A CHARACTERIZATION OF REAL ALMOST CONTINUOUS FUNCTIONS

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1. Introduction. The main purpose of this note is to give a characterization of almost continuous functions similar to that of approximately continuous functions ([3], § 235, p.312). The characterization of approximately continuous real functions has been used to show that each approximately continuous function is almost continuous ([4], § 6). Thus combining these results we see that the classes of almost continuous, approximately continuous and continuous functions form a descending chain.

2. The Characterization Theorem. Let f be a real valued function defined on the real line R. Then f is said to be almost continuous at $x_o \varepsilon R$ if for each arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\{x \varepsilon R: |f(x) - f(x_o)| < \varepsilon\}$ is dense in the open interval $(x_o - \delta, x_o + \delta) = (x \epsilon R: |x - x_o| < \delta\}$.

In view of the definition of continuity, it is quite clear that if f is continuous at x_o then it is also almost continuous. But the converse is not true (§ 3, Example 1).

THEOREM 1. Let f be a real valued function on R. A necessary and sufficient condition for f to be almost continuous at $x_o \varepsilon R$ is that there exists a subset G of R which is dense in some neighbourhood of x_o, and relative to which f is continuous at x_o, i.e., if x_n $(n \geq 1)$ is any sequence in G such that x_n converges to x_o then the sequence $f(x_n)(n \geq 1)$ converges to $f(x_o)$.

Proof. Sufficient part. Let $\varepsilon > 0$ be given. Then the

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continuity of f relative to G implies that there exists a $\delta_1 = \delta_1(\varepsilon) > 0$ such that $|f(x) - f(x_o)| < \varepsilon$, whenever $x \varepsilon G$ and $|x - x_o| < \delta_1$. Let U_o be a neighbourhood of x_o in which the set G is dense. Then there exists a $\delta_2 > 0$ such that the open interval $(x_o - \delta_2, x_o + \delta_2) \subset U_o$. Let $\delta = \min.[\delta_1, \delta_2]$. Then clearly for all $x \varepsilon G \cap (x_o - \delta, x_o + \delta)$, $|x - x_o| < \delta$ implies $|f(x) - f(x_o)| < \varepsilon$. Since G is dense in U_o and therefore dense in $(x_o - \delta, x_o + \delta)$, it follows that,

$$G \supset \overline{G \cap (x_o - \delta, x_o + \delta)} \supset (x_o - \delta, x_o + \delta),$$

where $\bar{G}$ denotes the closure of G. Thus

$$\{x \varepsilon R: |f(x) - f(x_o)| < \varepsilon\} \supset (x_o - \delta, x_o + \delta) \cap G$$

and

$$\overline{G \cap (x_o - \delta, x_o + \delta)} \supset (x_o - \delta, x_o + \delta),$$

both taken together show that f is almost continuous.

Necessary part: Suppose f is almost continuous at x_o. For each positive integer n, there exists a $\delta_n > 0$ such that $G_n = \{x \varepsilon R: |f(x) - f(x_o)| < \frac{1}{n}\}$ is dense in $(x - \delta_n, x + \delta_n)$.

By induction we can choose a strictly monotonically decreasing sequence $\{\delta_n\}_{n \geq 1}$ such that $\delta_n \to 0$.

For each n, define

$$F_n = G_n \cap [(x_o + \delta_{n+1}, x_o + \delta_n) \cup (x_o - \delta_n, x_o - \delta_{n+1})].$$

Then $F_m \subset F_n$ for $m > n$ because $G_m \subset G_n$.

Let $G = \bigcup_{n=1}^{\infty} F_n$. Since G_n is dense in $(x_o - \delta_n, x_o + \delta_n)$, so is F_n in $[(x_o + \delta_{n+1}, x_o + \delta_n) \cup (x_o - \delta_n, x_o - \delta_{n+1})]$ for each $n \geq 1$. Hence G is dense in

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$$\bigcup_{n=1}^{\infty} [(x_o + \delta_{n+1}, x_o + \delta_n) \cup (x_o - \delta_n, x_o - \delta_{n+1})]$$

$$= (x_o, x_o + \delta_1) \cup (x_o - \delta_1, x_o) .$$

But x_o being a limit point of $(x_o - \delta_1, x + \delta_1)$, G is dense in $(x_o - \delta_1, x_o + \delta_1)$ which is a neighbourhood of x_o .

Now to show the continuity of f relative to G at x_o , let $\{x_m\}_{m \geq 1}$ be a monotonic sequence in G such that $x_m \to x_o$. Clearly for each $m \geq 1$, $x_m \varepsilon F_{n(m)}$ for some $n(m)$. As $m \to \infty$, $n(m) \to \infty$ because $\{x_m\}$ is monotonic. But since $F_n \subset G_n$ for each $n \geq 1$, we have $|f(x_m) - f(x_o)| < \frac{1}{n(m)}$. As $m \to \infty$, $f(x_m) \to f(x_o)$. This completes the proof.

3. Examples.

1. There exists an almost continuous function which is not continuous. For example, let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational.} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then it is very well known that f is totally discontinuous, i.e., it is continuous at no points of R . To show that f is almost continuous, let $\varepsilon > 0$ be given. Then, $\{x \varepsilon R: |f(x) - 1| < \varepsilon = Q$ or R , according as $\varepsilon < 1$ or $\varepsilon > 1$, where Q is the set of rationals. In each case Q and R being dense in R it follows that f is almost continuous at each rational point. Similarly one can show that f is almost continuous at each irrational point.

2. There exists an almost continuous function which is not approximately continuous.

The real valued function f in 1. is almost continuous but not approximately continuous at 0 . For, the metric density of the rationals at 0 is zero. Clearly f is continuous only relative to the rationals at 0 . Thus by the characterization theorem of approximately continuous functions ([3], p.312) it follows that f is not approximately continuous.

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3. There exists an approximately continuous real function which is not continuous. For example see ([2], p.190).

Thus we see that the class of all real almost continuous functions is strictly larger than that of approximately continuous functions and the latter is strictly larger than the class of all real continuous functions.

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