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SUBGROUPS OF THE POWER SEMIGROUP OF A FINITE SEMIGROUP

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Throughout this paper, S will denote a finite semigroup and $\mathbf{Z}^+$ the set of positive integers. $E = E(S)$ denotes the set of idempotents of S. Let $\mathscr{P}(S) = \{A | A \subseteq S,\ A \neq \emptyset\}$. If $A,\ B \in \mathscr{P}(S)$, then let $AB = \{ab | a \in A, b \in B\}$. $\mathscr{P}(S)$ has been studied by many authors, including [**2**, **3**, **5**, **6**, **7**]. If X is a set, then $|X|$ denotes the cardinality of X. For undefined terms in this paper, see [**1**, **4**].

THEOREM 1. *Let I be an ideal of S, $\mathscr{G}$ a subgroup of $\mathscr{P}(S)$. Then $\mathscr{G}$ has a normal subgroup $\mathscr{N}$ such that $\mathscr{N}$ is isomorphic to a subgroup of $\mathscr{P}(I)$ and $\mathscr{G}/\mathscr{N}$ is isomorphic to a subgroup of $\mathscr{P}(S/I)$.*

Proof. Let T denote the identity element of $\mathscr{G}$. First assume $T \subseteq S\backslash I$. Let $A \in \mathscr{G}$. Then $T = AB$ for some $B \in \mathscr{G}$. So $A \subseteq S\backslash I$. It then follows that $\mathscr{G}$ is isomorphic to a subgroup of $\mathscr{P}(S/I)$ and the theorem is trivial. So assume $T \cap I \neq \emptyset$. Let $\phi: S \to S/I$ denote the natural homomorphism. Let $\hat{\phi}: \mathscr{P}(S) \to \mathscr{P}(S/I)$ denote the obvious extension of ϕ. Let ψ denote the restriction of $\hat{\phi}$ to $\mathscr{G}$. Then $\bar{\mathscr{G}} = \psi(\mathscr{G})$ is a subgroup of $\mathscr{P}(S/I)$. Let $\mathscr{N}$ denote the kernel of ψ. It suffices to show that $\mathscr{N}$ is isomorphic to a subgroup of $\mathscr{P}(I)$. Let $T_1 = T \cap I \neq \emptyset$. Then $T = V \cup T_1$ where $V = T\backslash T_1$. So $\psi(T) = V \cup \{0\}$ is the identity element of $\bar{\mathscr{G}}$. If $V = \emptyset$, then $\psi(T) = \{0\}$ and $\mathscr{N} \subseteq \mathscr{P}(I)$. We are then trivially done. So assume $V \neq \emptyset$. Then for $A \in \mathscr{N}$, $\phi(A) = V \cup \{0\}$. So $A = V \cup A_1$ for some $A_1 \in \mathscr{P}(I)$. Now $T^2 = T$. So

(1) $\quad V \cup T_1 = V^2 \cup VT_1 \cup T_1V \cup T_1{}^2.$

Comparing the 'I-part' and the '$S\backslash I$-part' of both sides of (1), we have,

(2) $\quad V^2 \cap I \subseteq T_1,\ V \subseteq V^2,\ VT_1 \cup T_1V \cup T_1{}^2 \subseteq T_1.$

Now let $A \in \mathscr{N}$. Then $A = V \cup A_1$, $\emptyset \neq A_1 \subseteq I$. There exists $n \in \mathbf{Z}^+$ such that $A^n = T$. So $A^{n+1} = A$. Then

$$T_1 \subseteq T = A^n = [V \cup A_1]^n.$$

Let $a \in T_1$. Then $a = x_1 \ldots x_n$ for some $x_1, \ldots, x_n \in V \cup A_1$. First assume some $x_i \in V$. Then by (2), $x_i \in V^2 \subseteq A^2$. So $a = x_1 \ldots x_n \in A^{n+1} = A$. But then $a \in A \cap I = A_1$. If $x_i \notin V$ for all i, then $a \in A_1{}^n$. Thus

(3) $\quad T_1 \subseteq A_1 \cup A_1{}^n.$

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We claim that $A_1T_1 = T_1A_1$. By symmetry, it suffices to show that $A_1T_1 \subseteq T_1A_1$. So let $u \in A_1T_1$. Then $u = ab$ for some $a \in A_1, b \in T_1$.

Case 1. $a \in T_1$. If $b \in A_1$, then $u = ab \in T_1A_1$. Next assume $b \notin A_1$. By (3), $b \in A_1^n$. So $u = ab \in A_1^{n+1} = A_1^nA_1$. But $A_1^n \subseteq A^n = T$. So $A_1^n \subseteq T \cap I = T_1$. Thus $u \in T_1A_1$.

Case 2. $a \notin T_1$. Now $AT = A = TA$. Since $A = V \cup A_1, T = V \cup T_1$, we have

(4) $VT_1 \cup A_1T_1 \subseteq A_1$

(5) $V \cup A_1 = V^2 \cup VA_1 \cup T_1A_1 \cup T_1V$.

There exists $C \in \mathcal{N}$ such that $CA = T$. Since $V \subseteq C$, we have

(6) $VA_1 \subseteq T_1$.

Since $a \in A_1 \backslash T_1$, we have $a \in I \backslash T_1$. So by (2), (5), (6) we have $a \in T_1A_1$. So $u = ab \in T_1A_1T_1$. But by (4), $A_1T_1 \subseteq A_1$. So $u \in T_1A_1$.

We have thus shown that

(7) $A_1T_1 = T_1A_1$.

By (2) $T_1^2 \subseteq T_1$. So

$$T_1 \supseteq T_1^2 \supseteq T_1^3 \supseteq \ldots$$

Hence there exists $k \in \mathbf{Z}^+$ such that $T_1^k = T_1^{k+1}$. Let $W = T_1^k$. Then $W = W^2$. By (4), $VT_1 \subseteq A_1$. So

$$AW = (V \cup A_1)T_1^k = VT_1^k \cup A_1T_1^k = VT_1^{k+1} \cup A_1T_1^k = A_1T_1^k = A_1W.$$

Similarly $WA = WA_1$. By (7), $WA_1 = A_1W$. So

(8) $WA = AW = A_1W = WA_1,\ W^2 = W \subseteq I$.

Let $f(A) = AW \in \mathscr{P}(I)$. If $A, B \in \mathcal{N}$, then by (8)

(9) $f(A)f(B) = AWBW = ABW^2 = ABW = f(AB)$.

Let $\bar{\mathcal{N}} = f(\mathcal{N}) \subseteq \mathscr{P}(I)$. By (8), $f: \mathcal{N} \to \bar{\mathcal{N}}$ is a surjective homomorphism. Thus $\bar{\mathcal{N}}$ is a subgroup of $\mathscr{P}(I)$. We claim that f is an isomorphism. So let $A \in \mathcal{N}$ and suppose

(10) $f(A) = f(T)$

where $A = V \cup A_1, \emptyset \neq A_1 \subseteq I$. First suppose $A_1 \not\subseteq T_1$. Let $a \in A_1 \backslash T_1$. By (2), (5), (6), $a \in T_1A_1$. So $a = bc$ for some $b \in T_1, c \in A_1$. If $c \in T_1$, then $a \in T_1$, a contradiction. Thus $c \in A_1 \backslash T_1$. Hence $a \in T_1(A_1 \backslash T_1)$. Therefore

$$A_1 \backslash T_1 \subseteq T_1(A_1 \backslash T_1).$$

So

$$A_1 \backslash T_1 \subseteq T_1^i(A_1 \backslash T_1) \quad \text{for all } i \in \mathbf{Z}^+.$$

In particular, by (8), (10),

$$A_1 \backslash T_1 \subseteq T_1^k(A_1\backslash T_1) \subseteq WA_1 = f(A) = f(T) = T_1^k \subseteq T_1,$$

a contradiction. Thus $A_1 \subseteq T_1$. Hence $A \subseteq T$. So

$$A^2 \subseteq TA = A.$$

Therefore

(11) $T \supseteq A \supseteq A^2 \supseteq A^3 \supseteq \ldots$

There exists $n \in \mathbf{Z}^+$ such that $A^n = T$. By (11), $T = A^n \subseteq A \subseteq T$. So $A = T$. Hence $\mathcal{N} \cong \bar{\mathcal{N}}$, $\bar{\mathcal{N}}$ is a subgroup of $\mathcal{P}(I)$. Since $\mathcal{G}/\mathcal{N} \cong \bar{\mathcal{G}}$ is a subgroup of $\mathcal{P}(S/I)$, the theorem is proved.

Example 1. In the proof of Theorem 1, it is tempting to look at $\mathcal{N}_1 = \{A \cap I | A \in \mathcal{N}\}$ and see if $\mathcal{N}_1$ is in fact a subgroup of $\mathcal{P}(I)$. However, this is not always true. For example, let $I = \{0, a\}$ be the null semigroup, $S = I^1$. Let $\mathcal{G} = \{\{1, 0, a\}\}$. Then $\mathcal{N} = \{\{1, 0, a\}\}$, $\mathcal{N}_1 = \{\{0, a\}\}$. $\mathcal{N}_1$ is not a subgroup, $\mathcal{N}_1^2 = \{\{0\}\}$. However $\bar{\mathcal{N}} = \mathcal{N}\{\{0\}\} = \{\{0\}\}$ is a group which is isomorphic to $\mathcal{N}$. So the construction of $\bar{\mathcal{N}}$ in the proof of Theorem 1 is necessary.

Example 2. Let G_1, G_2 be disjoint groups with identities e_1, e_2, respectively. Let $S = G_1 \cup G_2 \cup \{0\}$ with $g_1g_2 = g_2g_1 = g_10 = 0g_1 = g_20 = 0g_2 = 00 = 0$ for $g_1 \in G_1$, $g_2 \in G_2$. Let

$$I = G_2 \cup \{0\},\ \mathcal{G} = \{\{g_1, g_2, 0\} | g_1 \in G_1, g_2 \in G_2\}.$$

Then $\mathcal{G}$ is a subgroup of $\mathcal{P}(S)$. If $\mathcal{N} = \{\{e_1, g_2, 0\} | g_2 \in G_2\}$, then $\mathcal{N} < G$, $G_2 \cong \mathcal{N} \cong \bar{\mathcal{N}} = \{\{g_2, 0\} | g_2 \in G_2\} \subseteq \mathcal{P}(I)$. Also $\mathcal{G}/\mathcal{N} \cong G_1$ and is also isomorphic to a subgroup of $\mathcal{P}(S/I)$.

If J is a $\mathcal{J}$-class of S, then in J^0 we define

$$a \cdot b = \begin{cases} ab & \text{if } ab \in J \\ 0 & \text{if } ab \notin J. \end{cases}$$

Then J^0 is a semigroup [**4**; p. 151].

THEOREM 2. *Let $\mathcal{G}$ be a subgroup of $\mathcal{P}(S)$. Then $\mathcal{G}$ admits a normal series $\{1\} = \mathcal{G}_0 \lhd \mathcal{G}_1 \lhd \ldots \lhd \mathcal{G}_m = \mathcal{G}$ such that each factor group $\mathcal{G}_i/\mathcal{G}_{i-1}$ $(i = 1, \ldots, m)$ is isomorphic to a subgroup of $\mathcal{P}(J^0)$ for some $\mathcal{J}$-class J of S.*

Proof. We prove the theorem by induction on $|S|$. Suppose S has an ideal I, $|I| \neq |S|$, $|I| \neq 1$. If J is a $\mathcal{J}$-class of S/I, other than $\{0\}$, then it is a $\mathcal{J}$-class of S. If J is a regular $\mathcal{J}$-class of I, then J is a $\mathcal{J}$-class of S. If J is a non-regular $\mathcal{J}$-class of I, then J^0 is null and $\mathcal{P}(J^0)$ has only trivial subgroups. We are thus done by Theorem 1 and the induction hypothesis. Next assume S has no proper ideals. Then $S = J$ or J^0 for some $\mathcal{J}$-class J of S. We are then trivially done.

A semigroup with only trivial subgroups is called a *combinatorial* semigroup.

THEOREM 3. *$\mathscr{P}(S)$ is combinatorial if and only if S is combinatorial and for all $e, f \in E(S)$, $e\mathscr{J}f$ implies $e\mathscr{J}ef$ or $e\mathscr{J}fe$.*

Proof. First suppose $\mathscr{P}(S)$ is combinatorial. If H is a subgroup of S, then H is a subgroup of $\mathscr{P}(H) \subseteq \mathscr{P}(S)$. So H must be trivial. Hence S is combinatorial. Suppose there exist $e, f \in E(S)$ such that $e\mathscr{J}f$, $e\not\mathscr{J}ef$, $e\not\mathscr{J}fe$. We will obtain a contradiction. Let J denote the $\mathscr{J}$-class of e. Let $T = J^0$. If J is the kernel of S, then $\mathscr{P}(J)$ and hence $\mathscr{P}(J^0) = \mathscr{P}(T)$ is combinatorial. Otherwise by [**4**, p. 151], there exist ideals I_1, I_2 of S such that $I_2 \subseteq I_1$, $T \cong I_1/I_2$. Since $\mathscr{P}(I_1)$ is an ideal of $\mathscr{P}(S)$, it is combinatorial. The natural homomorphism from I_1 onto I_1/I_2 extends naturally to a homomorphism from $\mathscr{P}(I_1)$ onto $\mathscr{P}(I_1/I_2)$. So $\mathscr{P}(I_1/I_2) \cong \mathscr{P}(T)$ is combinatorial. Thus in all cases, $\mathscr{P}(T)$ is combinatorial. In particular, T is combinatorial. Since $e \in T$, T is isomorphic to a regular Rees matrix semigroup. Since T is combinatorial, we can assume, without loss of generality, that there exist non-empty sets A, B, $P: A \times B \to \{0, 1\}$ such that $T = (A \times B) \cup \{0\}$ and in T,

$$(12) \quad (i, j)(k, l) = \begin{cases} (i, l) & \text{if } P(j, k) = 1 \\ 0 & \text{if } P(j, k) = 0. \end{cases}$$

Let $e = (\alpha, \beta)$, $f = (\gamma, \delta)$. Then $ef = fe = 0$. So

$$(13) \quad P(\beta, \alpha) = P(\delta, \gamma) = 1,\ P(\beta, \gamma) = P(\delta, \alpha) = 0.$$

In particular, $\beta \neq \delta$, $\alpha \neq \gamma$. Let $L = \{(\alpha, \beta), (\gamma, \delta), 0\}$, $K = \{(\alpha, \delta), (\gamma, \beta), 0\}$. Then $K \neq L$, $L^2 = L$, $KL = LK = K$, $K^2 = L$. So $\{K, L\}$ is a two element subgroup of $\mathscr{P}(T)$, a contradiction.

Conversely assume S is combinatorial and for all $e, f \in E(S)$,

$$(14) \quad e\mathscr{J}f \text{ implies } e\mathscr{J}ef \text{ or } e\mathscr{J}fe.$$

Let J be a $\mathscr{J}$-class and let $T = J^0$. By Theorem 2, it suffices to show that $\mathscr{P}(T)$ is combinatorial. If T is null, this is trivial. So assume T is a regular Rees matrix semigroup. Since S is combinatorial, so is T. Se we can assume that T has the structure given by (12). By (14),

$$(15) \quad e, f \in E(T),\ e, f \neq 0 \text{ implies } ef \neq 0 \text{ or } fe \neq 0.$$

Let $(i, j), (k, l) \in T$ such that $P(j, i) = P(l, k) = 1$. Then $(i, j), (k, l) \in E(T)$. By (15), $P(j, k) = 1$ or $P(l, i) = 1$. Thus we have that if $i, k \in A$, $j, l \in B$, then

$$(16) \quad P(j, i) = P(l, k) = 1 \text{ implies } P(j, k) = 1 \text{ or } P(l, i) = 1.$$

Let $K \in \mathscr{P}(T)$. Suppose K lies in a subgroup of $\mathscr{P}(T)$. Then $K^m = K$ for some $m \in \mathbf{Z}^+$, $m > 1$. We claim that $K^3 \subseteq K^2$. So let $u \in K^3$. First assume $u = 0$. Then $0 \in K^3$. So $0 \in K^r$ for $r \geqq 3$. In particular $0 \in K^{m+1} = K^2$. So $u \in K^2$. Next assume $u \neq 0$. So there exist $i_1, i_2, i_3 \in A$, $j_1, j_2, j_3 \in B$ such that $(i_1, j_1), (i_2, j_2), (i_3, j_3) \in K$,

$$(17) \quad u = (i_1, j_1)(i_2, j_2)(i_3, j_3) = (i_1, j_3).$$

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Since $u \neq 0$, $P(j_1, i_2) = P(j_2, i_3) = 1$. By (16)

(18) $P(j_1, i_3) = 1$ or $P(j_2, i_2) = 1$.

First assume $P(j_1, i_3) = 1$. Then by (17), $u = (i_1, j_1)(i_3, j_3) \in K^2$. Next assume $P(j_2, i_2) = 1$. Then (i_2, j_2) is idempotent. So by (17),

$$u = (i_1, j_1)(i_2, j_2)^r(i_3, j_3) \text{ for all } r \in \mathbf{Z}^+.$$

So $u \in K^r$ for all $r \in \mathbf{Z}^+$, $r \geqq 3$. In particular $u \in K^{m+1} = K^2$. Thus we have shown that $K^2 \supseteq K^3$. So

$$K^2 \supseteq K^3 \supseteq K^4 \supseteq \ldots .$$

In particular $K^2 \supseteq K^m \supseteq K^{m+1} = K^2$. So $K^2 = K^m = K$. Thus $\mathscr{P}(T)$ is combinatorial. This proves the theorem.

If S_1, S_2 are semigroups, then $S_1|S_2$ (S_1 *divides* S_2) if S_1 is a homomorphic image of a subsemigroup of S_2. In the following let

$$Y = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right\}$$

be the Rees matrix semigroup with sandwich matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

COROLLARY 1. *Suppose S is combinatorial. Then $\mathscr{P}(S)$ is combinatorial if and only if $Y \nmid S$.*

Proof. Suppose $Y|S$. Then it is obvious that $\mathscr{P}(Y)|\mathscr{P}(S)$. Since Y does not satisfy the hypothesis of Theorem 3, $\mathscr{P}(Y)$ is not combinatorial. Hence $\mathscr{P}(S)$ is not combinatorial.

Conversely, assume $\mathscr{P}(S)$ is not combinatorial. By Theorem 3, there exist $e, f \in E(S)$ such that $e\mathscr{J}f$, $e\not\mathscr{J}ef$, $e\not\mathscr{J}fe$. In particular e is not in the kernel of S. Let J denote the $\mathscr{J}$-class of e. Then J is not the kernel of S. So by [**4**, p. 151], $T = J^0|S$. T, of course, must have the structure given by (12). As in the proof of Theorem 3, there must exist (α, β), $(\gamma, \delta) \in T$ such that

(19) $P(\beta, \alpha) = P(\delta, \gamma) = 1, P(\beta, \gamma) = P(\delta, \alpha) = 0.$

Let $Y' = \{(\alpha, \beta), (\gamma, \delta), (\alpha, \delta), (\gamma, \beta), 0\}$. Using (19) it is easy to see that $Y \cong Y'$. So $Y|T|S$. Hence $Y|S$. This proves the corollary.

Example 3. Let

$$Y_1 = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\right\}$$

be the Rees matrix semigroup with sandwich matrix $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$. Then $Y \nmid Y_1$ and so by Corollary 1, $\mathscr{P}(Y_1)$ is combinatorial.

THEOREM 4. *S is a band if and only if $\mathscr{P}(S)$ has the property that $A \subseteq A^2$ for all $A \in \mathscr{P}(S)$. Suppose S is a band. Then $\mathscr{P}(S)$ has the following properties.*

(i) *$\mathscr{P}(S)$ is a combinatorial semigroup which is a disjoint union of nil semigroups.*

(ii) *Let $A, B \in \mathscr{P}(S)$, $A^i = K$, $B^j = L$ where $K^2 = K$, $L^2 = L$. If $KL = L$ then there exists $r \in \mathbf{Z}^+$ such that $(AB)^r = L$. If $KL = K$, then there exists $r \in \mathbf{Z}^+$ such that $(AB)^r = K$.*

(iii) *If T is a subsemigroup of $\mathscr{P}(S)$ and if T has a zero, then the nilpotent elements of T form an ideal of T.*

Proof. Suppose S is a band, $A \in \mathscr{P}(S)$. If $e \in A$, then $e = e^2 \in A^2$. So $A \subseteq A^2$. Conversely, assume $A \subseteq A^2$ for all $A \in \mathscr{P}(S)$. Then for $e \in S$, $\{e\} \subseteq \{e\}^2$ and so $e = e^2$. Now let S be a band, $A \in \mathscr{P}(S)$. Then $A \subseteq A^2$. So

$$A \subseteq A^2 \subseteq A^3 \subseteq \ldots.$$

There exists $n \in \mathbf{Z}^+$ such that $A^n = A^{n+1}$. So $\mathscr{P}(S)$ is combinatorial. The second part of (i) clearly follows from (ii) if we let $K = L$. We now prove (ii).

By symmetry we can assume $KL = L$. Let $b \in B$. Then $b \in B^j = L = KL$. So $b = eb$ for some $e \in K$. Now $e = a_1 \ldots a_i$ for some $a_1, \ldots, a_i \in A$. So $a_1 e = e$ whence $a_1 b = b$. So $b \in AB$. Thus $B \subseteq AB$. There exists $r \in \mathbf{Z}^+$ such that $AB \subseteq (AB)^r = (AB)^{r+1}$. So $L = B^j \subseteq (AB)^r$. Since $A \subseteq K$, $B \subseteq L$, $AB \subseteq L$. So $(AB)^r \subseteq L$. Thus $(AB)^r = L$. Next we prove (iii). Suppose 0 is the zero of T. T, being a subsemigroup of $\mathscr{P}(S)$, satisfies (ii). Let $b \in T$ be nilpotent, say $b^j = 0$. Let $a \in T$. Then $a^i = e \in E(T)$ for some $i \in \mathbf{Z}^+$. Since $e0 = 0$, we see by (ii) that $(ab)^r = 0$ for some $r \in \mathbf{Z}^+$. Similarly $(ba)^s = 0$ for some $s \in \mathbf{Z}^+$. This proves the theorem.

Example 4. The power semigroup of a rectangular band is an inflation of a rectangular band [**6**]. The structure of the power semigroup of a band can be considerably more complicated. Let $\mathscr{B}$ be the free band on letters e, f, g. Let $S = \mathscr{B}^1$. Let $A = \{1, e, f, fe\}$, $L = \{eg, egfeg, egefeg\}$. Then $L^2 = L$, $A^2 = K = K^2 = \{1, e, f, ef, fe, efe, fef\}$. $KL = M = M^2 = \{eg, egfeg, feg, efeg, egefeg\}$. However, $AL = P = P^2 = \{eg, egfeg, egefeg, feg\}$. Clearly $P \neq M$. Thus even though, by Theorem 4, $\mathscr{P}(S)$ must be a disjoint union of nil semigroups, it is not a band of nil semigroups. Also note that in $\mathscr{P}(S)$, a product of idempotents need not be an idempotent. For instance, $\{1, e\}$, $\{1, f\}$ are idempotents, but their product $\{1, e, f, ef\}$ is clearly not idempotent.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, Amer. Math. Soc. (Providence, Rhode Island, 1961).

2. D. J. McCarthy and D. L. Hayes, *Subgroups of the power semigroup of a group*, J. of Comb. Theory, ser. A. *14* (1973), 173–186.

3. M. S. Putcha, *On the maximal semilattice decomposition of the power semigroup of a semigroup*, Semigroup Forum *15* (1978), 263–267.

4. J. L. Rhodes and B. R. Tilson, *Local structure of finite semigroups*, in "Algebraic Theory of Machines, Languages, and Semigroups," M. A. Arbib, ed., Chapter 7 (Academic Press, 1968).

5. T. Tamura and J. Shafer, *Power semigroups*, Math. Japan. *12* (1967), 25–32.

6. ——— *Power semigroups II*, Notices AMS, *15* (1968), 395.

7. V. Trnkova, *On a representation of commutative semigroups*, Semigroup Forum *10* (1975), 203–214.

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