *Let $\Gamma \subset \Sigma(H)$ and $\Lambda \subset \Sigma(G)$. Then the following conditions are equivalent:*

(i) $C^*_\Gamma(H) \le C^*_\Lambda(G)$;

(ii) $\|f\|_\Gamma = \|f^\sigma\|_\Lambda$ *for all* $f \in L^1(H)$;

(iii) $\Gamma \cong \Lambda \mid H$;

(iv) $P_\Gamma(H) = P_\Lambda(G) \mid H$.

Proof. Proposition 2.1 gives (i) $\Leftrightarrow$ (ii). For (ii) $\Leftrightarrow$ (iii), we observe that $\|f\|_\Gamma = \|f\|_{\Lambda|H}$ for all $f \in L^1(H)$ is equivalent to $\Gamma \cong \Lambda \mid H$ by [10; 1.24]. So it suffices to show that $\|f\|_{\Lambda|H} = \|f^\sigma\|_\Lambda$. Indeed, for $\pi \in \Lambda$ and $f \in L^1(H)$, $(\pi \mid H)(f) = \int_H \pi(x) f(x)\, dx = \int_G \pi(x) f^\sigma(x)\, dx = \pi(f^\sigma)$; thus $\|f\|_{\Lambda|H} = \|f^\sigma\|_\Lambda$.

Assume (i), and let $u \in P_\Gamma(H)$. We may treat u as a positive linear functional on $C^*_\Gamma(H)$, and identify $C^*_\Gamma(H)$ as a C^*-subalgebra of $C^*_\Lambda(G)$. Since positive functionals on the subalgebra extend to positive functions on the containing algebra, we obtain a $v \in P_\Lambda(G)$ and $v \mid H = u \in P_\Lambda(G) \mid H$. This shows that $P_\Gamma(H) \subset P_\Lambda(G) \mid H$. Conversely, let $v \in P_\Lambda(G)$; then for $f \in L^1(H)$,

$$|\langle v \mid H, f \rangle| = \left| \int_H v(x) f(x)\, dx \right| = \left| \int_G v(x) f^\sigma(x)\, dx \right| \le v(e) \|f^\sigma\|_\Lambda.$$

Since $\|f\|_\Gamma = \|f^\sigma\|_\Lambda$ by the assumption, $v \mid H \in P_\Gamma(H)$ by [10; Proposition 1.21(iii)]. Thus $P_\Lambda(G) \mid H \subset P_\Gamma(H)$ and we have shown that (i) $\Rightarrow$ (iv).

To prove that (iv) $\Rightarrow$ (ii), we note that it suffices to show that (ii) holds for hermitian elements in $L^1(H)$. Let $f \in L^1(H)$ and f is hermitian. By [18; 1.5.4], $\|f\|_\Gamma = \sup\{|\langle u, f \rangle| : u \in P_\Gamma(H),\ \|u\| = 1\}$. Since $P_\Gamma(H) = P_\Lambda(G) \mid H$ by assumption, and for $v \in P_\Lambda(G)$ $\langle v, f^\sigma \rangle = \langle v \mid H, f \rangle$; it is easy to see that $\|f\|_\Gamma = \|f^\sigma\|_\Lambda$. The proof of the Theorem is now complete.

We let $\Lambda^{\sim}$ be the class of all representation weakly contained in $\Lambda \subset \Sigma(G)$.

PROPOSITION 2.6. *Let* $\Lambda \subset \Sigma(G)$. *Then* $\Lambda^{\sim}$ *is the class of all representations of* $C^*_\Lambda(G)$.

Proof. Let N'_Λ be the kernel of the quotient map from $C^*(G)$ onto $C^*_\Lambda(G)$ as in [10; 1.15]. Then π is a representation of $C^*_\Lambda(G)$ if and only if $\pi \in \Sigma(G)$ and $N'_\Lambda \subset \ker \pi$, i.e., $\pi \in \Lambda^{\sim}$.

COROLLARY 2.7. *Let* $\Gamma \subset \Sigma(H)$, $\Lambda \subset \Sigma(G)$ *and* $\Gamma \cong \Lambda \mid H$. *Then for any* $\pi \in \Gamma^{\sim}$, *there exists a* $\pi' \in \Lambda^{\sim}$ *such that* $\pi = (\pi' \mid H) \mid \mathcal{H}_\pi$.

Proof. By Theorem 2.5, we may consider $C^*_\Gamma(H)$ as a C^*-subalgebra of $C^*_\Lambda(G)$. The corollary follows immediately from Proposition 3.5 and [8; 2.10.2].

LEMMA 2.8. *If* $u \in P(H)$, *then* $u^\sigma \in P(G)$.

Proof. It follows from [13; vol. II, 32.43] and the assumption that H is open.

THEOREM 2.9. $C^*(H) \leq C^*(G)$ *if H is an open subgroup of G.*

Proof. By Lemma 2.8, we have $P(H) \subset P(G) \mid H$. Since $P(G) \mid H \subset P(H)$ is obvious, $P(H) = P(G) \mid H$ follows; and Theorem 2.5 asserts $C^*(H) \leq C^*(G)$.

REMARK 2.10. From [9; Remarks, pp. 476–477], for any locally compact group G, we may embed G, $L^1(G)$, $M^1(G)$ and $C^*(G)$ into $W^*(G)$. If H is an open subgroup of G, then Theorem 2.9 together with Theorem 2.4 enables us to identify $W^*(H)$ as a W^*-subalgebra of $W^*(G)$ and this identification extends the natural embeddings from $H(L^1(H)$, $M^1(H)$, $C^*(H)$ resp.) into $G(L^1(G)$, $M^1(G)$, $C^*(G)$ resp.)

§3. **Embedding theorems for the duals.** For $\Lambda \subset \Sigma(G)$, recall that the dual $B_\Lambda(G)$ of $C^*_\Lambda(G)$ is a Banach subalgebra of the Fourier–Stieltjes algebra $B(G)$. This section centers at the question when $B_\Gamma(H)$ can be embedded in $B_\Lambda(G)$ via $u \to u^\sigma$ for $u \in B_\Gamma(H)$.

THEOREM 3.1. *For* $\Gamma \subset \Sigma(H)$ *and* $\Lambda \subset \Sigma(G)$, *the following conditions are equivalent*:

(i) $\|f \mid H\|_\Gamma \leq \|f\|_\Lambda$ *for* $f \in L^1(G)$;

(ii) *There exists a bounded *-linear map* τ *from* $C^*_\Lambda(G)$ *to* $C^*_\Gamma(H)$ *which extends* $f \to f \mid H$ *for* $f \in L^1(G)$;

(iii) *For every* $u \in P_\Gamma(H)$, *we have* $u^\sigma \in P_\Lambda(G)$.

Proof. (i) $\Rightarrow$ (ii) is clear. Assume (ii) and consider the transpose ${}^t\tau : B_\Gamma(H) \to$

$B_\Lambda(G)$. Let $u \in B_\Gamma(H)$. Then for $f \in L^1(G)$,

$$\langle {}^t\tau(u), f\rangle = \langle u, \tau(f)\rangle = \int_H u(x)f(x)\,dx = \langle u^\sigma, f\rangle.$$

Thus ${}^t\tau(u) = u^\sigma$; and if $u \in P_\Gamma(H)$, then

$$|\langle u^\sigma, f\rangle| = |\langle u, \tau(f)\rangle| \le u(e)\,\|\tau(f)\|_\Gamma.$$

Since τ is bounded, $u^\sigma \in P_\Lambda(G)$ by [10; Proposition 1.21(iii)].

To show that (iii) implies (i), we need the following lemma which can be shown by straight-forward calculation.

LEMMA 3.2. *If we write $H^c = G\backslash H$, then for $f \in L^1(G)$, $g \in L^p(G)$ and $x \in H$, $f * g(x) = (f \mid H) * (g \mid H)(x) + (f \mid H^c) * (g \mid H^c)(x)$.*

Now assume (iii), and let $u \in P_\Gamma(H)$. For $f \in L^1(G)$,

$$\begin{aligned}\langle u^\sigma, f^* * f\rangle &= \langle u, (f^* * f) \mid H\rangle \\ &= \langle u, (f^* \mid H) * (f \mid H)\rangle + \langle u^\sigma, (f^* \mid H^c)^\sigma * (f \mid H^c)^\sigma\rangle.\end{aligned}$$

Since $u^\sigma \in P_\Lambda(G)$ and $(f^* \mid H^c)^\sigma * (f \mid H^c)^\sigma$ is positive in $C^*_\Lambda(G)$, it follows that $\langle u^\sigma, f^* * f\rangle \ge \langle u, (f^* \mid H) * (f \mid H)\rangle$. Note that $f^* \mid H = (f \mid H)^*$, we have

$$\|(f \mid H)^* * (f \mid H)\|_\Gamma \le \|f^* * f\|_\Lambda$$

in view of [18; 1.5.4]; thus $\|f \mid H\|_\Gamma \le \|f\|_\Lambda$ and the proof of Theorem 3.1 is complete.

The equivalent conditions in Theorem 3.1 show when the map $u \to u^\sigma$ defines a homomorphism from $B_\Gamma(H)$ into $B_\Lambda(G)$. In order to assure that it is an isometry, we have

THEOREM 3.3. *If the conditions in Theorem 3.1 are satisfied and $C^*_\Gamma(H) \le C^*_\Lambda(G)$, then the map $u \to u^\sigma$ is an isometry from $B_\Gamma(H)$ into $B_\Lambda(G)$. Furthermore, the map $f \to f \mid H$ from $L^1(G)$ onto $L^1(H)$ induces an expectation Φ from $W^*_\Lambda(G)$ onto $W^*_\Gamma(H)$ such that $\Phi(C^*_\Lambda(G)) = C^*_\Gamma(H)$.*

Proof. Let $u \in B_\Gamma(H)$, $f \in L^1(H)$ and $g \in L^1(G)$. We have $|\langle u^\sigma, g\rangle| = |\langle u, g \mid H\rangle| \le \|u\|\,\|g \mid H\|_\Gamma \le \|u\|\,\|g\|_\Lambda$ and $|\langle u, f\rangle| = |\langle u^\sigma, f^\sigma\rangle| \le \|u^\sigma\|\,\|f^\sigma\|_\Lambda = \|u^\sigma\|\,\|f\|_\Gamma$, thus $\|u\| = \|u^\sigma\|$ and $u \to u^\sigma$ is an isometry from $B_\Gamma(H)$ into $B_\Lambda(G)$. Let Φ be the dual of this isometry, then Φ maps $W^*_\Lambda(G)$ onto $W^*_\Gamma(H)$. It is easy to see that $\Phi(f) = f \mid H$ for $f \in L^1(G)$; thus if we identify $W^*_\Gamma(H)$ as a W^*-subalgebra of $W^*_\Lambda(G)$ via the natural embedding, we have $\Phi \circ \Phi(a) = \Phi(a)$ and $\|\Phi(a)\| \le \|a\|$ for all $a \in C^*_\Lambda(G)$. Since $C^*_\Lambda(G)$ is weakly dense in $W^*_\Lambda(G)$, we have $\Phi \circ \Phi = \Phi$ and $\|\Phi(a)\| \le \|a\|$ for $a \in W^*_\Lambda(G)$; by a result of Tomiyama [19], it follows that Φ is an expectation. It is clear that $\Phi(C^*_\Lambda(G)) \subset C^*_\Gamma(H)$; hence $\Phi(C^*_\Lambda(G)) = C^*_\Gamma(H)$. This completes the proof.

THEOREM 3.4. *If H is an open subgroup of G, then the map $u \to u^\sigma$ is an isometry of $B(H)$ into $B(G)$, which maps $B_{\rho_1}(H)$ into $B_\rho(G)$. Furthermore, the map $f \to f \mid H$ extends to an expectation Φ(Ψ resp.) from $W^*(G)$($W^*_\rho(G)$ resp.) onto $W^*(H)$($W^*_{\rho_1}(H)$ resp.) and $\Phi(C^*(G)) = C^*(H)$($\Psi(C^*_\rho(G)) = C^*_{\rho_1}(H)$ resp.).*

Proof. Since $C^*(H) \le C^*(G)$ by Theorem 2.9 and $u \in P(H)$ implies $u^\sigma \in P(G)$ by Lemma 2.8, it follows that $u \to u^\sigma$ is an isometry of $B(H)$ into $B(G)$ by Theorem 3.3. To show the rest of the Theorem, it is sufficient to prove $\|f \mid H\|_{\rho_1} \le \|f\|_\rho$ for $f \in L^1(G)$ in view of Theorem 3.3. Let $f \in L^1(G)$ and $g \in L^2(H)$. Then $f * g = f \mid H * g + f \mid H^c * g$, and $f \mid H^c * g \in L^2(H^c)$ by Lemma 3.2; thus $\|f * g\|_2 \ge \|(f \mid H) * g\|_2$. It follows that $\|f \mid H\|_{\rho_1} = \sup\{\|(f \mid H) * g\|_2 \mid g \in L^2(H) \text{ and } \|g\|_2 \le 1\} \le \|f\|_\rho$ for all $f \in L^1(G)$.

COROLLARY 3.5 (cf. [10; 3.21]). *Let H be an open subgroup of G. Then*

(1) *The map $v \to v \mid H$ is a norm decreasing homomorphism of $A(G)$ onto $A(H)$. Its transpose is an isometric *-homomorphism of* VN(H) *into* VN(G) *which extends the embedding of $C^*_\rho(H)$ into $C^*_\rho(G)$;*

(2) *The map $u \to u^\sigma$ is an isometric homomorphism of $A(H)$ into $A(G)$. Its transpose is an expectation of* VN(G) *onto* VN(H) *which maps $C^*_\rho(G)$ onto $C^*_\rho(H)$.*

Proof. (1) By Theorem 2.3 and 2.4, the map $v \to v \mid H$ is a norm decreasing homomorphism of $B_\rho(G)$ onto $B_{\rho_1}(H)$. It follows from [10; Proposition 3.4] that $v \to v \mid H$ maps $A(G)$ onto $A(H)$. Clearly, its transpose is injective and extends $f \to f^\sigma$ of $L^1(H)$ into $L^1(G)$; hence the transpose extends the embedding of $C^*_\rho(H)$ into $C^*_\rho(G)$. Hence the transpose is a *-homomorphism from VN(H) into VN(G) since $C^*_{\rho_1}(H)$ and $C^*_\rho(G)$ are weakly dense in VN(H) and VN(G) respectively. It is an isometry since it is injective.

(2) It follows from Theorem 3.4 that $u \to u^\sigma$ maps $A(H)$ into $A(G)$ isometrically, and its transpose is an expectation of VN(G) onto VN(H) if we identify VN(H) as a subalgebra of VN(G) by the isometry in (1).

§4. **Applications and examples.** First we generalize a result of Paschke and Salinas [16; Proposition 1.6].

COROLLARY 4.1. *Let G be a locally compact group with a non-trivial amenable normal open subgroup H. Then there is a tracial state τ on $C^*_\rho(G)$ such that $\tau(f) = \int_H f\, d\lambda$ for all $f \in L^1(G)$. Furthermore, $C^*_\rho(G)$ is not simple.*

Proof. Since H is amenable, the positive definite function τ_1, defined by $\tau_1(x) = 1$ for all $x \in H$, is in $P_{\rho_1}(H)$. By Theorem 3.4, $\tau_1^\sigma \in P_\rho(G)$. Write $\tau = \tau_1^\sigma$. Then τ can be considered as a state of $C^*_\rho(G)$ and $\tau(f) = \langle \tau_1^\sigma, f \rangle = \langle \tau_1, f \mid H \rangle = \int_H f\, d\lambda$ for $f \in L^1(G)$. For $f, g \in L^1(G)$, $\tau(f * g) = \int_H f * g\, d\lambda = \int_G \int_G \tau(xy) f(x) g(y)\, dx\, dy$. Since H is normal in G, $\tau(xy) = \tau(yx)$ for $x, y \in G$; it

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follows that $\tau(f*g)=\tau(g*f)$; hence τ is a tracial state. Let $I=\{T: \tau(T^* * T)=0$ for $T\in C^*_\rho(G)\}$. Then I is an ideal of $C^*_\rho(G)$ as τ is tracial. Observe that τ is multiplicative on the subalgebra $C^*_{\rho_1}(H)$ of $C^*_\rho(G)$, $\ker(\tau \mid C^*_{\rho_1}(H))$ is not trivial and contained in I. Thus $C^*_\rho(G)$ is not simple. This completes the proof.

Discrete groups have been used to construct examples (counterexamples) of different types of operator algebras ever since Murry and von Neumann began the serious study of operator algebras. Among them, the free group F_2 on two generators was the most prominent. Recently, attention has been paid to groups containing F_2 as a subgroup (see [1], [2], [5] and [6]). For the rest of this section, G will denote a countable discrete group with identity e. F_n denotes the free group n generators and F_∞ the free group on infinite generators. We shall write elements of $L^2(G)$ as formal sums $\sum_{x\in G}\alpha_x x$ where $\sum_{x\in G}|\alpha_x|^2<\infty$. For $y\in G$, let $\beta(y)$ be the operator is $L^2(G)$ defined by

$$\beta(y)\left(\sum_{x\in G}\alpha_x x\right)=\sum_{x\in G}\alpha_x yxy^{-1}.$$

It is immediate that β is a unitary representation of G in $L^2(G)$. We shall call β the inner representation of G. Let H be a subgroup of G with β_1, the inner representation of H in $L^2(H)$. Unlike the left regular representation, $\beta \mid H$ in general is not weakly equivalent to β_1 as the following example shows.

EXAMPLE 4.2. Let $G=F_2$ and $H=F_1$. Then β_1 is just the trivial representation, however, $\beta \mid H$ is not. Therefore, $C^*_{\beta_1}(H)$ in general cannot be embedded in $C^*_\beta(G)$ as a C^*-subalgebra in the canonical way.

By a similar argument used by Akemann in [1; Theorem 3] and [4; Theorem 1], we prove the following theorem.

THEOREM 4.3. *Let H be a subgroup of G such that $C^*_\rho(H)$ is simple and $\{z\in H: zw=wz\}$ is amenable for $e\neq w\in G$. Then $C^*_{\beta_1}(H)\leq C^*_\beta(G)$.*

Proof. It suffices to show that $\|f\|_{\beta_1}=\|f^\sigma\|_\beta$ for $f\in L^1(H)$. For $w\in G$ let $C_w=\{xwx^{-1}: x\in H\}$. We denote by L_w the closed subspace of $L^2(G)$ spanned by C_w.

We claim that $\|f^\sigma\|_{\beta|L_w}=\|f\|_\rho$ for $e\neq w\in G$ and $f\in L^1(H)$ where ρ is the left regular representation of H. To prove the claim, we write $f=\sum_{x\in H}\alpha_x x$ and fixed $e\neq w\in G$. Write $T=\{x\in H: xw=wx\}$. Let H/T be the left coset space and let θ be the unitary representation of H in $L^2(H/T)$ defined by left translation. Since T is amenable by assumption, the trivial representation is weakly contained in the left regular representation of T. By [11; Theorem 4.2] and [14; p. 121], θ is weakly contained in ρ. By [10; Lemma 1.23], $\|f\|_\theta\leq\|f\|_\rho$. Therefore, the identity map $i: L^1(H)\to L^1(H)$ extends to a *-homomorphism $\tilde{\imath}: C^*_\rho(H)\to C^*_\theta(H)$. Since $C^*_\rho(H)$ is simple, $\tilde{\imath}$ is injective; hence $\|f\|_\theta=\|f\|_\rho$. Now let $g \in L_w$ and $g = \sum_{x \in H} \alpha_x xwx^{-1}$, define $g' = \sum_{x \in H} \alpha_x (xT)$. It is clear that $g \to g'$ is an isomorphism of L_w onto $L^2(H/T)$, and we have, for every $y \in H$,

$$(\beta(y)(g))' = \left(\beta(y)\left(\sum_{x \in H} \alpha_x xwx^{-1}\right)\right)' = \sum_{x \in H} \alpha_x yxT = \theta(y)(g').$$

Hence $\beta \mid L_w$ is unitary equivalent to θ on H. It follows that $\|f\|_{\beta|L_w} = \|f\|_\theta$, hence $\|f\|_{\beta|L_w} = \|f\|_\rho$, and the claim is proved.

It is clear that $\{C_w: w \in G\}$ partitions G and $\beta(f^\sigma)$ reduces L_w for $w \in G$. Therefore, we have $\|\beta(f^\sigma)\| = \sup_{w \in G} \|\beta(f^\sigma) \mid L_w\|$. Since $\{C_w: w \in H\}$ partitions H and $\beta_1(f) \mid L_w$ is unitarily equivalent to $\beta(f^\sigma) \mid L_w$ for $w \in H$, we have $\|\beta_1(f)\| = \sup_{w \in H} \|\beta(f^\sigma) \mid L_w\|$. By definition (see §1), $\|\beta(f^\sigma) \mid L_w\| = \|f\|_{\beta|L_w}$; whence it follows from the claim that $\|\beta(f^\sigma) \mid L_w\| = \|f\|_\rho$ for $e \neq w \in G$. Thus $\|\beta(f^\sigma)\| = \max\{\|f \mid L_e\|, \|f\|_\rho\} = \|\beta_1(f)\|$. This completes the proof.

REMARK 4.4. Under the same hypothesis of Theorem 4.3, we have $\|f\|_{\beta_1} = \|f^\sigma\|_\beta = \max\{|\sum_{x \in H} \alpha_x|, \|f\|_\rho\}$ for $f = \sum_{x \in H} \alpha_x x$ since $\|f \mid L_e\| = |\sum_{x \in H} \alpha_x|$.

COROLLARY 4.5. *Let G_1 and G_2 be nontrivial groups (not both of order 2) with $G = G_1 * G_2$, their free product. Then $C^*_{\beta_n}(F_n)$ can be embedded in $C^*_\beta(G)$ as a C^*-subalgebra for $n > 1$ or equals ∞ where β_n denotes the inner representation of F_n on $L^2(F_n)$.*

Proof. By [2; Corollary 6], G contains a free non-abelian subgroup F for which $\{z \in F: wz = zw\}$ is abelian for each $e \neq w \in G$. As any non-abelian free group contains free subgroup of any rank, the corollary follows.

Using Corollary 4.5 and Remark 4.4, we can easily find examples of groups and group representations such that they satisfy the conditions in Theorem 2.5 but fail the conditions in Theorem 3.1.

EXAMPLE 4.6. Let F_3 be the free group on generators u, v and w. Let F_2 be the subgroup of F_3 generated by u and v. Let β_i denote the inner representation of F_i on $L^2(F_i)$ for $i = 2, 3$. Then by Corollary 4.5, we have $C^*_{\beta_2}(F_2) \leq C^*_{\beta_3}(F_3)$. Now let $\{x_i \mid i = 1, 2, \ldots\}$ be a free set of F_2. Let $f_n = (\sum_{i=1}^n x_i) - w$. Then $f_n \in L^1(F_3)$ and $f_n \mid F_2 = \sum_{i=1}^n x_i$. By Remark 4.4, $\|f_n\|_{\beta_3} = \max\{n-1, \|f_n\|_\rho\}$ and $\|f_n \mid F_2\|_{\beta_2} = \max\{n, \|f_n \mid F_2\|_\rho\}$. By Theorem 1V.K. of [3], $\|f_n\|_\rho = 2\sqrt{n}$ and $\|f_n \mid F_2\|_\rho = 2\sqrt{(n-1)}$. Therefore, by choosing a sufficiently large n, we have $\|f_n\|_{\beta_3} < \|f_n \mid F_2\|_{\beta_2}$. It follows by Theorem 3.1 that $u \to u^\sigma$ is not an embedding of $B_{\beta_2}(F_2)$ into $\beta_{\beta_3}(F_3)$.

REMARK 4.7. In [6], [7] and [20], some unusual aspects of $C^*(F_2)$ and $C^*_\rho(F_2)$ have been shown. We remark that by using embedding theorems for group C^*-algebras and their duals those results can easily be extended to $C^*(F_n)$ and $C^*_{\rho_n}(F_n)$ for $n > 1$ or equals ∞.

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