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Hyperplanes in the Space of Convergent Sequences and Preduals of ℓ_1

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Abstract. The main aim of this paper is to investigate various structural properties of hyperplanes of c, the Banach space of the convergent sequences. In particular, we give an explicit formula for the projection constants and prove that an hyperplane of c is isometric to the whole space if and only if it is 1-complemented. Moreover, we obtain the classification of those hyperplanes for which their duals are isometric to ℓ_1 and give a complete description of the preduals of ℓ_1 under the assumption that the standard basis of ℓ_1 is weak*-convergent.

1 Introduction

This paper is mainly devoted to investigating the structural properties of the closed hyperplanes of the Banach space c of the convergent sequences. This study reveals that this class of spaces is very interesting, since it provides a complete isometric description of the preduals of the Banach space ℓ_1 when it is assumed that the standard basis of this space is weak*-convergent.

The starting point of our work is a result that lists some properties of the hyperplanes of c_0. More specifically, the following essentially known theorem summarizes some characterizations of the 1-complemented hyperplanes in c_0. We prefer to give a short proof of this result for the reader's convenience. Indeed, some of the quoted known facts are scattered throughout the literature, and a simple remark, based on a well known property of ℓ_∞, is easy but not immediate.

Theorem 1.1 *Let $f \in \ell_1$ be such that $\|f\|_{\ell_1} = 1$. Let us consider the hyperplane $V_f = \ker f \subset c_0$. The following statements are equivalent:*

(i) *V_f is 1-complemented;*
(ii) *V_f^* is isometric to ℓ_1;*
(iii) *there exists an index j_0 such that $|f_{j_0}| \geq \frac{1}{2}$;*
(iv) *V_f is isometric to c_0.*

Proof First, we recall that (i) is equivalent to (iii) (see [3]), and (i) implies (iv), since the 1-complemented infinite dimensional subspaces of c_0 are isometric to the whole c_0 (see, *e.g.,* [7]). Trivially, (iv) implies (ii). Finally, we show that (ii) implies (i). By (ii) there exists an isometry $T\colon V_f^* \to \ell_1$, hence $T^*\colon \ell_\infty \to V_f^{**}$ is also an isometry. By [5, Proposition 5.13, p. 142], there exists a norm-1 projection $P\colon \ell_\infty \to V_f^{**}$.

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Therefore, since $V_f^{**} = [f]^{\perp} = \{x^{**} \in \ell_\infty : x^{**}(f) = 0\}$, [2, Corollary 2] implies that V_f is 1-complemented in c_0. ∎

One may ask whether a similar result is true when the space c_0 is replaced by c. Therefore, the main aim of this paper is to investigate the properties of hyperplanes in c. In particular, we would like to determine whether the implications of Theorem 1.1 are preserved when we consider c instead of c_0. This study allows us to show that the behavior of hyperplanes in c is much richer than in c_0. Indeed, we will show that the counterpart in c of Theorem 1.1 is the following result.

Theorem 1.2 *Let $f \in \ell_1$ be such that $\|f\|_{\ell_1} = 1$, and let $W_f = \ker f \subset c$. Let us consider the following properties:*

(i) *W_f is 1-complemented;*
(ii) *W_f is isometric to c;*
(iii) *there exists $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$;*
(iv) *W_f^* is isometric to ℓ_1;*
(v) *there exists $j_0 \geq 1$ such that $|f_{j_0}| \geq \frac{1}{2}$;*
(vi) *W_f is isometric to c_0;*
(vii) $\inf_P \|P\| = 2$ *(where $P\colon c \to W_f$ is a projection);*
(viii) $|f_1| = 1$, $f_j = 0$ *for every $j \geq 2$.*

Then the following implications hold

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longleftrightarrow \text{(iii)} \Longrightarrow \text{(iv)} \Longleftrightarrow \text{(v)} \Longleftarrow \text{(vi)} \Longleftrightarrow \text{(vii)} \Longleftrightarrow \text{(viii)}.$$

The previous theorem is the main result of our paper, and in order to prove it we need a number of intermediate results that are interesting in themselves. First of all, we investigate the properties of the projections on the hyperplanes of c. Indeed, in Section 2, by following the approach outlined in [3] for c_0, we characterize the 1-complemented hyperplanes of c and establish a formula to compute the projection constant of a given hyperplane in c. The second step (Section 3) is to study the hyperplanes of c that are isometric to c and c_0, respectively. It is worth mentioning that such situations appear only when the projection constant of the hyperplane attains its minimum and maximum, respectively. Indeed we show that a hyperplane of c is isometric to c itself if and only if it is 1-complemented. Moreover, the only hyperplane of c that is an isometric copy of c_0 has projection constant 2, and is the "natural" one, *i.e.*, the subspace of c whose elements are the vanishing sequences. At the beginning of Section 4, Proposition 4.1 characterizes the hyperplanes of c whose duals are isometric copies of ℓ_1. The most interesting situations among the class of the spaces W_f such that its dual is ℓ_1 occur when W_f is isometric neither to c nor to c_0. Therefore, in these particular cases we compute the $\sigma(\ell_1, W_f)$-limit of the standard basis of ℓ_1 by explicitly describing the duality between W_f and ℓ_1 (see Theorem 4.3). This theorem allow us to obtain two interesting structural results. First, by using a result of [1], we show that a ℓ_1-predual space X is an isometric copy of W_f for a suitable choice of the functional $f \in \ell_1$, whenever the standard basis of $\ell_1 \simeq X^*$ is assumed to be weak*-convergent. Second, we characterize the hyperplanes W_f that are ℓ_1-preduals and are isometric to a quotient of some $C(\alpha)$ where $C(\alpha)$ denotes the space of all continuous real-valued functions on the ordinals less than or equal to α with the order topology.

In the sequel, whenever X is a Banach space, B_X denotes the closed unit ball of X and $[x]$ the linear span of a vector $x \in X$. We write $X \simeq Y$ when X and Y are isometrically isomorphic. We also use standard duality between c and ℓ_1, that is, for $x \in c$ and $f \in \ell_1$: $f(x) = \sum_{i=0}^{\infty} f_{i+1}x_i$, where $x_0 = \lim_{i\to\infty} x_i$. Throughout the paper, the hyperplane $W_f \subset c$ stands for the kernel of $f \in \ell_1$ with $\|f\|_{\ell_1} = 1$.

2 The Projections on the Hyperplanes of c

The aim of this section is to extend to c the study of the projections onto the hyperplanes of c_0 developed in [3].

First, the following lemma establishes a formula to compute the norm of a given projection on an hyperplane of c.

Lemma 2.1 *A projection of c onto W_f has the form $P_z(x) = x - f(x)z$ for some $z \in f^{-1}(1)$. Moreover,*

$$\|P_z\| = \sup_{i\geq 1}\{|1 - f_{i+1}z_i| + |z_i|(1 - |f_{i+1}|)\}. \tag{2.1}$$

Proof The first part is well known (see, *e.g.*, [3]). Now we will prove formula (2.1). We have

$$\|P_z\| = \sup_{x\in B_c}\sup_{i\geq 1}\Big|(P_z(x))_i\Big| = \sup_{x\in B_c}\sup_{i\geq 1}\Big|x_i - f(x)z_i\Big| = \sup_{x\in B_c}\sup_{i\geq 1}\Big|\sum_{j=0}^{+\infty}(\delta_{ij} - f_{j+1}z_i)x_j\Big|.$$

Therefore, it holds

$$\|P_z\| \leq \sup_{i\geq 1}\Big(\sum_{j=0}^{+\infty}|\delta_{ij} - f_{j+1}z_i|\Big). \tag{2.2}$$

Now let us consider, for every $i \geq 1$, the sequences $\{x^{(n,i)}\}_{n\geq 1} \subset B_c$ where $x^{(n,i)} = (x_1^{(n,i)}, x_2^{(n,i)}, \dots)$ is defined by

$$x_j^{(n,i)} = \begin{cases} \operatorname{sgn}(\delta_{ij} - f_{j+1}z_i) & \text{for } j \leq n, \\ \operatorname{sgn}(-f_1 z_i) & \text{for } j > n. \end{cases}$$

Then we have that for all integers $n \geq 1$,

$$\begin{aligned}\sup_{i\geq 1}\|P_z(x^{(n,i)})\| &= \sup_{i\geq 1}\Big|\sum_{j=0}^{+\infty}(\delta_{ij} - f_{j+1}z_i)x_j^{(n,i)}\Big| \geq \sup_{1\leq i\leq n}\Big|\sum_{j=0}^{+\infty}(\delta_{ij} - f_{j+1}z_i)x_j^{(n,i)}\Big| \\ &= \sup_{1\leq i\leq n}\Big|\sum_{j=0}^{n}|\delta_{ij} - f_{j+1}z_i| - z_i \operatorname{sgn}(-f_1 z_i)\sum_{j=n+1}^{+\infty} f_{j+1}\Big|.\end{aligned}$$

Therefore, we obtain that

$$\|P_z\| \geq \sup_{i\geq 1}\Big(\sum_{j=0}^{+\infty}|\delta_{ij} - f_{j+1}z_i|\Big). \tag{2.3}$$

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Hence, combining (2.2) and (2.3), we conclude that

$$\|P_z\| = \sup_{i\geq 1}\Big(\sum_{j=0}^{+\infty} |\delta_{ij} - f_{j+1}z_i|\Big).$$

Finally, an easy computation shows that

$$\|P_z\| = \sup_{i\geq 1}\Big\{|1-f_{i+1}z_i| + |z_i|\sum_{\substack{j=0\\ j\neq i}}^{+\infty}|f_{j+1}|\Big\} = \sup_{i\geq 1}\Big\{|1-f_{i+1}z_i| + |z_i|(1-|f_{i+1}|)\Big\}. \qquad \blacksquare$$

By means of the previous lemma, we are able to characterize the 1-complemented hyperplanes of c.

Proposition 2.2 *A norm-1 projection of c onto W_f exists if and only if $|f_j| \geq \frac{1}{2}$ for some $j \geq 2$. Moreover, there exists a unique norm-1 projection of c onto W_f if and only if there exists a unique index $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$.*

Proof By Lemma 2.1 W_f is the rank of a norm-1 projection if and only if there exists $z \in c$ such that

$$|1-f_{i+1}z_i| + |z_i|(1-|f_{i+1}|) \leq 1 \quad \forall i \geq 1, \tag{2.4}$$

$$\sum_{j=0}^{+\infty} f_{j+1}z_j = 1. \tag{2.5}$$

Inequality (2.4) implies that $\operatorname{sgn}(f_{i+1}) = \operatorname{sgn}(z_i)$ for every $i \geq 1$. Then (2.4) becomes

$$1 - f_{i+1}z_i + |z_i| - z_i f_{i+1} \leq 1 \quad \forall\, i \geq 1,$$

and hence

$$|z_i|(1-2|f_{i+1}|) \leq 0 \quad \forall i \geq 1.$$

Therefore, $z_i = 0$ for every i such that $|f_{i+1}| < \frac{1}{2}$. By equation (2.5) we conclude that there exists at least one index $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$.

Now let us consider an element $z^0 \in c$ such that

$$z^0_{j_0-1} = \frac{1}{f_{j_0}}, \quad z^0_j = 0 \quad \forall\, j \neq j_0 - 1. \tag{2.6}$$

It is easy to see that z^0 satisfies equations (2.4) and (2.5), $\|P_{z^0}\| = 1$. Finally, if there is a unique index j_0 such that $|f_{j_0}| \geq \frac{1}{2}$, we remark that a unique projection P_{z^0} exists (where z^0 is defined by (2.6)). If there are two indexes j_1 and j_2 such that $|f_{j_1}| = |f_{j_2}| = \frac{1}{2}$, then both the projections P_{z^1} and P_{z^2} (where z^1 and z^2 are defined by (2.6)) have norm 1. $\blacksquare$

Lemma 2.1 allows us to give an explicit formula to compute the projection constant of the hyperplane W_f.

Proposition 2.3 *Let $f \in \ell_1$ be such that $\|f\|_{\ell_1} = 1$ and $|f_j| < \frac{1}{2}$ for every $j \geq 2$. Then*

$$\inf_{z\in f^{-1}(1)} \|P_z\| = 1 + \Big(|f_1| + \sum_{j=1}^{+\infty}\frac{|f_{j+1}|}{1-2|f_{j+1}|}\Big)^{-1}.$$

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Proof Let us consider the quantity

$$\alpha_N = |f_1| + \sum_{j=1}^{N-1} \frac{|f_{j+1}|}{1-2|f_{j+1}|} + \operatorname{sgn}(f_1) \sum_{j=N}^{+\infty} f_{j+1}.$$

We first remark that there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$, $\alpha_N > 0$ and

$$\alpha_N \geq \frac{|f_{k+1}|}{1-2|f_{k+1}|} \tag{2.7}$$

for every $1 \leq k \leq N-1$.

Let us consider the sequence $\{z^N\}_{N \geq N_0} \subset c$ defined by

$$z^N = \lambda_N \Big(\underbrace{\frac{\operatorname{sgn}(f_2)}{1-2|f_2|}, \dots, \frac{\operatorname{sgn}(f_N)}{1-2|f_N|}}_{N-1}, \operatorname{sgn}(f_1), \operatorname{sgn}(f_1), \dots \Big),$$

where λ_N is a positive real number such that $f(z^N) = 1$. Therefore, it is $\lambda_N = \alpha_N^{-1}$.

Now we have

$$\|P_{z^N}\| = \sup_{i \geq 1} \Big\{ |1 - f_{i+1} z_i^N| + |z_i^N|(1 - |f_{i+1}|) \Big\} \leq 1 + \lambda_N \tag{2.8}$$

for every $N > N_0$. Indeed, by inequality (2.7) we obtain that $1 - \lambda_N \frac{|f_{i+1}|}{1-2|f_{i+1}|} \geq 0$, and hence, for $1 \leq i \leq N-1$, we have

$$1 - \lambda_N \frac{|f_{i+1}|}{1-2|f_{i+1}|} + \lambda_N \Big(\frac{1-|f_{i+1}|}{1-2|f_{i+1}|} \Big) = 1 + \lambda_N.$$

Moreover, for $i \geq N$

$$|1 - \lambda_N f_{i+1} \operatorname{sgn}(f_1)| + \lambda_N (1 - |f_{i+1}|) \leq 1 + \lambda_N |f_{i+1}| + \lambda_N - \lambda_N |f_{i+1}| \leq 1 + \lambda_N.$$

By (2.8) we have that $\inf_{z \in f^{-1}(1)} \|P_z\| \leq 1 + \lambda$, where

$$\lambda = \lim_N \lambda_N = \Big(|f_1| + \sum_{j=1}^{+\infty} \frac{|f_{j+1}|}{1-2|f_{j+1}|} \Big)^{-1}. \tag{2.9}$$

We will finish the proof by showing that $\inf_{z \in f^{-1}(1)} \|P_z\| = 1 + \lambda$. Let us consider two different cases.

First, let us suppose that $|f_1| = 1$, and hence $\lambda = 1$. In this case it is well known that $\inf_{z \in f^{-1}(1)} \|P_z\| = 2$ (see, *e.g.*, [5]).

Finally, let $|f_1| < 1$. By contradiction, let us suppose that there exists $\widehat{z} \in f^{-1}(1)$ such that

$$\|P_{\widehat{z}}\| = \sup_{i \geq 1} \Big\{ |1 - f_{i+1} \widehat{z}_i| + |\widehat{z}_i|(1 - |f_{i+1}|) \Big\} < 1 + \lambda,$$

hence

$$|1 - f_{i+1} \widehat{z}_i| + |\widehat{z}_i|(1 - |f_{i+1}|) < 1 + \lambda$$

for every $i \geq 1$, and then

$$1 - |f_{i+1}||\widehat{z}_i| + |\widehat{z}_i| - |\widehat{z}_i||f_{i+1}| < 1 + \lambda.$$

Therefore, for every $i \geq 1$, it holds that

$$(1 - 2|f_{i+1}|)|\widehat{z}_i| < \lambda. \tag{2.10}$$

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Moreover, the last relation gives that

$$|\widehat{z}_0| = \lim_i |\widehat{z}_i| \le \lim_i \frac{\lambda}{1-2|f_{i+1}|} = \lambda. \tag{2.11}$$

Since there exists at least one index $\widehat{j} \ge 1$ such that $f_{\widehat{j}+1} \ne 0$, by using inequalities (2.10) and (2.11) and by recalling (2.9), we conclude that

$$\sum_{j=0}^{+\infty} f_{j+1}\widehat{z}_j \le |f_1||\widehat{z}_0| + \sum_{j=1}^{+\infty} |f_{j+1}||\widehat{z}_j| < \lambda\Big(|f_1| + \sum_{j=1}^{+\infty} \frac{|f_{j+1}|}{1-2|f_{j+1}|}\Big) = 1.$$

The last inequality is a contradiction, because $\sum_{j=0}^{+\infty} f_{j+1}\widehat{z}_j = 1$. ∎

3 Isometries Between the Hyperplanes of c and the Spaces c and c_0

In this section we show that the isometric structure of the hyperplanes is completely described whenever the associated projection constant assumes the extreme values. Indeed, we will prove that W_f is isometric to c if and only if it is 1-complemented, whereas W_f is isometric to c_0 if and only if its projection constant is 2. We begin with the study of the 1-complemented hyperplanes of c. The first step shows that a 1-complemented hyperplane is isometric to c.

Proposition 3.1 *If $W_f \subset c$ is 1-complemented then W_f is isometric to c.*

Proof By Theorem 2.2, there exists $j_0 \ge 2$ such that $|f_{j_0}| \ge \frac{1}{2}$. Now let us consider $T: c \to W_f$ defined by

$$T(x_1, x_2, \dots) = (x_1, \dots, x_{j_0-2}, \underbrace{\alpha}_{j_0-1}, x_{j_0-1}, x_{j_0}, \dots)$$

where

$$\alpha = -\frac{1}{f_{j_0}}\Big(\sum_{j=0}^{j_0-2} f_{j+1}x_j + \sum_{j=j_0}^{+\infty} f_{j+1}x_{j-1}\Big).$$

The inverse of T is $T^{-1}: W_f \to c$; it acts on $y = (y_1, y_2, \dots) \in W_f$ by deleting the (j_0-1)-th component of y. Moreover, if $x \in c$, then

$$\begin{aligned}|\alpha| &\le \frac{1}{|f_{j_0}|}\Big(\sum_{j=0}^{j_0-2} |f_{j+1}||x_j| + \sum_{j=j_0}^{+\infty} |f_{j+1}||x_{j-1}|\Big) \le \frac{1}{|f_{j_0}|}\Big(\sum_{j=0}^{j_0-2} |f_{j+1}| + \sum_{j=j_0}^{+\infty} |f_{j+1}|\Big)\|x\| \\ &= \frac{1}{|f_{j_0}|}\Big(\sum_{\substack{j=0\\ j\ne j_0-1}}^{+\infty} |f_{j+1}|\Big)\|x\| = \frac{1}{|f_{j_0}|}(1-|f_{j_0}|)\|x\| \le \|x\|.\end{aligned}$$

Therefore T is an isometry between W_f and c. ∎

In order to prove the reverse implication, we need to investigate the family of the isometries on c with 1-codimensional range. To this aim, we adapt to our framework some results from [6] (Theorem 2.1 and Lemma 2.2) about the isometries on the space of continuous functions defined on a compact set. It is worth noting that the mentioned results in [6] do not refer to general isometries, but they only consider shift operators. Nevertheless, by considering the proofs of these results, it is easy to see that they hold for general isometries with 1-codimensional range.

As is well known, the space c can be seen as the space $\mathcal{C}(\mathbb{N}^*)$ of continuous function on $\mathbb{N}^*$, where $\mathbb{N}^*$ denotes the Alexandroff one-point compactification of the set of positive integers. For the sake of convenience, we denote by 0 the unique limit point of $\mathbb{N}^*$.

Theorem 3.2 ([6, Theorem 2.1 and Lemma 2.2]) *Let $T\colon \mathcal{C}(\mathbb{N}^*) \to \mathcal{C}(\mathbb{N}^*)$ be an isometry with 1-codimensional range. Then there exist a closed subset M of $\mathbb{N}^*$, a continuous and surjective function $\phi\colon M \to \mathbb{N}^*$, where $\phi^{-1}(n)$ has at most two elements for each $n \in \mathbb{N}^*$, and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}^*}$, where $|\varepsilon_n| = 1$ such that*

$$(Tx)_n = \varepsilon_n x_{\phi(n)} \quad \text{for every } n \in M. \tag{3.1}$$

Moreover, only one of the two following alternatives holds:

(i) *$M = \mathbb{N}^* \setminus \{\bar{n}\}$, where $\bar{n}$ is a positive integer (in addition, in this case ϕ is also injective);*

(ii) *$M = \mathbb{N}^*$, and if there exists $n' \in \mathbb{N}^*$ such that the set $\phi^{-1}(n')$ has two elements, then $\phi^{-1}(n)$ is a singleton for every $n \in \mathbb{N}^* \setminus \{n'\}$.*

Now, by means of the previous theorem, we will show that W_f is 1-complemented whenever it is isometric to c.

Proposition 3.3 *If $W_f \subset c$ is isometric to c, then W_f is 1-complemented.*

Proof By recalling Proposition 2.2, in order to prove the theorem, it is sufficient to show that there exists $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$ whenever there exists an isometry $T\colon c \to W_f$.

First of all, by the continuity of ϕ it is easy to see that $\phi(0) = 0$. Now let us consider Theorem 3.2(i) and let us suppose that $M = \mathbb{N}^* \setminus \{1\}$ without loss of generality. Therefore equation (3.1) is true for every $n \geq 2$. Hence, all the components of Tx are known except $(Tx)_1 = z$. Since $Tx \in W_f$ for every $x \in c$,

$$f_1\varepsilon_0 x_0 + f_2 z + \sum_{j=2}^{+\infty} f_{j+1}\varepsilon_j x_{\phi(j)} = 0. \tag{3.2}$$

By the injectivity of ϕ, we can choose $x_N \in c$ such that

$$\varepsilon_n (x_N)_{\phi(n)} = \operatorname{sgn} f_{n+1}$$

for every $2 \leq n \leq N$. All the other components of x_N are equal to a value x_0 given by

$$\varepsilon_0 x_0 = \operatorname{sgn} f_1.$$

By (3.2), we have

$$|f_1| + f_2 z_N + |f_3| + \cdots + |f_{N+1}| + x_0 \sum_{j=N+1}^{+\infty} f_{j+1}\varepsilon_j = 0,$$

where $z_N = (Tx_N)_1$. Since T is an isometry $|z_N| \leq 1$, and hence, up to a subsequence, we can suppose that z_N converges to $\widehat{z}$. Therefore, as $N \to \infty$, we have

$$|f_1| + f_2\widehat{z} + \sum_{j=2}^{+\infty} |f_{j+1}| = f_2\widehat{z} + 1 - |f_2| = 0,$$

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and hence $\widehat{z} = -\frac{1-|f_2|}{f_2}$. Since $|\widehat{z}| \le 1$, we conclude that $|f_2| \ge \frac{1}{2}$.

Now let us study Theorem 3.2(ii), where $M = \mathbb{N}^*$. We consider three different situations.

• Let $\phi^{-1}(n)$ be a pair for an element $n \in \mathbb{N}^*$, $n \neq 0$. Without loss of generality, we can suppose that the map $\phi\colon \mathbb{N}^* \to \mathbb{N}^*$ is one-to-one everywhere except at the point 1 where $\phi^{-1}(1) = \{1, 2\}$. Since $Tx \in W_f$, we have

$$f_1\varepsilon_0 x_0 + f_2\varepsilon_1 x_1 + f_3\varepsilon_2 x_1 + \sum_{j=3}^{+\infty} f_{j+1}\varepsilon_j x_{\phi(j)} = 0. \tag{3.3}$$

By the injectivity of ϕ, we can choose $x_N \in c$ such that $\varepsilon_n(x_N)_{\phi(n)} = \operatorname{sgn} f_{n+1}$ for every $3 \le n \le N$. Moreover, the component $(x_N)_1$ is chosen to satisfy $\varepsilon_1(x_N)_1 = \operatorname{sgn} f_2$. All the other components of x_N are equal to the value x_0 given by $\varepsilon_0 x_0 = \operatorname{sgn} f_1$. Since $Tx_N \in W_f$, by (3.3), we have

$$|f_1| + |f_2| + f_3\varepsilon_1\varepsilon_2 \operatorname{sgn} f_2 + \cdots + |f_{N+1}| + x_0 \sum_{j=N+1}^{+\infty} f_{j+1}\varepsilon_j = 0.$$

Therefore, as $N \to \infty$, we have

$$|f_1| + |f_2| + f_3\varepsilon_1\varepsilon_2 \operatorname{sgn} f_2 + \sum_{j=3}^{+\infty} |f_{j+1}| = 0,$$

and hence

$$1 - |f_3| + f_3\varepsilon_1\varepsilon_2 \operatorname{sgn} f_2 = 0. \tag{3.4}$$

Similarly, by choosing $(x_N)_1$ to satisfy $\varepsilon_2(x_N)_1 = \operatorname{sgn} f_3$, we obtain an analogous equation for f_2:

$$1 - |f_2| + f_2\varepsilon_1\varepsilon_2 \operatorname{sgn} f_3 = 0. \tag{3.5}$$

Equations (3.4) and (3.5) hold true simultaneously only if $|f_2| = |f_3| = \frac{1}{2}$.

• Let $\phi^{-1}(0)$ be a pair. Without loss of generality we can suppose that

$$\phi^{-1}(0) = \{0, 1\}.$$

Analysis similar to that in the previous point shows that $|f_1| = |f_2| = \frac{1}{2}$.

• Finally, let ϕ be injective. The same argument applied above yields the contradiction

$$\sum_{j=1}^{+\infty} |f_j| = 0. \qquad \blacksquare$$

Now we study the hyperplanes of c that are isometric to c_0. First, we prove that a hyperplane of c with projection constant equal to 2 is an isometric copy of c_0.

Proposition 3.4 *If W_f is such that $\inf_z \|P_z\| = 2$ (where $P_z\colon c \to W_f$ is a projection), then W_f is isometric to c_0.*

Proof Let us recall that, by Proposition 2.3,

$$\inf_z \|P_z\| = 1 + \left(|f_1| + \sum_{j=1}^{+\infty} \frac{|f_{j+1}|}{1 - 2|f_{j+1}|} \right)^{-1}.$$

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Then, since $\inf_z \|P_z\| = 2$, we have

$$|f_1| + \sum_{j=1}^{+\infty} \frac{|f_{j+1}|}{1 - 2|f_{j+1}|} = 1.$$

Since $\|f\|_{\ell_1} = 1$,

$$\sum_{j=1}^{+\infty} \frac{|f_{j+1}|}{1 - 2|f_{j+1}|} = \sum_{j=1}^{+\infty} |f_{j+1}|,$$

and hence

$$\sum_{j=1}^{+\infty} \Big(\frac{|f_{j+1}|}{1 - 2|f_{j+1}|} - |f_{j+1}| \Big) = \sum_{j=1}^{+\infty} \Big(\frac{2|f_{j+1}|^2}{1 - 2|f_{j+1}|} \Big) = 0.$$

Since $|f_{j+1}| < \frac{1}{2}$ for every $j = 1, 2, 3, \dots$ (otherwise $\inf_z \|P_z\| = 1$ by Proposition 2.2), the last equality holds if and only if $f_{j+1} = 0$ for every $j = 1, 2, \dots$. Therefore, $f = \pm(1, 0, 0, \dots)$, and hence $W_f \simeq c_0$. ∎

Now we prove that there exists a unique hyperplane of c isometric to c_0. This assertion follows directly from a simple lemma that can be stated in a more general setting.

Lemma 3.5 *Let V be a subspace of $\mathcal{C}(K)$ where K is a compact metric space. If V is isometric to c_0, then there exists $p \in K$ such that*

$$V \subseteq \big\{ f \in \mathcal{C}(K) : f(p) = 0 \big\}.$$

Proof Let $T\colon V \to c_0$ be an isometry. Therefore, $T^*\colon \ell_1 \to V^*$ is also an isometry, and hence $T^*(e_n) \in \operatorname{ext} B_{V^*}$, where e_n is the n-th element of the standard basis of ℓ_1. By [4, Lemma 6, p. 441], we have $T^*(e_n) = \pm x^*_{p_n}$ where x^*_p is the evaluation functional. Since $\{e_n\}$ is a weak*-null sequence, $|x^*_{p_n}(f)| \to 0$ for every $f \in V$. Now, by the compactness of K, there exists an element $p \in K$ such that $f(p) = 0$ for every $f \in V$. ∎

If we take $\mathcal{C}(K) = \mathcal{C}(\mathbb{N}^*) = c$ and $V = W_f$ in the previous lemma, then, necessarily,

$$W_f = \{x \in c : x_0 = \lim_i x_i = 0\}$$

since W_f has codimension 1.

4 Duality Between W_f and ℓ_1 and Some Applications

We start this section by characterizing the hyperplanes of c such that their duals are isometric to ℓ_1. Despite its simple proof, this result plays a central role in our approach, since it allows us conclude the proof of our main result, Theorem 1.2.

Proposition 4.1 *There exists $j_0 \in \mathbb{N}$ such that $|f_{j_0}| \geq \frac{1}{2}$ if and only if $W_f^* \simeq \ell_1$.*

Proof Let us denote by V_f the subspace of c_0 defined as $V_f = \ker f \subset c_0$. Since

$$V_f^* \simeq \ell_1/[f],$$

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then $W_f^* \simeq V_f^*$. By Theorem 1.1 we have that $W_f^* \simeq \ell_1$ if and only if there exists $j_0 \in \mathbb{N}$ such that $|f_{j_0}| \geq \frac{1}{2}$. ∎

A consequence of Theorem 1.2 is the classification of the hyperplanes of c such that their duals are isometric copies of ℓ_1.

Remark 4.2 The hyperplanes W_f of c such that $W_f^* \simeq \ell_1$ can be divided into three distinct classes:

- $W_f \simeq c$ (or, equivalently, there exists $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$);
- W_f is isometric neither to c nor to c_0 (or, equivalently, $\frac{1}{2} \leq |f_1| < 1$ and $|f_j| < \frac{1}{2}$ for every $j \geq 2$);
- $W_f \simeq c_0$ (or, equivalently, $|f_1| = 1$).

The most interesting situations occur when W_f is isometric neither to c nor to c_0. In these cases we study the $\sigma(\ell_1, W_f)$-limit of the standard basis of ℓ_1. It is worth mentioning that in order to reach this aim we explicitly describe the duality between W_f and ℓ_1.

Theorem 4.3 *Let $W_f \subset c$ be such that $W_f^* \simeq \ell_1$, $\frac{1}{2} \leq |f_1| < 1$, and $|f_j| < \frac{1}{2}$ for every $j \geq 2$. If $\{e_n\}$ is the standard basis of ℓ_1, then*

$$e_n \overset{\sigma(\ell_1, W_f)}{\longrightarrow} \widehat{e},$$

where $\widehat{e} = \left(-\frac{f_2}{f_1}, -\frac{f_3}{f_1}, -\frac{f_4}{f_1}, \dots\right)$.

Proof Consider the map $\phi\colon \ell_1 \to W_f^*$ defined by

$$(\phi(y))(x) = \sum_{j=1}^{+\infty} x_j y_j,$$

where $y = (y_1, y_2, \dots) \in \ell_1$ and $x = (x_1, x_2, \dots) \in W_f$.

It is easy to see that $\phi(\ell_1) = W_f^*$, and for any $y \in \ell_1$,

$$\|\phi(y)\|_{W_f^*} \leq \|y\|_{\ell_1}. \tag{4.1}$$

Now, for a given $y = (y_1, y_2, \dots) \in \ell_1$, consider the points x^N, $N = 1, 2, \dots$, defined as

$$x^N = \left(\operatorname{sgn}(y_1), \operatorname{sgn}(y_2), \dots, \operatorname{sgn}(y_N), x_0^N, x_0^N, \dots\right),$$

where x_0^N satisfies the following equation:

$$f_1 x_0^N + \sum_{j=1}^{N} f_{j+1} \operatorname{sgn}(y_j) + x_0^N \sum_{j=N+1}^{+\infty} f_{j+1} = 0.$$

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It is clear that $x^N \in W_f$ for all N. Moreover, for any $N \geq N_0$, where N_0 is such that $\sum_{j=N_0+1}^{+\infty} |f_{j+1}| < \frac{1}{2}$, we have $x^N \in B_{W_f}$. Indeed, for every $N \geq N_0$, we have

$$|x_0^N| = \Big| \frac{-\sum_{j=1}^N f_{j+1} \operatorname{sgn}(y_j)}{f_1 + \sum_{j=N+1}^{+\infty} f_{j+1}} \Big| \leq \frac{\sum_{j=1}^N |f_{j+1}|}{|f_1 + \sum_{j=N+1}^{+\infty} f_{j+1}|}$$
$$= \frac{1 - |f_1| - \sum_{j=N+1}^{+\infty} |f_{j+1}|}{|f_1 + \sum_{j=N+1}^{+\infty} f_{j+1}|} \leq \frac{\frac{1}{2} - \sum_{j=N+1}^{+\infty} |f_{j+1}|}{\frac{1}{2} - \sum_{j=N+1}^{+\infty} |f_{j+1}|} = 1.$$

Now, for every $N \geq N_0$, we have

$$\|\phi(y)\|_{W_f^*} \geq |(\phi(y))(x^N)| = |\sum_{j=1}^{N} |y_j| + x_0^N \sum_{j=N+1}^{+\infty} y_j|.$$

Letting $N \to +\infty$ we get

$$\|\phi(y)\|_{W_f^*} \geq \|y\|_{\ell_1}. \tag{4.2}$$

By (4.1) and (4.2) we conclude that the map ϕ is an isometry.

Finally, for any $x = (x_1, x_2, \dots) \in W_f$, we obtain

$$\lim_{n\to\infty} (\phi(e_n))(x) = \lim_{n\to\infty} x_n = x_0 = -\frac{1}{f_1} \sum_{j=1}^{\infty} f_{j+1} x_j = (\phi(\widehat{e}))(x). \qquad \blacksquare$$

The last result has some interesting consequences.

First of all, by recalling Lemma 2 in [1], that asserts "that the w^*-closure of the ℓ_1-(standard) basis is the only thing that is important" to describe the structure of an ℓ_1-predual space, we obtain a complete isometric descriptions of the preduals of ℓ_1 under the additional assumption that its standard basis is $\sigma(\ell_1, X)$-convergent.

Corollary 4.4 *Let X be a Banach space such that $X^* = \ell_1$. If the standard basis $\{e_n\}$ of ℓ_1 is a $\sigma(\ell_1, X)$-convergent sequence, then there exists $f \in \ell_1$ with $\|f\|_{\ell_1} = 1$ such that X is isometric to W_f.*

Proof Let $\widehat{e} = (\widehat{e}_1, \widehat{e}_2, \dots) \in \ell_1$ be the $\sigma(\ell_1, X)$-limit of $\{e_n\}$. If $|\widehat{e}_j| < 1$ for every $j \geq 1$, then by Theorem 4.3 and [1, Lemma 2] we have that $X \simeq W_f$ where $f = (f_1, f_2, \dots) \in \ell_1$ is defined by

$$f_1 = \frac{1}{1 + \sum_{n=1}^{\infty} |\widehat{e}_n|}, \quad f_n = -\frac{\widehat{e}_{n-1}}{1 + \sum_{n=1}^{\infty} |\widehat{e}_n|} \text{ for every } n \geq 2.$$

Now it remains to consider the case where $\widehat{e} = e_m$ for a fixed $m \geq 1$. Under this assumption it is easy to show that $X \simeq c$, and hence $X \simeq W_f$ for every $f \in \ell_1$ such that there exists $j_0 \geq 2$ such that $|f_{j_0}| \geq \frac{1}{2}$. $\blacksquare$

Finally, we obtain some additional information about the isometric structure of the hyperplanes W_f. Let us introduce the following notation. For a countable ordinal α, let $C(\alpha)$ be the space of all continuous real-valued functions on the ordinals less than or equal to α with the order topology.

Corollary 4.5 *There exists a countable ordinal α such that W_f is isometric to a quotient of $C(\alpha)$ if and only if one of the following conditions holds:*

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(i) *there exists an index* $j_0 \geq 2$ *such that* $|f_{j_0}| \geq \frac{1}{2}$;

(ii) $\frac{1}{2} \leq |f_1| \leq 1$ *and* $|f_j| < \frac{1}{2}$ *for all* $j \geq 2$ *and* $f = (f_1, f_2, \dots, f_n, 0, 0, \dots, 0, \dots)$ *for some* $n \in \mathbb{N}$.

Proof First of all, by Theorem 1.2 we have that condition (i) is satisfied if and only if $W_f \simeq c$. Now it remains to consider the case where $\frac{1}{2} \leq |f_1| \leq 1$ and $|f_j| < \frac{1}{2}$ for all $j \geq 2$. Let us suppose that $f = (f_1, f_2, \dots, f_n, 0, 0, \dots)$ for some $n \geq 2$. Then $W_f \subset c$ is isometric to a quotient of $C(\omega \cdot n)$. To see it, consider the sequence of measures μ_i defined by $\mu_i = \delta_{\omega \cdot i}$ for $i = 1, 2, \dots, n-1$, and for $i = n, n+1, n+2, \dots$, (we put $\delta_{\omega \cdot 0 + i} := \delta_i$)

$$\mu_i = -\frac{1}{f_1} \sum_{j=2}^{n} f_j \cdot \delta_{\omega \cdot (j-2)+i} + \frac{2|f_1| - 1}{|f_1| \cdot i} \sum_{j=1}^{i} (-1)^j \cdot \delta_{\omega \cdot (n-1) + \frac{i \cdot (i-1)}{2} + j}.$$

Now it is enough to apply Theorem 4.3 and [1, Proposition 3], with the mapping ϕ given by $\phi(e_i) = \mu_i$. To show that condition (ii) implies that W_f is isometric to a quotient of $C(\alpha)$ for some α, it is sufficient to consider [1, Proposition 6] and Theorem‘4.3. ∎

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