

ON DIFFERENTIAL INVARIANTS OF HOLOMORPHIC PROJECTIVE CURVES

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1. Canonical forms

Let $(\varphi_1(u), \dots, \varphi_n(u))$ be a system of holomorphic functions whose Wronskian does not vanish at origin, where holomorphic functions mean functions holomorphic around origin.

A variable transformation

$$(u, y) \mapsto (u(z), \lambda(z)y)$$

induces a map

$$\begin{aligned} (\varphi_1(u), \dots, \varphi_n(u)) &\mapsto (\phi_1(z), \dots, \phi_n(z)) \\ &= (\lambda(z)\varphi_1(u(z)), \dots, \lambda(z)\varphi_n(u(z))), \end{aligned}$$

where $\frac{du}{dz}(0) \neq 0$ and $\lambda(0) \neq 0$.

We associate linear differential operators of rank n which is a projective invariant of a holomorphic curve¹⁾: $u \mapsto (\varphi_1(u), \dots, \varphi_n(u))$ in \mathbf{P}^{n-1} .

$$\begin{aligned} L_n(p|u, y) &= \sum_{l=0}^n \binom{n}{l} p_l(u) \left(\frac{d}{du}\right)^{n-l} y \\ &= (-1)^{n-1} \begin{vmatrix} \varphi_1 & \cdots & \varphi_n \\ \vdots & & \\ \left(\frac{d}{du}\right)^{n-1} \varphi_1 & \cdots & \left(\frac{d}{du}\right)^{n-1} \varphi_n \end{vmatrix}^{-1} \begin{vmatrix} y & \varphi_1 & \cdots & \varphi_n \\ \vdots & & & \\ \left(\frac{d}{du}\right)^n y & \left(\frac{d}{du}\right)^n \varphi_1 & \cdots & \left(\frac{d}{du}\right)^n \varphi_n \end{vmatrix}, \end{aligned}$$

$$L_n(q|z, y) = \sum_{l=0}^n \binom{n}{l} q_l \left(z \left(\frac{d}{dz}\right)\right)^{n-l} y$$

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1) We mean holomorphic maps by holomorphic curves.

$$= (-1)^{n-1} \begin{vmatrix} \phi_1, & \cdots, & \phi_n \\ \vdots & & \vdots \\ \left(\frac{d}{dz}\right)^{n-1} \phi_1, \cdots, \left(\frac{d}{dz}\right)^{n-1} \phi_n \end{vmatrix}^{-1} \begin{vmatrix} y, & \phi_1, & \cdots, & \phi_n \\ \vdots & & & \vdots \\ \left(\frac{d}{dz}\right)^n y, \left(\frac{d}{dz}\right)^n \phi_1, \cdots, \left(\frac{d}{dz}\right)^n \phi_n \end{vmatrix}.$$

The variable transformation $(u, y) \mapsto (u(z), \lambda(z)y)$ induces a transformation:

$$\begin{aligned} L_n(p|u, y) &\mapsto L_n(q|z, y) \\ &= \lambda(z)^{-1} \left(\frac{du}{dz}\right)^{-n} \sum \binom{n}{l} p_l(u(z)) \left(\frac{d}{du(z)}\right)^{n-l} (\lambda(z)y). \end{aligned}$$

DEFINITION (Lagurre-Forsyth). A linear differential operator $L_n(Q|z, y)$ is called to be canonical, if

$$Q_1 \equiv Q_2 \equiv 0.$$

We call z a canonical independent variable of a canonical form $L_n(Q|z, y)$.

THEOREM 1 (Forsyth)²⁾. For each $L_n(p|u, z)$ there exists a variable transformation

$$(u, y) \mapsto (u(z), \lambda(z)y)$$

such that $L_n(p|u, z)$ is transformed to a canonical form. Moreover a variable transformation maps a canonical form to a canonical form, if and only if

$$u(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \lambda(z) = \frac{cy}{(\gamma z + \delta)^{n-1}} \quad \left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \in SL(2, \mathbf{C}), c \in \mathbf{C} - \{0\} \right).$$

Forsyth's theorem means that for each holomorphic projective curve:

$$u \mapsto (\varphi_1(u), \cdots, \varphi_n(u)) \in \mathbf{P}^{n-1}$$

we may associate a canonical variable and a canonical form which are unique up to Möbius transformations.

2. Differential invariants

Similarly as classical invariant theory, we define differential invariants of $L_n(P|u, y)$ first for generic coefficients $P_i(u)$ ($1 \leq i \leq n$), and then specialization

2) See Theorem 6.1 and 6.2, Chap 6 [1], or Theorem 2.4, Chap 2 [2].

$$\left(\frac{d}{du}\right)^l P_i(u) \mapsto \left(\frac{d}{du}\right)^l p_i(u) \quad (1 \leq i \leq n; 0 \leq l < \infty)$$

gives the definition for $L_n(p|u, y)$.

DEFINITION. A differential invariant of weight p of $L_n(P|u, y)$ (with generic coefficients) is a polynomial

$$\Phi\left(\dots, \left(\frac{d}{dz}\right)^l P_j(u), \dots\right)$$

such that

$$\Phi\left(\dots, \left(\frac{d}{du}\right)^l P_j(u), \dots\right)(du)^p$$

is invariant for every variable transformation

$$(u, y) \mapsto (u(z), \lambda(z)y),$$

where

$$\frac{du(0)}{dz} \neq 0, \quad \lambda(0) \neq 0.$$

Forsyth gave the following fundamental system of differential invariants of a canonical form

$$L_n(Q|z, y) = \left(\frac{d}{dz}\right)^n y + \sum_{l=3}^n \binom{n}{l} Q_l(z) \left(\frac{d}{dz}\right)^{n-l} y;$$

$$(\theta_3(z), \dots, \theta_n(z)),$$

$$\theta_p(z) = \frac{1}{2} \sum \frac{(-1)^s (p-2)! p! (2p-s-2)!}{(p-s-1)! (p-s)! (2p-3)! s!} \left(\frac{d}{dz}\right)^s Q_{p-s}(z) \quad (3 \leq p \leq n),$$

where weight of $\theta_p(z)$ is p .

For a complex number $w (\neq 0, 1, 2, \dots)$ we denote

$$\binom{w}{l} = \frac{w(w-1)\dots(w-l+1)}{l!}.$$

Let w_1, \dots, w_n be complex numbers ($\neq 0, 1, 2, \dots$),

$$\xi_1 = (\xi_1^{(0)}, \xi_1^{(1)}, \xi_1^{(2)}, \dots), \dots, \xi_n = (\xi_n^{(0)}, \xi_n^{(1)}, \xi_n^{(2)}, \dots)$$

be variable vectors of infinite length and denote

$$f_j(\xi_j | z) = \sum_{l=0}^{\infty} \binom{w_j}{l} \xi_j^{(l)} z^l \quad (1 \leq j \leq n).$$

Germ of $SL(2, \mathbb{C})$ at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts on $f_j(\xi_j | z)$ as follows

$$f_j\left(\begin{pmatrix} \delta\beta \\ \gamma\alpha \end{pmatrix} \xi_j | z\right) = (\gamma z + \delta)^{w_j} f_j\left(\xi_j \left| \frac{\alpha z + \beta}{\gamma z + \delta} \right.\right) \quad (1 \leq j \leq n),$$

where $(\gamma z + \delta)^{w_j} = \delta^{w_j} \sum_{l=0}^{\infty} \binom{w_j}{l} (\delta^{-1} \gamma z)^l$.

DEFINITION. A formal power series $F(\xi; z)$ with coefficients in $\mathbb{C}[\xi]$ is called a covariant of index u (a complex number), if

$$F(\xi; z) = \sum_{l=0}^{\infty} \binom{u}{l} c_l(\xi) z^l,$$

$$F\left(\begin{pmatrix} \delta\beta \\ \gamma\alpha \end{pmatrix} \xi; z\right) = (\gamma z + \delta)^u F\left(\xi; \frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad \left(\begin{pmatrix} \delta\beta \\ \gamma\alpha \end{pmatrix} \in \text{Germ of } SL(2, \mathbb{C})\right).$$

DEFINITION. A polynomial $\varphi(\xi)$ is called a semi-invariant of index u , if

$$\sum_{j=1}^n \sum_{l=0}^{\infty} l \xi_j^{(l-1)} \frac{\partial}{\partial \xi_j^{(l)}} \varphi(\xi) = 0$$

and

$$\sum_{j=1}^n \sum_{l=0}^{\infty} (w_j - 2l) \xi_j^{(l)} \frac{\partial}{\partial \xi_j^{(l)}} \varphi(\xi) = u \varphi(\xi).$$

THEOREM 2³⁾. *The following three conditions are equivalent;*

- i) $F(\xi; z)$ is a covariant of index u ,
- ii) $F(\xi; z) = \exp(z\Delta)\varphi(\xi)$ with a semi-invariant of $\varphi(\xi)$ index u , where

$$\Delta = \sum_{j=1}^n \sum_{l=0}^{\infty} (w_j - l) \xi_j^{(l+1)} \frac{\partial}{\partial \xi_j^{(l)}},$$

iii) $F(\xi; z) = \varphi\left(\dots, \frac{(d/dz)^l f_j(\xi_j | z)}{w_j(w_j - 1) \cdots (w_j - l + 1)}, \dots\right)$ with a semi-invariant $\varphi(\dots, \xi_j^{(l)}, \dots)$ of index u .

THEOREM 3⁴⁾.

3) See Theorem 5.1 (Robert's Theorem), Chap. 5 [1], or Theorem 1.3, Chap 1 [2].

4) See Theorem 6.5, Chap. 6 [1], or Theorem 2.6, Chap 2 [2].

$$\begin{aligned} & \{ \text{Differential invariants of } L_n(Q|z, y) \} \\ & = \{ \text{covariants of } \theta_3(z), \dots, \theta_n(z) \}, \end{aligned}$$

where $\theta_p(z) = \sum_{l=0}^{\infty} \binom{-2p}{l} \eta_p^{(l)} z^l$ and $(w_3, \dots, w_n) = (-6, -8, \dots, -2n)$.

3. Defining differential equations of moduli

Two projective holomorphic curves in P^{n-1}

$$C_\varphi: u \rightarrow (\varphi_1(u), \dots, \varphi_n(u))$$

and

$$C_\phi: z \rightarrow (\phi_1(z), \dots, \phi_n(z))$$

are called to be equivalent, if a projective transformation in P^{n-1} and a variable transformation $(u, y) \rightarrow (u(z), \lambda(z)y)$ map C_φ to C_ϕ .

By virtue of Forsyth's theorem equivalence classes of projective holomorphic curves correspond bijectively to equivalence classes of canonical forms with respect to transformations

$$(z, y) \rightarrow \left(\frac{\alpha z + \beta}{\gamma z + \delta}, \frac{cy}{(\gamma z + \delta)^{n-1}} \right).$$

Here we shall give the moduli spaces of projective holomorphic curves in P^{n-1} and stable projective holomorphic curves by means of non-linear differential equations. The answer is not so difficult, that is a consequence of properties of Schwarzian derivatives.

Schwarzian

$$\{z, \tau\} = \frac{\frac{d^3 z}{d\tau^3}}{\frac{d^2 z}{d\tau^2}} - \frac{3}{2} \left[\frac{\frac{d^2 z}{d\tau^2}}{\frac{dz}{d\tau}} \right]^2$$

is naturally generalized to pairs of differential forms as follows

$$\{(df)^m, dg\} = \{df, dg\} = \{f, g\}(dg)^2.$$

LEMMA 1. *If we put*

$$\chi_j(z) = \begin{vmatrix} \theta_j(z) & \frac{-1}{2j} \frac{d\theta_j(z)}{dz} \\ \frac{-1}{2j} \frac{d\theta_j(z)}{dz} & \frac{1}{2j(2j+1)} \frac{d^2\theta_j(z)}{dz^2} \end{vmatrix},$$

we have

$$(1) \quad \chi_j(z)(dz)^{2j+2} = \frac{(\theta_j(z)(dz)^j)^2}{2(2j+1)} \{\theta_j(z)(dz)^j, dz\}.$$

Proof. Putting $\theta_j(z)(dz)^j = (df)^j$, we have

$$\begin{aligned} \chi_j(z)(dz)^{2j+2} &= \left(\frac{(df/dz)^j}{2j(2j+1)} \left(\frac{d}{dz} \right)^2 \left(\frac{df}{dz} \right)^j - \frac{1}{(2j)^2} \left(\frac{d}{dz} \left(\frac{df}{dz} \right)^j \right)^2 \right) (dz)^{2j+2} \\ &= \frac{1}{2(2j+1)} \left(\frac{df}{dz} \right)^{2j} \{f, z\} dz^{2j+2}. \end{aligned}$$

LEMMA 2. Let $\theta_j(u)(du)^j, \chi_j(u)(du)^{2j+2}$ ($3 \leq j \leq n$) be holomorphic differential forms around origin. Then there exists a variable z such that

$$\chi_j(u)(du)^{2j+2} = \frac{(\theta_j(u)(du)^j)^2}{2(2j+1)} \{\theta_j(u)(du)^j, dz\} \quad (3 \leq j \leq n),$$

if and only if

$$(2) \quad \begin{aligned} &2(2j+1)\chi_j(u)\theta_k(u)^2 - 2(2k+1)\chi_k(u)\theta_j(u)^2 \\ &= \frac{1}{j}\theta_j(u)\theta_k(u)^2 \frac{d^2\theta_j(u)}{du^2} - \frac{1}{k}\theta_k(u)\theta_j(u)^2 \frac{d^2\theta_k(u)}{du^2} \\ &\quad - \frac{2j+1}{2j^2}\theta_k(u)^2 \left(\frac{d\theta_j(u)}{du} \right)^2 + \frac{2k+1}{2k^2}\theta_j(u)^2 \left(\frac{d\theta_k(u)}{du} \right)^2 \\ &\quad (3 \leq j < k \leq n). \end{aligned}$$

If $(\theta_3(u)(du)^3, \dots, \theta_n(u)(du)^n) \neq (0, \dots, 0)$, then the variable z is uniquely determined up to Möbius transformations.

Proof. If $\theta_j(u), \theta_k(u) \not\equiv 0$, then from properties of Schwarzian it follows.

$$\begin{aligned} &\{\theta_j(u)(du)^j, dz\} - \{\theta_k(u)(du)^k, dz\} \\ &= \{\theta_j(u)(du)^j, du\} + \{du, dz\} - \{\theta_k(u)(du)^k, du\} - \{du, dz\} \\ &= \{\theta_j(u)(du)^j, du\} - \{\theta_k(u)(du)^k, du\} \\ &= \left(\frac{1}{j} \frac{d^2\theta_j(u)}{d^2u} \theta_j(u)^{-1} - \frac{2j+1}{2j^2} \left(\frac{d\theta_j(u)}{du} \right)^2 \theta_j(u)^{-2} \right) (du)^2 \\ &\quad - \left(\frac{1}{k} \frac{d^2\theta_k(u)}{d^2u} \theta_k(u)^{-1} - \frac{2k+1}{2k^2} \left(\frac{d\theta_k(u)}{du} \right)^2 \theta_k(u)^{-2} \right) (du)^2. \end{aligned}$$

For a fixed j the equation on z

$$\{\theta_j(u)(du)^j, du\} - 2(2j + 1)\theta_j(u)^{-2}\chi_j(u)(du)^2 = \{dz, du\}$$

is solvable, and

$$\{\theta_j(u)(du)^j, dz\} = 2(2j + 1)\theta_j(u)^{-2}\chi_j(u)(du)^2 .$$

Hence we can choose a variable z such that

$$\chi_j(u)(du)^{2j+2} = \frac{(\theta_j(u)(du)^j)^2}{2(2j + 1)}\{\theta_j(u)(du)^j, dz\}$$

if and only if (2). Assume that $\theta_j(u)du \neq 0$. Then

$$\{\theta_j(u)(du)^j, dz\} - \{\theta_j(u)(du)^j, dz'\} = \{dz, dz'\} = 0$$

and thus z is uniquely determined by $\{\theta_j(u)(du)^j, dz\}$ up to Möbius transformations.

Now we have the next theorem:

THEOREM 4. *Equivalence classes of holomorphic projective curves in P^{n-1} correspond bijectively to system of holomorphic differential forms*

$$(\theta_3(u)(du)^3, \dots, \theta_n(u)(du)^n, \chi_3(u)(du)^3, \dots, \chi_n(u)(du)^{2n+2})$$

such that

$$\begin{aligned} & 2(2j + 1)\chi_j(u)\theta_k(u)^2 - 2(2k + 1)\chi_k(u)\theta_j(u)^2 \\ (3) \quad & = \frac{1}{j} \theta_j(u)\theta_k(u)^2 \frac{d^2\theta_3(u)}{du^2} - \frac{1}{k} \theta_k(u)\theta_j(u)^2 \frac{d^2\theta_k(u)}{du^2} \\ & - \frac{2j + 1}{2j^2} \theta_k(u)^2 \frac{d\theta_j(u)^2}{du} + \frac{2k + 1}{2k^2} \theta_j(u)^2 \left(\frac{d_k(u)}{du}\right)^2 \\ & \hspace{15em} (3 \leq j < k \leq n) . \end{aligned}$$

Let $L_n(Q|z, y)$ be a canonical form associating with a holomorphic projective curve C , and $(\theta_3(z), \dots, \theta_n(z))$ be the system of fundamental differential invariants of $L_n(Q|z, y)$. Then the bijective correspondence is given by

$$(4) \quad \begin{aligned} & \theta_j(u)(du)^j = \theta_j(z)(dz)^j \\ & \chi_j(u)(du)^{2j+2} = \frac{(\theta_j(z)(dz)^j)^2}{2(2j + 1)}\{\theta_j(z)(dz)^j, dz\} \quad (3 \leq j \leq n) \end{aligned}$$

Proof. It follows from the following equivalence

$$L_n(Q|z, y) \leftrightarrow (\theta_3(z), \dots, \theta_n(z)) \leftrightarrow (\theta_3(z)(dz)^3, \dots, \theta_n(z)(dz)^n, z) .$$

THEOREM 5. Denote

$$\theta_{jh}(z) = \begin{vmatrix} \theta_j(z) & \theta_h(z) \\ -\frac{1}{2j} \frac{d}{dz} \theta_j(z) & -\frac{1}{2h} \frac{d}{dz} \theta_h(z) \end{vmatrix} \quad (3 \leq j < h \leq n),$$

$$\chi_{jk}(z) = \begin{vmatrix} \chi_j(z) & \theta_k(z) \\ -\frac{1}{4j+4} \frac{d}{dz} \chi_j(z) & -\frac{1}{2k} \frac{d}{dz} \theta_k(z) \end{vmatrix} \quad (3 \leq j, k \leq n).$$

Let \tilde{C} be the portrait (equivalence class with respect to independent variable transformations) of the curve

$$z \mapsto (\dots, \theta_j(z), \dots, \theta_{jk}(z), \dots, \chi_j(z), \dots, \chi_{jk}(z), \dots)$$

in weighted projective space with weight system

$$(\dots, j, \dots, j + k + 1, \dots, 2j + 2, \dots, 2j + k + 3, \dots).$$

Then, if $(\dots, \theta_{jh}, \dots, \chi_{jk}, \dots) \neq (0, \dots, 0)$, the correspondence:

$$[\text{equivalence class of } C] \mapsto \tilde{C}$$

is bijective. If $(\dots, \theta_{jh}, \dots, \chi_{jk}, \dots) \equiv (0, \dots, 0)$, then \tilde{C} is a point and C is equivalent to a curve

$$(z^{\lambda_1}, z^{\lambda_1} \log z, \dots, z^{\lambda_1} (\log z)^{m_1}, \dots, z^{\lambda_r}, z^{\lambda_r} \log z, \dots, z^{\lambda_r} (\log z)^{m_r})$$

with $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ and $\sum m_i = n$.

Proof. Let $(\theta'_s(z'), \dots, \theta'_n(z'))$ be the system of fundamental differential invariants of a canonical form $L_n(Q'|z', y)$ corresponding to the same curve \tilde{C} in weighted projective space. Then there exists $\mu(z) \neq 0$ such that

$$\begin{aligned} \theta'_j(z') &= \mu(z)^j \theta_j(z), \\ \theta'_{jk}(z') &= \mu(z)^{j+k+1} \theta_{jk}(z), \\ \chi'_j(z') &= \mu(z)^{2j+2} \chi_j(z), \\ \chi'_{jk}(z') &= \mu(z)^{2j+k+3} \chi_{jk}(z). \end{aligned}$$

On the other hand

$$\mu(z)^{j+k+1} \theta_{jk}(z) = \theta'_{jk}(z') = \begin{vmatrix} \theta'_j(z') & \theta'_k(z') \\ -\frac{1}{2j} \frac{d}{dz'} \theta'_j(z') & -\frac{1}{2k} \frac{d}{dz'} \theta'_k(z') \end{vmatrix}$$

$$\begin{aligned} &= \mu(z)^{j+k} \frac{dz}{dz'} \theta_{jk}(z), \\ \mu(z)^{2j+k+3} \chi_{jk}(z) = \chi'_{jk}(z') &= \begin{vmatrix} \chi'_j(z) & \theta'_k(z') \\ -1 & \frac{d}{dz'} \chi'_j(z') \end{vmatrix} \begin{vmatrix} \theta'_k(z') \\ -1 & \frac{d}{dz'} \theta'_k(z') \end{vmatrix} \\ &= \mu(z)^{2j+k+2} \frac{dz}{dz'} \chi_{jk}(z). \end{aligned}$$

If $(\dots \theta_{jk}, \dots, \chi_{jk} \dots) \neq (0, \dots, 0)$ then we have

$$\mu(z) \frac{dz'}{dz} \equiv 1$$

and

$$\begin{aligned} \theta'_j(z')(dz')^j &= \theta_j(z)(\mu(z)dz')^j = \theta_j(z)(dz)^j, \\ \chi'_j(z')(dz')^{2j+2} &= \chi_j(z)(\mu(z)dz')^{2j+2} = \chi_j(z)(dz)^{2j+2}. \end{aligned}$$

By virtue of (4) this shows

$$\{\theta_j(z)(dz)^j, dz\} = \{\theta_j(z)(dz)^j, dz'\} \quad (3 \leq j \leq n),$$

and thus z' is a Möbius transformation of z . Namely $L_n(Q'|z', y)$ is equivalent to $L_n(Q|z, y)$. Assume that $(\dots, \theta_{jk}, \dots, \chi_{j,k}, \dots) = (0, \dots, 0)$. Then by virtue of Lemma 2 there exist a constant c and a function $u(z)$ such that

$$\begin{aligned} \theta_j(z) &= c_j \left(\frac{du}{dz} \right)^j, \\ \chi_j(z) &= \frac{c_j^2 (du/dz)^{2j}}{2(2j+1)} \{u, z\} = \frac{-c_j^2 (du/dz)^{2j}}{2(2j+1)} \{z, u\}, \\ \chi_{jk}(z) &= \begin{vmatrix} \chi_j & \theta_k \\ -1 & \frac{d\chi_j}{dz} \end{vmatrix} \begin{vmatrix} \theta_k \\ -1 & \frac{d\theta_k}{dz} \end{vmatrix} \\ &= \frac{-c_j^2 (du/dz)^{2j+2}}{4(2j+1)(2j+2)} \frac{d}{dz} \{z, u\} = 0. \end{aligned}$$

This shows $\{z, u\} = -\frac{\alpha}{2}$ with α in C , and thus

$$\begin{aligned} &(\dots \theta_j(z), \dots, \theta_{jk}(z), \dots, \chi_j(z), \dots, \chi_{jk}(z), \dots) \\ &= \left(\dots, c_j, \dots, 0 \dots, \frac{-c_j^2 \alpha}{4(2j+1)}, \dots 0 \dots 0 \right). \end{aligned}$$

Hence \tilde{C} is a point. We may assume $z = e^{\alpha u}$ within Möbius transformation. This means

$$\frac{du}{\alpha z} = \frac{1}{\alpha z}, \quad \theta_j(z) = c_j(\alpha z)^{-j}$$

$$z^{-n}L_n(Q|z, y) = \sum \binom{n}{l} \gamma_l \left(\frac{1}{z} \frac{d}{dz}\right)^{n-l} y$$

with $\gamma_3, \dots, \gamma_n$ in C . The fundamental solution of this type of linear differential operator is given by

$$(z^{i_1}, z^{i_1} \log z, \dots, z^{i_1} (\log z)^{m_1}, \dots, z^{i_r}, z^{i_r} \log z, \dots, z^{i_r} (\log z)^{m_r}).$$

4. Coordinate-free formulation

We shall reformulate the above results in terms of bundles. Let u be an independent variable, and let

$$d^l u \quad (l = 1, 2, 3, \dots)$$

be independent variables over ring $C\{u\}$ of convergent power series in u . A derivation d is defined in polynomial algebra

$$C\{u\}[\dots, d^l u, \dots]$$

as follows

$$df = \frac{df}{du} du \quad (f \in C\{u\}),$$

$$d(d^l u) = d^{l+1} u \quad (l = 1, 2, 3, \dots).$$

Wronskian of $(\varphi_1(u), \dots, \varphi_n(u))$ is defined by

$$(3) \quad W_{(\varphi_1, \dots, \varphi_n)} = \begin{vmatrix} \varphi_1 & \dots & \varphi_n \\ d\varphi_1 & \dots & d\varphi_n \\ \vdots & & \vdots \\ d^{n-1}\varphi_1 & \dots & d^{n-1}\varphi_n \end{vmatrix}.$$

The relation between this Wronskian and the usual one is given by

$$(4) \quad W_{(\varphi_1, \dots, \varphi_n)} = \begin{vmatrix} \varphi_1, & \dots, & \varphi_n \\ \frac{d}{du}\varphi_1, & \dots, & \frac{d}{du}\varphi_n \\ \vdots & & \vdots \\ \left(\frac{d}{du}\right)^{n-1}\varphi_1, & \dots, & \left(\frac{d}{du}\right)^{n-1}\varphi_n \end{vmatrix} (du)^{n(n-1)/2}.$$

We denote

$$\begin{aligned} \tilde{L}_n(\tilde{p}|y) &= d^n y + \sum_{l=1}^n \binom{n}{l} \tilde{p}_l(\dots, d^n \varphi_g, \dots) d^{n-l} y \\ &= (-1)^n W_{(\varphi_1, \dots, \varphi_n)}^{-1} \begin{vmatrix} y, & \varphi_1, & \dots, & \varphi_n \\ dy, & d\varphi_1, & \dots, & d\varphi_n \\ \vdots & \vdots & & \vdots \\ d^n y, & d^n \varphi_1, & \dots, & d^n \varphi_n \end{vmatrix}, \end{aligned}$$

then

$$L_n(\tilde{p}|y) = L_n(p|u, y)(du)^n,$$

$\tilde{L}_n(\tilde{p}|y) (L_n(p|u, y))$ is called to be semi-canonical if $\tilde{p}_1 \equiv 0$. $\tilde{L}_n(\tilde{p}|u, y)$ is semi-canonical if and only if

$$W_{(\varphi_1, \dots, \varphi_n)} \equiv \gamma(du)^n$$

with $\gamma \neq 0$ in \mathbb{C} .

Changing the dependent variable

$$y \mapsto \lambda(u)y$$

with a suitable $\lambda(u)$ ($\lambda(0) \neq 0$), we may transform any $\tilde{L}_n(\tilde{p}|u, y)$ to a unique semi-canonical form.

A differential invariant of weight p of $\tilde{L}_n(\tilde{p}|u, y)$ is defined by

$$\Phi \left(\dots, \left(\frac{d}{du}\right)^i p_j, \dots \right) (du)^m$$

with a differential invariant of weight m

$$\Phi \left(\dots, \left(\frac{d}{du}\right)^i p_j, \dots \right).$$

For a semi-canonical form $\tilde{L}_n(\tilde{p}|y)$ the set of differential invariants of weight m coincides with the set of differential polynomial with coeffi-

cients in C

$$\Phi(\dots, d^l \tilde{p}_j, \dots)$$

such that

$$\Phi(\dots, d^l \tilde{p}_j, \dots) = \Phi\left(\dots, \left(\frac{d}{du}\right)^l p_j, \dots\right)(du)^m.$$

For each differential invariant Φ of weight m of $L_n(p|u, y)$ we denote

$$\tilde{\Phi} = \Phi(\dots, d^l \tilde{p}_j, \dots) = \Phi\left(\dots, \left(\frac{d}{du}\right)^l p_j, \dots\right)(du)^m,$$

then we get the system of fundamental differential invariants

$$(\tilde{\theta}_3, \dots, \tilde{\theta}_n)$$

of $\tilde{L}_n(\tilde{p}|y)$. Moreover the system of differential invariants

$$(\tilde{\theta}_3, \dots, \tilde{\theta}_n, \dots, \tilde{\theta}_{jk}, \dots, \tilde{\chi}_j, \dots, \tilde{\chi}_{jk}, \dots)$$

corresponding to

$$(\theta_3, \dots, \theta_n, \dots, \theta_{jk}, \dots, \chi_j, \dots, \chi_{jk}, \dots).$$

5. Several variable case

Let u_1, \dots, u_r be independent variables and let

$$d^l u_j \quad (l = 1, 2, 3, \dots; j = 1, 2, \dots, r)$$

be independent variables over ring of convergent power series $C\{u_1, \dots, u_r\}$.

We define a derivation d on commutative polynomial algebra

$$C\{u_1, \dots, u_r\}[\dots, d^l u_j, \dots]$$

over $C\{u_1, \dots, u_r\}$ as follows

$$df = \sum_{j=1}^r \frac{\partial f}{\partial u_j} du_j \quad (f \in C\{u_1, \dots, u_r\}),$$

$$d(d^l u_j) = d^{l+1} u_j \quad (l = 1, 2, 3, \dots; j = 1, 2, \dots, r).$$

Remark. For any holomorphic curve

$$t \mapsto (u_1(t), \dots, u_r(t))$$

we can associate a differential algebra homomorphism

$$\mathcal{C}\{u_1, \dots, u_r\}[\dots, d^l u_j, \dots] \rightarrow \mathcal{C}\{t\}[\dots, d^l t, \dots]$$

such that

$$d^l \varphi(u_1, \dots, u_r) \mapsto d^l \varphi(u_1(t), \dots, u_r(t)).$$

For a system $(\varphi_1(u_1, \dots, u_r), \dots, \varphi_n(u_1, \dots, u_r))$ Wronskian of $(\varphi_1, \dots, \varphi_n)$ is defined by

$$(5) \quad W_{(\varphi_1, \dots, \varphi_n)} = \begin{vmatrix} \varphi_1 & \dots & \varphi_n \\ d\varphi_1 & \dots & d\varphi_n \\ \vdots & & \vdots \\ d^{n-1}\varphi_1 & \dots & d^{n-1}\varphi_n \end{vmatrix}.$$

We denote

$$(6) \quad \begin{aligned} \tilde{L}_n(\tilde{p}|y) &= (-1)^n W_{(\varphi_1, \dots, \varphi_n)}^{-1} \begin{vmatrix} y & \varphi_1 & \dots & \varphi_n \\ dy & d\varphi_1 & \dots & d\varphi_n \\ \vdots & \vdots & & \vdots \\ d^n y & d^n \varphi_1 & \dots & d^n \varphi_n \end{vmatrix} \\ &= d^n y + \sum_{l=1}^n \binom{n}{l} \tilde{p}_l(\dots, d^l \varphi, \dots) d^{n-l} y. \end{aligned}$$

We may define differential invariants of weight p of $\tilde{L}_n(p|y)$ by the same differential polynomials as differential invariants of one variable case.

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