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ON FANO VARIETIES WITH TORUS ACTION OF COMPLEXITY 1

ELAINE HERPPICH

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany (elaine.herppich@uni-tuebingen.de)

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Abstract In this work we provide effective bounds and classification results for rational $\mathbb{Q}$-factorial Fano varieties with a complexity-one torus action and Picard number 1 depending on the two invariants dimension and Picard index. This complements earlier work by Hausen *et al.*, where the case of a free divisor class group of rank 1 was treated.

Keywords: Fano varieties; group actions on varieties; factorially graded rings

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1. Statement of the results

The subject of this paper is rational $\mathbb{Q}$-factorial Fano varieties X defined over an algebraically closed field $\mathbb{K}$ of characteristic 0 (see, for example, [**10**, **14**] for classical work). A more recent focus in this field was toric Fano varieties, where one uses the description in terms of lattice polytopes (see, for example, [**2**, **12**, **13**]). Here, we study the case where X comes, more generally, with an effective action of a torus T of complexity 1, i.e. $\dim X - \dim T = 1$; by Fano varieties we mean normal projective varieties with ample anticanonical divisor $-K_X$. We continue the work of [**8**], where classification results for the case $\mathrm{Cl}(X) = \mathbb{Z}$ were given. In this paper we study the more general case of Picard number 1, i.e. we allow torsion in the divisor class group. A first step is Theorem 3.2, where we provide effective bounds for the number of deformation types of Fano varieties X, as above, with fixed dimension d and Picard index $\mu := [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. As a consequence, we obtain restricting statements about the number $\delta(d, \mu)$ of different deformation types of $\mathbb{Q}$-factorial d-dimensional Fano varieties with a complexity-one torus action, Picard number 1 and Picard index μ. In the toric situation, $\delta(d, \mu)$ is bounded above by μ^{d^2}. For the non-toric case we get the following asymptotic results.

Theorem 1.1. *For fixed $d \in \mathbb{Z}_{>0}$, the number $\delta(d, \mu)$ is asymptotically bounded above by $\mu^{(1+\varepsilon)\mu^2}$ for $\varepsilon > 0$ arbitrarily small, and, for fixed $\mu \in \mathbb{Z}_{>0}$, it is asymptotically bounded above by d^{Ad} with a constant A depending only on μ.*

Table 1. $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
1	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2^3 + T_3^4 + T_4^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{pmatrix}$	1	1

Table 2. $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 3$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
2	$\mathbb{K}[T_1,\dots,T_4]/\langle T_1T_2^2 + T_3^3 + T_4^3\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{2} & \bar{0} \end{pmatrix}$	1	1

We turn to the classification. Our approach uses the Cox ring $\mathcal{R}(X)$, which is defined by

$$\mathcal{R}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Given this ring, the variety X can be realized as a quotient of an open subset in $\mathrm{Spec}(\mathcal{R}(X))$ by the action of a diagonalizable group.

According to [**7**, Theorem 1.3], the Cox ring of a normal complete rational variety with a complexity-one torus action is finitely generated. Furthermore, every such Fano variety is uniquely determined by its Cox ring (as a $\mathrm{Cl}(X)$-graded ring). In the case of Picard number 1, the toric varieties of this type correspond to the fake weighted projective spaces as defined in [**11**], and the Cox ring is polynomial. In the subsequent theorems we list non-toric complexity-one Fano varieties with Picard number 1 in the cases where $\mathrm{Cl}(X)$ has non-trivial torsion; for the non-toric results in the case of $\mathrm{Cl}(X) = \mathbb{Z}$ we refer the reader to [**8**]. The Cox rings are described in terms of generators and relations, and we specify the $\mathrm{Cl}(X)$-grading by giving the degrees of the generators. Additionally, we list the degree of the Fano varieties $d_X := (-K_X)^d$ and the Gorenstein index $\iota(X)$, i.e. the smallest positive integer such that $\iota(X)K_X$ is Cartier.

Theorem 1.2. *Let X be a non-toric Fano surface with an effective $\mathbb{K}^*$-action, Picard number* 1*, non-trivial torsion in the class group and* $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \leqslant 6$*. Its Cox ring is then precisely one of those listed in Tables 1–4, where the parameter λ occurring in the second relation of surface number* 7 *can be any element of $\mathbb{K}^* \setminus \{1\}$. Furthermore, the Cox rings listed in Tables 1–4 are pairwise non-isomorphic as graded rings.*

Remark 1.3. Gorenstein surfaces are well known to have ADE-singularities, which are, in particular, canonical. Consequently, the surfaces numbered 1–5, 7, 9 and 10 are canonical. Furthermore, in [**16**] all log terminal del Pezzo $\mathbb{K}^*$-surfaces of Gorenstein index up to 3 are classified. Comparing the surfaces listed in [**16**, Theorems 4.9, 4.10] with Table 4 shows that number 11 is not log terminal. The resolution of this surface can be

Table 3. $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 4$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
3	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^2 + T_2^2 + T_3^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	2	1
4	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^2 + T_4^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} \end{pmatrix}$	2	1
5	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1^2T_2 + T_3^2 + T_4^4\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 2 & 2 & 1 \\ \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix}$	2	1
6	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2^2 + T_3^6 + T_4^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 2 & 2 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{pmatrix}$	1	2
7	$\mathbb{K}[T_1, \dots, T_5]\Big/\left\langle \begin{matrix} T_1T_2 + T_3^2 + T_4^2, \\ \lambda T_3^2 + T_4^2 + T_5^2 \end{matrix} \right\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	1	1

Table 4. $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 6$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
8	$\mathbb{K}[T_1, T_2, T_3, S_1]/\langle T_1^3 + T_2^3 + T_3^2\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} 2 & 2 & 3 & 1 \\ \bar{1} & \bar{2} & \bar{0} & \bar{0} \end{pmatrix}$	$\frac{2}{3}$	3
9	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^3 + T_4^3\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{pmatrix} 1 & 2 & 1 & 1 \\ \bar{1} & \bar{2} & \bar{2} & \bar{0} \end{pmatrix}$	2	1
10	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^2 + T_4^4\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 3 & 1 & 2 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	3	1
11	$\mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2^5 + T_3^2 + T_4^8\rangle$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 3 & 1 & 4 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	$\frac{1}{3}$	3

explicitly computed by using the method of toric ambient modification, as demonstrated in [**5**, Examples 3.20, 3.21].

Theorem 1.4. *Let X be a three-dimensional non-toric Fano variety with an effective two-torus action, Picard number 1, non-trivial torsion in the class group and $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$. Its Cox ring is then precisely one of those in Table 5, where the parameter λ occurring in the second relation of 3-fold number 38 can be any element of $\mathbb{K}^* \setminus \{1\}$. Furthermore, the Cox rings listed in Table 5 are pairwise non-isomorphic as graded rings.*

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Table 5. $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
1	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2+T_3^2+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{pmatrix}$	27	1
2	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{pmatrix}$	8	2
3	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	8	1
4	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^3+T_3^2+T_4^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	8	2
5	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^5+T_3^2+T_4^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{pmatrix}$	1	1
6	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^5+T_3^2+T_4^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{pmatrix}$	1	1
7	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1^2T_2^4+T_3^2+T_4^3\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 3 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{pmatrix}$	4	1
8	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1^2T_2^4+T_3^2+T_4^3\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 3 & 2 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} & \bar{0} \end{pmatrix}$	4	1
9	$\mathbb{K}[T_1,\dots,T_4,S_1]/\langle T_1T_2^5+T_3^3+T_4^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{pmatrix}$	4	2
10	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4^2+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	8	2
11	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3+T_3^2T_4^2+T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix}$	2	1
12	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^2T_4^4+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	1	1
13	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5+T_3^2T_4^4+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{pmatrix}$	1	1
14	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^3T_4^3+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	1	2
15	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^3T_4^3+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$	1	2
16	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4+T_3^5T_4+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	1	2

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Table 5. (*cont.*) $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = 2$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
17	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1^2T_2^4 + T_3^5T_4 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{1} & \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$	1	2
18	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	27	2
19	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	27	1
20	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2 + T_3T_4^2 + T_5^3\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} & \bar{0} \end{pmatrix}$	12	1
21	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{1} & \bar{1} & \bar{0} & \bar{0} & \bar{0} \end{pmatrix}$	2	2
22	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	2	1
23	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	8	2
24	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^3 + T_3T_4^3 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	8	2
25	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	1	1
26	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5 + T_3T_4^5 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	1	2
27	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5 + T_3^3T_4^3 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	1	1
28	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^5 + T_3^3T_4^3 + T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	1	2
29	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2 + T_4^2 + T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	8	2
30	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2 + T_4^2 + T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{1} & \bar{1} \end{pmatrix}$	8	1
31	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^2 + T_4^2 + T_5^4\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{1} \end{pmatrix}$	8	1
32	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4 + T_4^2 + T_5^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix} 1 & 1 & 1 & 3 & 1 \\ \bar{0} & \bar{0} & \bar{1} & \bar{1} & \bar{0} \end{pmatrix}$	1	1

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Table 5. (*cont.*) $[\mathrm{Cl}(X):\mathrm{Pic}(X)]=2$.

no.	$\mathcal{R}(X)$	$\mathrm{Cl}(X)$	grading	d_X	$\iota(X)$
33	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^2+T_5^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&3&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{pmatrix}$	1	1
34	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^2+T_5^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&3&1\\ \bar{0}&\bar{1}&\bar{0}&\bar{1}&\bar{0}\end{pmatrix}$	1	1
35	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^2+T_5^6\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&3&1\\ \bar{0}&\bar{1}&\bar{0}&\bar{0}&\bar{1}\end{pmatrix}$	1	1
36	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2^2T_3^3+T_4^3+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&2&3\\ \bar{1}&\bar{0}&\bar{1}&\bar{0}&\bar{0}\end{pmatrix}$	4	1
37	$\mathbb{K}[T_1,\dots,T_5]/\langle T_1T_2T_3^4+T_4^3+T_5^2\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&2&3\\ \bar{0}&\bar{0}&\bar{1}&\bar{0}&\bar{1}\end{pmatrix}$	4	2
38	$\mathbb{K}[T_1,\dots,T_6]\Big/\Big\langle\begin{matrix}T_1T_2+T_3T_4+T_5^2,\\ \lambda T_3T_4+T_5^2+T_6^2\end{matrix}\Big\rangle$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\begin{pmatrix}1&1&1&1&1&1\\ \bar{0}&\bar{0}&\bar{1}&\bar{1}&\bar{1}&\bar{0}\end{pmatrix}$	16	2

Let X be a normal complete rational variety coming with a complexity-one torus action of T. Consider the T-invariant open subset X_0 consisting of all points $x\in X$ having finite isotropy group. According to [**15**, Corollary 3] there exists a geometric quotient $q\colon X_0\to X_0/T$ such that X_0/T is irreducible and normal but possibly not separated. The property of the orbit space X_0/T being separated is reflected in the Cox ring relations by the condition that each monomial depends on only one variable, e.g. surface number 3 in Theorem 1.2 (see [**7**, Theorem 1.2]). Geometrically, this means that every orbit is contained in the closure of either exactly one maximal orbit or of infinitely many maximal orbits. For such varieties we have the following general finiteness statement.

Theorem 1.5. *The number of d-dimensional normal complete rational varieties of Picard number* 1 *with a complexity-one torus action of T and Picard index μ, such that X_0/T is separated, is finite.*

2. Description of the Cox ring

We briefly recall from [**6**] a construction of $\mathbb{Q}$-factorial normal rational projective varieties with a complexity-one torus action. Here, we specialize to the case of Picard number 1; the details are given in [**6**, Proposition 2.4].

Construction 2.1. For $r\geqslant 1$, consider a sequence $A=(a_0,\dots,a_r)$ of pairwise linearly independent vectors in $\mathbb{K}^2$, a sequence $\mathfrak{n}=(n_0,\dots,n_r)$ of positive integers, a non-negative integer m and a family $L=(l_{ij})$ of positive integers, where $0\leqslant i\leqslant r$ and $1\leqslant j\leqslant n_i$. Set

$$R(A,\mathfrak{n},L,m):=\mathbb{K}[T_{ij},S_k]/\langle g_0,\dots,g_{r-2}\rangle,$$

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where the T_{ij} are indexed by $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$, the S_k by $1 \leqslant k \leqslant m$ and the relations g_i are defined as follows. Set $T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}$ and

$$g_i := \det \begin{pmatrix} a_i & a_{i+1} & a_{i+2} \\ T_i^{l_i} & T_{i+1}^{l_{i+1}} & T_{i+2}^{l_{i+2}} \end{pmatrix}.$$

Define $n := n_0 + \cdots + n_r$ and let $K := \mathbb{Z} \oplus K^t$ be an abelian group with torsion part K^t. Suppose that $R(A, \mathfrak{n}, L, m)$ is positively K-graded via

$$\deg T_{ij} = w_{ij} \in K, \qquad \deg S_k = u_k \in K,$$

i.e. $w_{ij}, u_k \in \mathbb{Z}_{\geqslant 0} \otimes K^t$, and that any $n + m - 1$ of these degrees generate K as a group. The K-grading defines a diagonal action of $H := \operatorname{Spec} \mathbb{K}[K]$ on $\mathbb{K}^{n+m}$. By construction,

$$\bar{X} := V(g_i;\ 0 \leqslant i \leqslant r-2) = \operatorname{Spec} R(A, \mathfrak{n}, L, m)$$

is invariant under this H-action. The open set $\mathbb{K}^{n+m} \setminus \{0\}$ allows for a geometric quotient of this H-action, which is denoted by $p\colon \mathbb{K}^{n+m} \setminus \{0\} \to Z$, where the toric variety Z is a fake weighted projective space. Furthermore, we get a geometric quotient $p\colon \hat{X} \to X$ of the embedded open subset $\hat{X} := \bar{X} \setminus \{0\}$:

$$\begin{array}{ccc} \hat{X} & \hookrightarrow & \mathbb{K}^{n+m} \setminus \{0\} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X & \hookrightarrow & Z \end{array}$$

The quotient space $X := \hat{X} /\!\!/ H$ is a $\mathbb{Q}$-factorial normal projective variety of dimension

$$\dim(X) = n + m - r.$$

It has divisor class group $\operatorname{Cl}(X) = K$, Cox ring $\mathcal{R}(X) = R(A, \mathfrak{n}, L, m)$ and a complexity-one torus action. This torus is given by the stabilizer of X under the action of the maximal torus T_Z of Z.

Note that, if there is an index $0 \leqslant i \leqslant r$ such that $l_{i1} = 1$ and $n_i = 1$, there is at least one relation containing a linear term. In this case the ring is isomorphic to the polynomial ring that we get if we omit the relations of this type. Consequently, we may always assume that $l_{i1} n_i \neq 1$.

Remark 2.2. Varieties with complexity-one action, as formulated in Construction 2.1, can be considered as a generalized version of well-formed complete intersections in weighted projective spaces, in the sense of [**9**].

According to [**6**, Theorem 1.5], every $\mathbb{Q}$-factorial normal complete rational variety X with a complexity-one torus action and Picard number 1 has a Cox ring $R(X)$ that is isomorphic as a graded ring to some K-graded algebra $R(A, \mathfrak{n}, L, m)$ with $K \cong \operatorname{Cl}(X)$.

We collect some geometric properties of the varieties X just constructed. Every element $w \in K = \mathbb{Z} \oplus K^t$ can be written as $w = w^0 + w^t$, where $w^0 \in \mathbb{Z}$ and $w^t \in K^t$. Furthermore, every $\bar{x} = (\bar{x}_{ij}, \bar{x}_k) \in \hat{X} \subseteq \mathbb{K}^{n+m}$ defines a point $x \in X$ by $x := p(\bar{x})$; the points $\bar{x} \in \hat{X}$ are called Cox coordinates of x. We define the set of all weights corresponding to a non-zero coordinate of $\bar{x}$ by

$$W_{\bar{x}} := \{w_{ij};\ \bar{x}_{ij} \neq 0\} \cup \{u_k;\ \bar{x}_k \neq 0\}.$$

Moreover, let $\mathrm{Cl}(X, x)$ denote the local divisor class group in x, i.e. the group of all divisor classes that are principal near x.

Proposition 2.3. *Let X be a $\mathbb{Q}$-factorial complete normal variety with complexity-one torus action and Picard number 1, as formulated in Construction 2.1, and set $\gamma_i := \deg(g_i)$, $0 \leqslant i \leqslant r$. The following statements then hold.*

(i) *For any $\bar{x} \in \hat{X}$, the local divisor class group $\mathrm{Cl}(X, x)$ of $x := p(\bar{x})$ is finite and $\gcd(w^0;\ w \in W_{\bar{x}})$ always divides the order of the group.*

(ii) *The Picard group $\mathrm{Pic}(X)$ is free and the Picard index is given by*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mathrm{lcm}_{x \in X}(\gcd(w^0;\ w \in W_{\bar{x}}))|\mathrm{Cl}(X)^t|.$$

In particular, $|\mathrm{Cl}(X)^t|$ is a divisor of $[\mathrm{Cl}(X) : \mathrm{Pic}(X)]$ and we have that $|\mathrm{Cl}(X)^t| \leqslant [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$.

(iii) *For the anticanonical class $-K_X \in \mathrm{Cl}(X)$ and its self-intersection number $d_X := (-K_X)^d$, one has*

$$-K_X = \sum_{i=0}^{r} \sum_{j=1}^{n_i} w_{ij} + \sum_{k=1}^{m} u_k - \sum_{i=0}^{r-2} \gamma_i,$$
$$d_X = \left(\sum_{i=0}^{r} \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{k=1}^{m} u_k^0 - \sum_{i=0}^{r-2} \gamma_i^0 \right)^d \frac{\gamma_0^0 \cdots \gamma_{r-2}^0}{\prod_{i=0}^{r} \prod_{j=1}^{n_i} w_{ij}^0 \prod_{k=1}^{m} u_k^0 |\mathrm{Cl}(X)^t|}.$$

(iv) *The variety X is Fano if and only if the following inequality holds:*

$$(r-1)\deg(g_0)^0 = \sum_{i=0}^{r-2} \deg(g_i)^0 < \sum_{i=0}^{r} \sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_k^0.$$

Proof. Let $\bar{x}(i, j)$ (respectively, $\bar{x}(k)$) be a point in $\hat{X}$ having the ijth (respectively, $(n+k)$th) entry 1 and all other entries 0. With $\hat{Z} := \mathbb{K}^{n+m} \setminus \{0\}$, we obtain a commutative diagram:

$$\begin{array}{ccc} \hat{X} & \hookrightarrow & \hat{Z} \\ {\scriptstyle /\!\!/ H} \downarrow & & \downarrow {\scriptstyle /\!\!/ H} \\ X & \hookrightarrow & Z \end{array}$$

where the induced map embeds X into a toric variety Z such that $\mathrm{Cl}(X) \cong \mathrm{Cl}(Z)$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$ holds (see [**1**, Corollary III.3.1.7]). By choice, $\bar{x}(i, j)$ (respectively, $\bar{x}(k)$)

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is a toric fixed point. Consequently, the Picard group $\mathrm{Pic}(Z)$, and also $\mathrm{Pic}(X)$, is free [**3**, Theorem VII 2.16]. According to [**4**, Corollary 4.9], we obtain that

$$\mathrm{Pic}(X) = \bigcap_{\bar{x}\in\hat{X}} \langle w;\ w \in W_{\bar{x}}\rangle \cong \bigcap_{\bar{x}\in\hat{X}} \langle w^0;\ w \in W_{\bar{x}}\rangle,$$

where the last equality follows from the fact that $\mathrm{Pic}(X)$ is free. This proves assertions (i) and (ii). The remaining statements are special cases of [**4**, Proposition 4.15 and Corollary 4.16]. The self-intersection number can be easily computed by using toric intersection theory in the ambient toric variety (cf. [**1**, Construction III 3.3.4]). □

Corollary 2.4. *Let X be a $\mathbb{Q}$-factorial complete normal variety with complexity-one torus action and Picard number* 1*. If X is locally factorial, then the divisor class group $\mathrm{Cl}(X)$ is free.*

The following example shows that one can use Proposition 2.3 (iv) to create series of Fano varieties by altering the torsion part of the divisor class group $\mathrm{Cl}(X)$.

Example 2.5. Set $l_{01} = 7$, $l_{02} = 1$, $l_{11} = 5$ and $l_{21} = 2$, as well as $w^0_{01} = 1$, $w^0_{02} = 3$, $w^0_{11} = 2$ and $w^0_{21} = 5$. According to Construction 2.1, these data define one single Cox ring relation of the form $g_0 = T_{01}^7 T_{02} + T_{11}^5 + T_{21}^2$. Since we have that

$$w^0_{01} + w^0_{02} + w^0_{11} + w^0_{21} = 11 > 10 = \deg(g_0)^0,$$

one can use these data to create Cox rings of Fano varieties. We provide some possible $\mathrm{Cl}(X)$-gradings, given by the matrices Q_i, defining del Pezzo $\mathbb{K}^*$-surfaces with fixed grading in the free part of the divisor class group and varying torsion part of the class group $\mathrm{Cl}(X)^t$:

$$Q_1 = \begin{pmatrix} 1 & 3 & 2 & 5 \end{pmatrix}, \qquad \mathrm{Cl}(X_1) = \mathbb{Z};$$

$$Q_2 = \begin{pmatrix} 1 & 3 & 2 & 5 \\ \bar{0} & \bar{2} & \bar{1} & \bar{1} \end{pmatrix}, \qquad \mathrm{Cl}(X_2) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z};$$

$$Q_3 = \begin{pmatrix} 1 & 3 & 2 & 5 \\ \bar{2} & \bar{1} & \bar{3} & \bar{3} \end{pmatrix}, \qquad \mathrm{Cl}(X_3) = \mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z};$$

$$Q_4 = \begin{pmatrix} 1 & 3 & 2 & 5 \\ \bar{0} & \bar{1} & \bar{9} & \bar{6} \end{pmatrix}, \qquad \mathrm{Cl}(X_4) = \mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z};$$

$$Q_5 = \begin{pmatrix} 1 & 3 & 2 & 5 \\ \bar{0} & \bar{3} & \bar{11} & \bar{8} \end{pmatrix}, \qquad \mathrm{Cl}(X_5) = \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z};$$

$$Q_6 = \begin{pmatrix} 1 & 3 & 2 & 5 \\ \bar{0} & \bar{7} & \bar{15} & \bar{12} \end{pmatrix}, \qquad \mathrm{Cl}(X_6) = \mathbb{Z} \oplus \mathbb{Z}/17\mathbb{Z}.$$

Note that in this situation not every group of the form $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, $k \in \mathbb{N}_{>0}$, can be realized as a divisor class group.

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In Example 2.5 the numbers $\ell_i := \gcd(l_{i1}, \dots, l_{in_i})$ are pairwise coprime, namely, $\ell_0 = 1$, $\ell_1 = 2$ and $\ell_2 = 5$. This allows for the case $\mathrm{Cl}(X_1) = \mathbb{Z}$ (see [**8**, Theorem 1.9]). If the numbers ℓ_i are not pairwise coprime, then there is always non-trivial torsion in the divisor class group, as the following lemma shows.

Lemma 2.6. *Set $\ell_i := \gcd(l_{i1}, \dots, l_{in_i})$. All numbers $\gcd(\ell_i, \ell_j)$, $0 \leqslant i \neq j \leqslant r$, then divide $|\mathrm{Cl}(X)^t|$ and the Picard index μ. In particular, this holds for $\mathrm{lcm}_{j\neq i}(\gcd(\ell_i, \ell_j))$.*

Proof. According to [**6**, Theorem 1.5], the divisor class group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ where P^* is dual to $P \colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m-1}$ given by a matrix of the form

$$P = \begin{pmatrix} -l_0 & l_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \cdots & l_r & 0 \\ d_0 & d_1 & \cdots & d_r & d' \end{pmatrix},$$

with $l_i = (l_{i0}, \dots, l_{in_i})$ and some integral block matrices d_i and d'. Consequently, $|\mathrm{Cl}(X)^t|$ is the product of all elementary divisors of P, which implies that $\gcd(\ell_0, \ell_j)$ divides $|\mathrm{Cl}(X)^t|$. By an elementary row transformation, we obtain the analogous result for $\gcd(\ell_i, \ell_j)$ where $0 \leqslant i$, $j \leqslant r$, $i \neq j$. Since $|\mathrm{Cl}(X)^t|$ divides the Picard index μ, the assertion follows. □

Remark 2.7. One can even prove that $\mathrm{lcm}_{0\leqslant j\leqslant r}(\prod_{i\neq j} \gcd(\ell_i, \ell_j))$ divides $|\mathrm{Cl}(X)^t|$ (see, for example, surface number 3 in Table 3).

3. Effective bounds

First we consider the case $n_0 = \cdots = n_r = 1$, which means that each relation g_i of the Cox ring $\mathcal{R}(X)$ depends only on three variables. We then have $n = r + 1$ and, consequently, $m = d - 1$. Furthermore, we may write T_i instead of T_{i1} and w_i instead of w_{i1}, etc. In this setting, we obtain the following bounds for the numbers of possible varieties X (Fano or not).

Proposition 3.1. *For any pair $(d, \mu) \in \mathbb{Z}^2_{>0}$ there are, up to deformation equivalence, only a finite number of complete d-dimensional varieties with Picard number 1, Picard index $[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mu$ and Cox ring of the form*

$$\mathbb{K}[T_0, \dots, T_r, S_1, \dots, S_m]/\langle \alpha_i T_i^{l_i} + \alpha_{i+1} T_{i+1}^{l_{i+1}} + \alpha_{i+2} T_{i+2}^{l_{i+2}};\ 0 \leqslant i \leqslant r-2 \rangle.$$

In this situation we have $r < \mu + \xi(\mu) - 1$, where $\xi(\mu)$ denotes the number of primes smaller than μ. Moreover, for $w_i^0 \in \mathbb{Z}_{>0}$ and $u_k^0 \in \mathbb{Z}_{>0}$, where $0 \leqslant i \leqslant r$, $1 \leqslant k \leqslant m$, and the exponents l_i, one has

$$l_i \leqslant \mu, \qquad w_i^0 \leqslant \mu^r, \qquad u_k^0 \leqslant \mu.$$

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Proof. Consider the total coordinate space $\bar{X} \subseteq \mathbb{K}^{r+1+m}$ and the quotient $p\colon \hat{X} \to X$, as well as the points $\bar{x}(k) \in \hat{X}$ having $(r+k)$th coordinate 1 and all other coordinates 0. Set $x(k) := p(\bar{x}(k))$. Then, u_k^0 divides the order of the local class group $\mathrm{Cl}(X, x(k))$. In particular, we have that $u_k^0 \leqslant \mu$.

For each $0 \leqslant i \leqslant r$, fix a point $\bar{y}(i) = (\bar{y}_0, \ldots, \bar{y}_r, 0, \ldots, 0)$ in $\hat{X}$ such that $\bar{y}_i = 0$ and $\bar{y}_j \neq 0$ for $i \neq j$, and set $y_i := p(\bar{y}(i))$. We then obtain that

$$\gcd(w_j^0,\ j \neq i) \mid |\mathrm{Cl}(X, y(i))|.$$

By Lemma 2.6 we have that $\mathrm{lcm}_{j\neq i}(\gcd(l_i, l_j)) \mid |\mathrm{Cl}(X)^t|$. Now consider l_i' such that $l_i = \mathrm{lcm}_{j\neq i}(\gcd(l_i, l_j))l_i'$. The homogeneity condition $l_i w_i^0 = l_j w_j^0$ then gives $l_i' \mid w_j^0$ for all $j \neq i$, and, consequently, $l_i' \mid \gcd(w_j^0,\ j \neq i)$. Since $l_i = l_i' \,\mathrm{lcm}_{j\neq i}(\gcd(l_i, l_j))$, we can conclude that $l_i \leqslant \mu$ by using the formula

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] = \mathrm{lcm}_{x\in X}(\gcd(w^0;\ w \in W_{\bar{X}}))|\mathrm{Cl}(X)^t|$$

of Proposition 2.3 (ii). Since the l_i' are pairwise coprime, we obtain $l_0' \cdots l_r' \mid \gamma^0$ and $l_0' \cdots l_r' \mid \mu$, where $\gamma^0 := \deg(g_0)^0 = l_i w_i^0$. From $l_i w_i^0 = l_j w_j^0$ we deduce that

$$l_i = l_0 \frac{w_0^0}{w_i^0} = l_0 \frac{w_0^0 \cdots w_{i-1}^0}{w_1^0 \cdots w_i^0} = \eta_i \frac{\gcd(w_0^0, \ldots, w_{i-1}^0)}{\gcd(w_0^0, \ldots, w_i^0)} \leqslant \mu,$$

where $1 \leqslant \eta_i \leqslant \mu$. In particular, the last fraction is smaller than μ. All in all, this gives us

$$\begin{aligned} w_0^0 &= \frac{w_0^0}{\gcd(w_0^0, w_1^0)} \frac{\gcd(w_0^0, w_1^0)}{\gcd(w_0^0, w_1^0, w_2^0)} \cdots \frac{\gcd(w_0^0, \ldots, w_{r-2}^0)}{\gcd(w_0^0, \ldots, w_{r-1}^0)} \gcd(w_0^0, \ldots, w_{r-1}^0) \\ &\leqslant \mu^{r-1}\mu \\ &= \mu^r. \end{aligned}$$

Analogously, we get the boundedness for all w_i^0. Now let q be the number of l_i' that are greater than 1. Since all l_i', $0 \leqslant i \leqslant r$, are coprime, q is bounded by $\xi(\mu)$, the number of primes smaller than μ. To avoid the toric case we assume that $l_i \neq 1$ for all $0 \leqslant i \leqslant r$. Consequently, if $l_i' = 1$, then there exists at least one $0 \leqslant j \leqslant r$ such that $\gcd(l_i, l_j) > 1$. Since $\gcd(l_i, l_j)$ divides μ, we get $r + 1 - q < \mu$ as a rough bound. All in all, we get that $r + 1 = r + 1 - q + q < \mu + \xi(\mu)$. □

Proof of Theorem 1.5. Let X be a variety as in Theorem 1.5. Each monomial of the Cox ring relations then depends on only one variable, i.e. $n_i = 1$ for $0 \leqslant i \leqslant r$ (see [**7**, Theorem 1.2] for details). Consequently, Proposition 3.1 provides bounds for the discrete data, such as the non-torsion parts of the weights w_{ij}^0 and u_k^0, the exponents l_{ij} and the number of Cox ring relations r. Since $|\mathrm{Cl}(X)^t| \leqslant \mu$ holds, the number of possibilities for the torsion part of the grading is also restricted, which implies the assertion. □

Theorem 3.2. *Let X be a Fano variety with complexity-one torus action as introduced in Construction 2.1. Fix the dimension $d = \dim(X) = m + n + r$ and the Picard index*

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$\mu = [\mathrm{Cl}(X) : \mathrm{Pic}(X)]$. *The number of Cox ring relations* r, *the free part of the degree of the relations* γ^0, *the weights* w_{ij}^0, u_k^0 *and the exponents* l_{ij}^0, *where* $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$ *and* $1 \leqslant k \leqslant m$, *are then bounded. In particular, one obtains the following effective bounds. We have*

$$u_k^0 \leqslant \mu \quad \text{for } 1 \leqslant k \leqslant m \quad \text{and} \quad |\mathrm{Cl}(X)^t| \leqslant \mu.$$

Moreover, the handling of the remaining data can be organized into five cases, where $\xi(x)$ *denotes the number of primes smaller than* x.

(i) *Suppose that* $r = 0, 1$ *holds. Then,* $n + m \leqslant d + 1$ *holds and one has the bounds*

$$w_{ij}^0 \leqslant \mu \quad \text{for } 0 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant n_i,$$

and the Picard index is given by

$$\mu = \mathrm{lcm}(w_{ij}^0, u_k^0;\ 0 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n_i,\ 1 \leqslant k \leqslant m)|\mathrm{Cl}(X)^t|.$$

(ii) *Suppose that* $r \geqslant 2$ *and* $n_0 = 1$ *hold. Then,* $r \leqslant \mu + \xi(\mu) - 1$, $n = r + 1$ *and* $m = d - 1$ *hold, and one has that*

$$w_{i1}^0 \leqslant \mu^r, \quad l_{i1} \mid \mu \quad \text{for } 0 \leqslant i \leqslant r, \qquad \gamma^0 \leqslant \mu^{r+1},$$

and the Picard index is given by

$$\mu = \mathrm{lcm}(\gcd_i(w_{j1}^0;\ i \neq j), u_k^0;\ 0 \leqslant i \leqslant r,\ 1 \leqslant k \leqslant m)|\mathrm{Cl}(X)^t|.$$

(iii) *Suppose that* $r \geqslant 2$ *and* $n_0 > n_1 = 1$ *hold. We may then assume that* $l_{11} \geqslant \cdots \geqslant l_{r1} \geqslant 2$, *we have* $r \leqslant \mu + \xi(6d\mu) - 1$ *and* $n_0 + m = d$ *and the bounds*

$$\begin{gathered} w_{01}^0, \dots, w_{0n_0}^0 \leqslant \mu, \qquad l_{01}, \dots, l_{0n_0} \leqslant 6d\mu, \qquad \gamma^0 < 6d\mu, \\ w_{11}^0 < 2d\mu, \qquad w_{21}^0 < 3d\mu, \qquad w_{i1}^0, l_{i1} < 6d\mu \quad \text{for } 1 \leqslant i \leqslant r, \end{gathered}$$

and the Picard index is given by

$$\mu = \mathrm{lcm}(w_{0j}^0, \gcd(w_{11}^0, \dots, w_{r1}^0), u_k^0;\ 1 \leqslant j \leqslant n_0,\ 1 \leqslant k \leqslant m)|\mathrm{Cl}(X)^t|.$$

(iv) *Suppose that* $n_1 > n_2 = 1$ *holds. We may then assume that* $l_{21} \geqslant \cdots \geqslant l_{r1} \geqslant 2$, *we have* $r \leqslant \mu + \xi(2(d+1)\mu) - 1$ *and* $n_0 + n_1 + m = d + 1$ *and the bounds*

$$\begin{gathered} w_{ij}^0 \leqslant \mu \quad \text{for } i = 0, 1 \text{ and } 1 \leqslant j \leqslant n_i, \qquad w_{21}^0 < (d+1)\mu, \\ \gamma^0, w_{ij}^0, l_{ij} < 2(d+1)\mu \quad \text{for } 0 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant n_i, \end{gathered}$$

and the Picard index is given by

$$\mu = \mathrm{lcm}(w_{ij}^0, u_k^0;\ 0 \leqslant i \leqslant 1,\ 1 \leqslant j \leqslant n_i,\ 1 \leqslant k \leqslant m)|\mathrm{Cl}(X)^t|.$$

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(v) *Suppose that $n_2 > 1$ holds, and let s be the maximal number with $n_s > 1$. We may then assume that $l_{s+1,1} \geqslant \cdots \geqslant l_{r1} \geqslant 2$, we have $s \leqslant d$, $r \leqslant \mu + \xi((d+2)\mu) + d - 1$ and $n_0 + \cdots + n_s + m = d + s$ and the bounds*

$$w_{ij}^0 \leqslant \mu \quad \text{for } 0 \leqslant i \leqslant s, \qquad \gamma^0 < (d+2)\mu,$$
$$w_{ij}^0, l_{ij} < (d+2)\mu \quad \text{for } 0 \leqslant i \leqslant r \text{ and } 1 \leqslant j \leqslant n_i,$$

and the Picard index is given by

$$\mu = \operatorname{lcm}(w_{ij}^0, u_k^0;\ 0 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant n_i,\ 1 \leqslant k \leqslant m)|\mathrm{Cl}(X)^t|.$$

Note that assertions (i) and (ii) do not require the Fano condition.

The remaining part of this section is devoted to the proofs of the main statements of this paper. To prove Theorem 3.2 we need the following essential lemma.

Lemma 3.3. *Consider the ring $\mathbb{K}[T_{ij};\ 0 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant n_i][S_1, \dots, S_k]/\langle g\rangle$, where $n_0 \geqslant n_1 \geqslant n_2 \geqslant 1$ holds, and let K be a finitely generated abelian group of the form $K = \mathbb{Z} \oplus K^t$ with torsion part K^t. Suppose that g is homogeneous with respect to the K-grading of $\mathbb{K}[T_{ij}, S_k]$ given by $\deg T_{ij} =: w_{ij} = w_{ij}^0 + w_{ij}^t \in K$ with $w_{ij}^0 \in \mathbb{Z}_{>0}$, and $\deg S_k =: u_k = u_k^0 + u_k^t \in K$ with $u_k^0 \in \mathbb{Z}_{>0}$, and assume that*

$$\deg(g)^0 < \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_i^0.$$

Let $\mu \in \mathbb{Z}_{>1}$, assume that $w_{ij}^0 \leqslant \mu$ whenever $n_i > 1$, $1 \leqslant j \leqslant n_i$, and that $u_k^0 \leqslant \mu$ for $1 \leqslant k \leqslant m$, and set $d := n_0 + n_1 + n_2 + m - 2$. Depending on the shape of g, one obtains the following bounds.

(i) *Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} + \eta_2 T_{21}^{l_{21}}$ with $n_0 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds. If we have $l_{11} > l_{21} \geqslant 2$ and $\gcd(l_{11}, l_{21}) \mid \mu$, then we have*

$$w_{11}^0 < 2d\mu, \quad w_{21}^0 < 3d\mu, \qquad l_{22}, l_{21}, \deg(g)^0 < 6d\mu.$$

If we have $l_{11} = l_{21} \geqslant 2$, then we have

$$l_{11}, w_{11}^0, l_{21}, w_{21}^0, \deg(g)^0 \leqslant \mu.$$

(ii) *Suppose that $g = \eta_0 T_{01}^{l_{01}} \cdots T_{0n_0}^{l_{0n_0}} + \eta_1 T_{11}^{l_{11}} \cdots T_{1n_1}^{l_{1n_1}} + \eta_2 T_{21}^{l_{21}}$ with $n_1 > 1$ and coefficients $\eta_i \in \mathbb{K}^*$ holds, and we have $l_{21} \geqslant 2$. Then we have*

$$w_{21}^0 < (d+1)\mu, \qquad \deg(g)^0 < 2(d+1)\mu.$$

Proof. We prove (i). Set $c := (n_0 + m)\mu = d\mu$ for short. Using the homogeneity of g and the assumed inequality, we then obtain that

$$l_{11}w_{11}^0 = l_{21}w_{21}^0 = \deg(g)^0 < \sum_{i=0}^{2}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_i^0 \leqslant c + w_{11}^0 + w_{21}^0.$$

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First look at the case $l_{11} > l_{21} \geqslant 2$. Putting this into the above inequalities, we arrive at $2w_{11}^0 < c + w_{21}^0$ and $w_{21}^0 < c + w_{11}^0$. We conclude that $w_{11}^0 < 2c$ and $w_{21}^0 < 3c$. Consequently, we obtain that

$$\deg(g)^0 < c + w_{11}^0 + w_{21}^0 < 6c = 6d\mu.$$

If we have $l_{11} = l_{21}$, the homogeneity condition $l_{11}w_{11}^0 = l_{21}w_{11}^0$ gives us $w_{11}^0 = w_{21}^0$. Thus, we have that $\gcd(w_{11}^0, w_{21}^0) = w_{11}^0 = w_{21}^0 \mid \mu$ and, by assumption, $\gcd(l_{11}, l_{21}) = l_{21} = l_{11} \mid \mu$. Consequently, $l_{11}, w_{11}^0, l_{21}, w_{21}^0, \deg(g)^0 \leqslant \mu$.

We prove (ii). Here we set $c := (n_0 + n_1 + m)\mu = (d+1)\mu$. The assumed inequality then gives that

$$l_{21}w_{21}^0 = \deg(g)^0 < \sum_{i=0}^{1}\sum_{j=1}^{n_i} w_{ij}^0 + \sum_{i=1}^{m} u_i^0 + w_{21}^0 \leqslant c + w_{21}^0.$$

Since we assumed that $l_{21} \geqslant 2$, we can conclude that $w_{21}^0 < c$. This in turn gives us $\deg(g)^0 < 2c$. □

Proof of Theorem 3.2. As before, we denote by $\bar{X} \subseteq \mathbb{K}^{n+m}$ the total coordinate space, and we consider the quotient $p\colon \hat{X} \to X$.

We first discuss the case when X is a toric variety. The Cox ring is then a polynomial ring, $\mathcal{R}(X) = \mathbb{K}[S_1, \dots, S_m]$. For each $1 \leqslant k \leqslant m$, consider the point $\bar{x}(k) \in \hat{X}$ having kth coordinate 1 and all other coordinates 0, and set $x(k) := p(\bar{x}(k))$. By Proposition 2.3, the order of the local class group $\mathrm{Cl}(X, x(k))$ is then divisible by u_k^0. Together with Proposition 2.3 (ii) we get $u_k^0 \leqslant \mu$ for $1 \leqslant k \leqslant m$ and $|\mathrm{Cl}(X)^t| \leqslant \mu$, which settles assertion (i).

We now treat the non-toric case, which means that $r \geqslant 2$. Note that we have $n \geqslant 3$. The case $n_0 = 1$ is covered in Proposition 3.1, which proves assertion (ii). Hence, we are left with $n_0 > 1$. For every i with $n_i > 1$ and every $1 \leqslant j \leqslant n_i$, there exists the point $\bar{x}(i,j) \in \hat{X}$ with ij-coordinate T_{ij} equal to 1 and all other coordinates equal to 0, and, thus, we have the point $x(i,j) := p(\bar{x}(i,j)) \in X$. Moreover, for every $1 \leqslant k \leqslant m$, we have the point $\bar{x}(k) \in \bar{X}$ having the k-coordinate S_k equal to 1 and all other coordinates equal to 0; we set $x(k) := p(\bar{x}(k))$. Proposition 2.3 provides the bounds

$$w_{ij}^0 \leqslant \mu, \quad u_k^0 \leqslant \mu \quad \text{for } n_i > 1,\ 0 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n_i,\ 1 \leqslant k \leqslant m. \tag{3.1}$$

Let $0 \leqslant s \leqslant r$ be the maximal number with $n_s > 1$. Then g_{s-2} is the last polynomial such that each of its three monomials depends on more than one variable. For any $t \geqslant s$, we have the 'cut ring'

$$R_t := \mathbb{K}[T_{ij}, S_k]/\langle g_0, \dots, g_{t-2}\rangle,$$

where $0 \leqslant i \leqslant t$, $1 \leqslant j \leqslant n_i$, $1 \leqslant k \leqslant m$ and the relations g_i depend on only three variables when $i > s$ holds. For the free part of the degree γ^0 of the relations we have that

$$\begin{aligned}(r-1)\gamma^0 &= (t-1)\gamma^0 + (r-t)\gamma^0 \\ &= (t-1)\gamma^0 + l_{t+1,1}w^0_{t+1,1} + \cdots + l_{r1}w^0_{r1} \\ &< \sum_{i=0}^{r}\sum_{j=1}^{n_i} w^0_{ij} + \sum_{i=1}^{m} u^0_i \\ &= \sum_{i=0}^{t}\sum_{j=1}^{n_i} w^0_{ij} + w^0_{t+1,1} + \cdots + w^0_{r1} + \sum_{i=1}^{m} u^0_i.\end{aligned}$$

Note that the inequality is derived from the Fano condition of Proposition 2.3 (iv). Since $l_{i1}w^0_{i1} > w^0_{i1}$ holds, in particular, for $t+1 \leqslant i \leqslant r$, we derive from this the inequality

$$\gamma^0 < \frac{1}{t-1}\left(\sum_{i=0}^{t}\sum_{j=1}^{n_i} w^0_{ij} + \sum_{i=1}^{m} u^0_i\right). \tag{3.2}$$

To obtain the bounds in assertions (iii) and (iv), we consider the cut ring R_t with $t=2$ and apply Lemma 3.3 and Proposition 2.3; note that we have $d = n_0 + n_1 + n_2 + m - 2$ for the dimension $d = \dim(X)$ and that $l_{21} \geqslant 2$ due to the fact that X is non-toric. The bounds $w^0_{i1}, l_{i1} < 6d\mu$ for $3 \leqslant i \leqslant r$ in assertion (iii) follow from $\gamma^0 < 6d\mu$. Similarly, $w^0_{ij}, l_{ij} < 2(d+1)\mu$ for $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$ in assertion (iv) follow from $\gamma^0 < 2(d+1)\mu$. We still have to prove the restriction for the number of relations, which means bounding r. Recall from Lemma 2.6 the definition $\ell_i := \gcd(l_{i1}, \dots, l_{in_i})$, and set $\ell_i = \operatorname{lcm}_{0 \leqslant j \neq i \leqslant r}(\gcd(\ell_i, \ell_j))\ell'_i$. Then $\ell'_0, \dots, \ell'_r$ are coprime. For $i \geqslant 1$, we have $n_i = 1$. Thus, analogously to the proof of Proposition 3.1, we get that $r+1 = r+1-q+q \leqslant \mu + \xi(6d\mu)$, where q is the number of ℓ'_i that are greater than 1 and satisfy $n_i = 1$. For the bound in assertion (iv), the same argument yields $r+1 = r+1-q+q \leqslant \mu + \xi(2(d+1)\mu)$.

To obtain the bounds in assertion (v), we consider the cut ring R_t with $t = s$. Using $n_i = 1$ for $i \geqslant t+1$ and applying the inequalities (3.1) and (3.2), we can derive an upper bound for the degree of the relation as follows:

$$\gamma^0 < \frac{(n_0 + \cdots + n_t + m)\mu}{t-1} = \frac{(d+t)\mu}{t-1} \leqslant (d+2)\mu.$$

We have that $w^0_{ij} l_{ij} \leqslant \gamma^0$ for any $0 \leqslant i \leqslant r$ and any $1 \leqslant j \leqslant n_i$, which implies that all w^0_{ij} and l_{ij} are bounded by $(d+2)\mu$. Since $n_0, \dots, n_{s-1} > 1$ holds, the number s is bounded by $s = 2s - (s-1) - 1 \leqslant d$. Consequently, we obtain that

$$r+1 = r+1-s-q+s+q \leqslant \mu + \xi((d+2)\mu) + d,$$

where q is defined as above.

Finally, we have to express the Picard index μ in terms of the free part of the weights w^0_{ij}, u^0_k and the torsion part $\mathrm{Cl}(X)^t$, as claimed in the assertions. This is a direct application of the formula of Proposition 2.3. □

Proof of Theorem 1.1. Theorem 3.2 provides bounds for the exponents and the number of relations, as well as for the free part of the weights and the torsion part of Cl(X). Since we have that $|\mathrm{Cl}(X)^t| \leqslant \mu$, the possibilities for the torsion part of the weights are also restricted. One computes that the number $\delta(d, \mu)$ of different deformation types is bounded above by

$$\mu^{\mu^2+3\mu+\xi(\mu)^2+\xi(6d\mu)+5d}(6d\mu)^{2\mu+2\xi(6d\mu)+3d-2},$$

which leads to the results of Theorem 1.1. □

Proof of Theorems 1.2 and 1.4. For fixed d and μ, Theorem 3.2 bounds the number of possible data l_{ij}, w_{ij}^0, u_k^0 belonging to Fano varieties. We identify all these constellations by a computer-based algorithm. Since $|\mathrm{Cl}(X)^t| \leqslant \mu$ holds, there are only a finite number of possibilities for the torsion part of the weights that we have to check. By this procedure we obtain Tables 1–5.

We claim that no two of the listed Cox rings describe varieties that are isomorphic to each other. Two minimal systems of homogeneous generators of the Cox ring contain (up to reordering) the same free parts of generator degrees w_{ij}^0, $u_k^0 \in \mathbb{Z}$. Consequently, they are invariant under isomorphy. Furthermore, the exponents $l_{ij} > 1$ represent the orders of all finite non-trivial isotropy groups of one-codimensional orbits of the action T on X (see [**7**, Theorem 1.3]). Moreover, since none of the listed Cox rings is polynomial the varieties are all non-toric. This implies that every complexity-one action is maximal and, consequently, can be assigned to a maximal torus in Aut(X). Note that Aut(X) is also acting effectively on X. Since the maximal tori of Aut(X) are all conjugated, the varieties with complexity-one torus action are isomorphic if and only if they are T-equivariantly isomorphic. Thus, running through the exponents l_{ij} we see that no two of the varieties listed in Theorem 1.2 are isomorphic.

In the case of Theorem 1.4 there is some more work to do. There are no isomorphic 3-folds varying only in the torsion part of the weights; see, for example, the numbers 2, 3 and 4. In these cases, comparing the torsion parts of the gradings shows that it is not possible to install a Cl(X)-graded ring isomorphism between the Cox rings of two different 3-folds.

As an example we consider the 3-folds 2 and 3: let D_2 be a prime divisor, representing $\deg(T_2) \in \mathrm{Cl}(X)$, and let E_1 be a prime divisor, representing $\deg(S_1) \in \mathrm{Cl}(X)$. Then, D_2 has isotropy group of order $l_2 = 3$ and E_1 has infinite isotropy. In the case of 3-fold 2 the term $D_2 - E_1$ represents a non-trivial torsion element, whereas in the case of 3-fold 3 it is the zero element in Cl(X). Thus, these two varieties are not isomorphic. Analogously, we proceed with all other cases to finally obtain Table 5.

Finally, we apply [**4**, Corollary 4.9] to compute the Gorenstein index $\iota(X)$ for all listed varieties, i.e. we have to find the smallest integer $\iota(X)$ such that $\iota(X)K_X$ is contained in all local divisor class groups Cl(X, x) (see also Proposition 2.3). □

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References

1. I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface, Cox rings, eprint (arXiv: 1003.4229, 2010; see also extended version on the authors' webpages.)
2. V. V. Batyrev, Toric Fano threefolds, *Izv. Akad. Nauk SSSR Ser. Mat.* **45**(4) (1981), 704–717.
3. G. Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, Volume 168 (Springer, 1996).
4. J. Hausen, Cox rings and combinatorics, II, *Mosc. Math. J.* **8**(4) (2008), 711–757.
5. J. Hausen, Three lectures on Cox rings, in *Torsors, étale homotopy and applications to rational points*, London Mathematical Society Lecture Note Series, Volume 405, pp. 3–60 (Cambridge University Press, 2013).
6. J. Hausen and E. Herppich, Factorially graded rings of complexity one, in *Torsors, étale homotopy and applications to rational points*, London Mathematical Society Lecture Note Series, Volume 405, pp. 414–428 (Cambridge University Press, 2013).
7. J. Hausen and H. Süss, The Cox ring of an algebraic variety with torus action, *Adv. Math.* **225** (2010), 977–1012.
8. J. Hausen, E. Herppich and H. Süss, Multigraded factorial rings and Fano varieties with torus action, *Documenta Math.* **16** (2011), 71–109.
9. A. R. Iano-Fletcher, Working with weighted complete intersections, in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Note Series, Volume 281, pp. 101–173 (Cambridge University Press, 2000).
10. V. A. Iskovskih, Fano threefolds, II, *Izv. Akad. Nauk SSSR Ser. Mat.* **42**(3) (1978), 506–549.
11. A. Kasprzyk, Bounds on fake weighted projective spaces, *Kodai Math. J.* **32**(2) (2009), 197–208.
12. A. Kasprzyk, Canonical toric Fano threefolds, *Can. J. Math.* **62**(6) (2010), 1293–1309.
13. A. Kasprzyk, M. Kreuzer and B. Nill, On the combinatorial classification of toric log del Pezzo surfaces, *LMS J. Comput. Math.* **13** (2010), 33–46.
14. S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geqslant 2$, *Manuscr. Math.* **36**(2) (1981), 147–162.
15. H. Sumihiro, Equivariant completion, *J. Math. Kyoto Univ.* **14** (1974), 1–28.
16. H. Süss, Canonical divisors on T-varieties, eprint (arXiv:0811.0626v1, 2008).

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