Canad. J. Math. Vol. **54** (5), 2002 pp. 1086–1099

Combinatorics of the Heat Trace on Spheres

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Abstract. We present a concise explicit expression for the heat trace coefficients of spheres. Our formulas yield certain combinatorial identities which are proved following ideas of D. Zeilberger. In particular, these identities allow to recover in a surprising way some known formulas for the heat trace asymptotics. Our approach is based on a method for computation of heat invariants developed in [P].

1 Introduction and Main Results

1.1 Heat Trace Asymptotics on Spheres

Let S^d be a sphere with the standard Riemannian metric of curvature +1. The Laplace-Beltrami operator Δ on S^d has eigenvalues $\lambda_{k,d} = k(k+d-1)$, and each $\lambda_{k,d}$ has multiplicity $\mu_{k,d}$ given by

$$\mu_{k,d} = \frac{(2k+d-1)(k+d-2)!}{k!\,(d-1)!}, \quad k \geq 1 \text{ and } \mu_{0,d} = 1,$$

(see, for example, [Mü]). Consider an asymptotic expansion for the trace of the heat operator $e^{-t\Delta}$ as $t \to 0+$ (see [MP], [Se], [Be], [Gi]):

$$\sum_{\lambda} e^{-t\lambda} = \sum_{k=0}^{\infty} \mu_{k,d} e^{-t\lambda_{k,d}} \sim \sum_{n=0}^{\infty} a_{n,d} t^{n-\frac{d}{2}}. \tag{1.1.1}$$

Heat trace coefficients (or heat invariants) $a_{n,d}$ were calculated in [CW] (see (1.3.2) and (1.4.2) for similar formulas) by methods of Lie groups and representation theory (see also [Ca], [ELV], [DK] for related results). In this paper we present a different approach based on [P]. We obtain the following concise explicit expression for $a_{n,d}$.

Theorem 1.1.2 *For any $n \geq 1$ and any integer $\omega \geq 2n$ the heat invariants $a_{n,d}$ are equal to*

$$a_{n,d} = \sum_{j=1}^{\omega} \frac{2(-1)^n \Gamma(\omega + \frac{d}{2} + 1)}{(\omega - j)!\,(j+n)!\,(2j+d)!} \sum_{k=1}^{j} (-1)^k \binom{2j+d-1}{j-k} \mu_{k,d} \lambda_{k,d}^{j+n}. \tag{1.1.3}$$

Received by the editors November 30, 2001.
Supported by CRM-ISM and MSRI postdoctoral fellowships.
AMS subject classification: 05A19, 58J35.

There is some delicacy in the proof of Theorem 1.1.2. For $\omega \geq 3n$ it follows from a simple generalization of the main result of [P] and some facts about Legendre polynomials (see Sections 2.1 and 2.2). Theorem 1.1.2 for $2n \leq \omega < 3n$ follows from the proofs of Theorems 1.3.1 and 1.4.1 involving rather sophisticated combinatorial arguments due to Doron Zeilberger (see below).

Validity of formula (1.1.3) for $3n > \omega \geq 2n$ was suggested by computer experiments using [Wo]. Note that $2n$ is "sharp" in a sense that if $\omega < 2n$ then (1.1.3) is no longer true (see Section 3.1).

1.2 Combinatorial Identities

Taking $d = 1$ in (1.1.3) we should get zero since the heat trace coefficients $a_{n,1}$ of a circle S^1 vanish identically for $n \geq 1$. This gives rise to a surprising combinatorial identity:

Theorem 1.2.1 (S^1**-identity (D. Zeilberger, [Z]))**

$$\sum_{j=0}^{\omega} \frac{1}{(\omega - j)!\,(j+n)!\,(2j+1)} \sum_{k=0}^{j} \frac{(-1)^k k^{2j+2n}}{(j-k)!\,(j+k)!} = 0$$

for $n \geq 1$, $\omega \geq 2n$.

Theorem 1.2.1 was proved in [Z] (see also Section 3.1) by pure combinatorial methods.

Similarly, taking into account that

$$a_{n,3} = \frac{\sqrt{\pi}}{4 \cdot n!} \quad (\textit{cf.}\ [\text{MS}], [\text{CW}]),$$

we get:

Theorem 1.2.2 (S^3**-identity)**

$$\sum_{j=0}^{\omega} \frac{\Gamma(\omega + 5/2)}{(\omega - j)!\,(j+n)!\,(2j+3)} \sum_{l=0}^{j+1} \frac{(-1)^l l^2 (l^2 - 1)^{j+n}}{(j+l+1)!\,(j-l+1)!} = \frac{(-1)^{n+1}\sqrt{\pi}}{8 \cdot n!},$$

for $n \geq 1$, $\omega \geq 2n$.

A combinatorial proof of this theorem based on a generalization of Zeilberger's arguments is given in Section 3.2.

Interestingly enough, pushing forward this combinatorial approach one recovers the results of [CW] from Theorem 1.1.2 for $\omega \geq 3n$. We present them in a more concise form especially in some particular cases (see (1.3.4), (1.4.4)).

1.3 Odd-Dimensional Case

In odd dimensions formula (1.1.3) can be substantially simplified.

Theorem 1.3.1 *The heat invariants of odd-dimensional spheres $S^{2\alpha+1}$ are equal to*

$$a_{n,2\alpha+1} = \sum_{s=1}^{\alpha} \frac{\alpha^{2n-2\alpha+2s}\Gamma(s+\frac{1}{2})K_s^\alpha}{(n-\alpha+s)!\,(2\alpha)!}, \tag{1.3.2}$$

where the coefficients K_s^α are defined by

$$\prod_{\beta=0}^{\alpha-1}(z^2-\beta^2) = \sum_{s=1}^{\alpha} K_s^\alpha z^{2s}. \tag{1.3.3}$$

In particular,

$$a_{n,5} = \frac{4^{n-3}(6-n)\sqrt{\pi}}{3\cdot n!}, \quad a_{n,7} = \frac{3^{2n-6}(16n^2-286n+1215)\sqrt{\pi}}{640\cdot n!}. \tag{1.3.4}$$

1.4 Even-Dimensional Case

Formulas for even-dimensional spheres have a more intricate combinatorial structure due to a certain hypergeometric expression vanishing only for d odd (see Section 4.2).

Theorem 1.4.1 *The heat invariants of even-dimensional spheres $S^{2\nu}$ are equal to*

(1.4.2)

$$a_{n,2\nu} = \frac{1}{(2\nu-1)!}\Bigg(\sum_{t=0}^{\nu-1}\frac{(\nu-1-t)!}{(n-t)!}\left(\nu-\frac{1}{2}\right)^{2n-2t}K_t^\nu + \sum_{t=0}^{\nu-1}K_t^\nu\sum_{p=\nu-t}^{n-t}(-1)^{p+\nu-t-1}\frac{(\nu-\frac{1}{2})^{2n-2t-2p}B_{2p}}{p(n-t-p)!\,(p-\nu+t)!}\left(\frac{1}{2^{2p-1}}-1\right)\Bigg)$$

where B_{2p} are the Bernoulli numbers (see [GKP]) and the constants K_t^ν are defined by

$$\prod_{\beta=1/2}^{\nu-3/2}(z^2-\beta^2) = \sum_{t=0}^{\nu-1} K_t^\nu z^{2\nu-2-2t}. \tag{1.4.3}$$

In particular (*cf.* [Ca]),

$$a_{n,2} = \frac{1}{n!\,2^{2n}}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(2-2^{2r})B_{2r}. \tag{1.4.4}$$

Note that the second sum in (1.4.2) vanishes for $\nu > n$.

https://doi.org/10.4153/CJM-2002-040-4 Published online by Cambridge University Press

1.5 Structure of the Paper

In Section 2.1 we present a generalization of the main result of [P] which allows to prove Theorem 1.1.2 for $\omega \geq 3n$ using some properties of Legendre polynomials, see Section 2.2. In Section 3.1 we review Zeilberger's proof of Theorem 1.2.1 which leads to the proof of Theorem 1.2.2 in Section 3.2. Theorems 1.3.1 and 1.4.1 are proved in Sections 4.1 and 4.2 using Theorem 1.1.2 for $\omega \geq 3n$. Theorem 1.1.2 for $3n > \omega \geq 2n$ follows from Theorems 1.3.1 and 1.4.1 by taking the arguments in their proofs in the reverse order, see Section 4.3. Two auxiliary combinatorial lemmas are proved in Sections 5.1 and 5.2.

Acknowledgments I am very grateful to Doron Zeilberger for helpful advice concerning combinatorial identities proved in this paper, and especially for his proof of Theorem 1.2.1. I would like to thank Dmitry Jakobson, Yakar Kannai, Leonid Polterovich, Andrei Reznikov and Joseph Wolf for stimulating discussions. The author is also indebted to Klaus Kirsten and Francois Lalonde for useful remarks on the first draft of this paper.

This research was partially conducted during my stay at the Mathematical Sciences Research Institute in Berkeley whose hospitality and support are gratefully acknowledged.

2 Heat Invariants and Spherical Harmonics

2.1 Computation of Heat Invariants

For any d-dimensional closed Riemannian manifold M the coefficients $a_{n,d}$ can be obtained from the local heat invariants $a_{n,d}(x)$ (see [Gi]):

$$a_{n,d} = \int_M a_{n,d}(x)\, d\,\mathrm{vol}(x).$$

In particular, if $M = S^d$ the coefficients $a_{n,d}(x)$ are constants and therefore for any $x \in S^d$

$$a_{n,d} = \mathrm{vol}(S^d) a_{n,d}(x), \tag{2.1.1}$$

where the volume of a d-sphere is given by (see [Mü]):

$$\mathrm{vol}(S^d) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}. \tag{2.1.2}$$

Let us prove the following modification of the main result of [P]:

Theorem 2.1.3 *For any integer $\omega \geq 3n$ the local heat invariants $a_{n,d}(x)$ of a d-dimensional closed Riemannian manifold M are equal to:*

(2.1.4)

$$a_{n,d}(x) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{\omega} \binom{\omega + \frac{d}{2}}{j + \frac{d}{2}} \frac{1}{4^j\, j!\, (j+n)!} \Delta^{j+n}\Big(f\big(r_x(y)^2\big)^{\,j}\Big)\Big|_{y=x},$$

where $f(r_x^2)$ is a smooth function in some neighborhood of $x \in M$, $f(s) = s + O(s^2)$, $f\colon [0,\varepsilon] \to [0,\varepsilon]$.

Proof The result follows from Theorem 1.2.1 in [P]) (if $f(r_x^2) = r_x^2$ we get precisely the statement of that theorem). Indeed, let $(y_1, \dots, y_d)$ be normal coordinates in a neighborhood of the point $x = (0, \dots, 0) \in M$. The Riemannian metric at the point x has the form $ds^2 = dy_1^2 + \cdots + dy_d^2$ and the square of the distance function is locally given by

$$r_x(y)^2 = y_1^2 + \cdots + y_d^2, \tag{2.1.5}$$

where $y = (y_1, \dots, y_d)$. Let us note that the point $x \in M$ is a non-degenerate critical point of index 0 of the function f and hence due to Morse lemma [Mi] the function $f\bigl(r_x^2(y)\bigr)$ can be locally written as the sum of squares (2.1.5) in some new coordinate system $(y_1', \dots, y_d')$. Moreover, this new system can be chosen in such a way that $y_1' = y_1 + O(|y|^2), \dots, y_d' = y_d + O(|y|^2)$, and hence the Riemannian metric remains Euclidean at the point x. Repeating the proof of Theorem 1.2.1 in [P] with the coordinates $(y_1', \dots, y_d')$ taken instead of normal coordinates we complete the proof of (2.1.4). ■

Remark As was recently observed in [We], for $f(r^2) = r^2$ one could in fact take $\omega \geq n$ in (2.1.4).

2.2 Application of Legendre Polynomials

Recall that the Laplacian on S^d has eigenvalues $\lambda_{k,d} = k(k + d - 1)$ and the corresponding eigenfunctions are the Legendre polynomials $L_{k,d}(\cos r)$ (see [Mü]):

$$\Delta L_{k,d} = \lambda_{k,d} L_{k,d} = k(k + d - 1) L_{k,d}. \tag{2.2.1}$$

Proof of Theorem 1.1.2 for $\omega \geq 3n$ Take $f(r^2) = 2 - 2\cos(r) = r^2 + O(r^4)$ as the function f in Theorem 2.1.3. We express its powers in terms of the Legendre polynomials $L_{k,d}(\cos r)$. Denote $t = \cos r$. Then $f(r^2)^j = 2^j (1 - t)^j$. Let

$$f(r^2) = 2^j (1 - t)^j = 2^j \sum_{k=0}^{j} c_{jk} L_{k,d}(t). \tag{2.2.2}$$

Since Legendre polynomials are orthogonal with weight $(1 - t^2)^{\frac{d-2}{2}}$ we have

$$c_{jk} = \frac{\int_{-1}^{1} (1 - t)^j L_{k,d}(t) (1 - t^2)^{\frac{d-2}{2}}\, dt}{\int_{-1}^{1} L_{k,d}(t)^2 (1 - t^2)^{\frac{d-2}{2}}\, dt}. \tag{2.2.3}$$

The denominator of (2.2.3) is equal to (see [Mü]):

$$\frac{\operatorname{vol}(S^d)}{\operatorname{vol}(S^{d-1}) \mu_{k,d}} = \frac{\Gamma(\frac{d}{2}) \sqrt{\pi}}{\Gamma(\frac{d+1}{2}) \mu_{k,d}},$$

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where the last equality follows from (2.1.2). The numerator of (2.2.3) is computed using the Rodrigues rule [Mü] and the following integral (see [Er]):

$$\int_{-1}^{1}(1+t)^{\frac{d}{2}+k-1}(1-t)^{\frac{d}{2}+j-1}=\frac{2^{k+j+d-1}\Gamma(\frac{d}{2}+k)\Gamma(\frac{d}{2}+j)}{\Gamma(k+j+d)},$$

Finally we get:

$$c_{jk}=\frac{(-1)^k 2^j\Gamma(j+\frac{d}{2})\,j!}{(j-k)!\,(j+k+d-1)!}\frac{(4\pi)^{d/2}\mu_{k,d}}{\mathrm{vol}(S^d)}.$$

Let us substitute this into (2.2.2) and further on into (2.1.4). Note that $L_{k,d}(\cos 0)=L_{k,d}(1)=1$ for all k (see [Mü]). Taking into account (2.2.1) and (2.1.1) we obtain (1.1.3) after some easy combinatorial transformations. This completes the proof of Theorem 1.1.2 for $\omega\geq 3n$. ■

As we mentioned in Section 1.1, it follows from the proof of Theorems 1.3.1 and 1.4.1 that in fact one can take $\omega\geq 2n$ (see Section 4.3).

3 Proofs of the Identities

3.1 Proof of Theorem 1.2.1

In this section we follow [Z]. We will prove a more general statement:

$$\sum_{j=0}^{\omega}\frac{1}{(\omega-j)!\,(j+n)!\,(2j+1)}\sum_{k=-j}^{j}\frac{(-1)^k(x+k)^{2j+2n}}{(j-k)!\,(j+k)!}=0, \tag{3.1.1}$$

for $x\in\mathbb{R}$ and $\omega\geq 2n$. If $x=0$ we get the original S^1-identity. Note that we have symmetrized the summation limits in the inner sum—this is equivalent to multiplying the left-hand side by factor 2. Our aim is to make (3.1.1) hypergeometric, *i.e.*, to represent it as a function

$${}_2F_1(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}, \tag{3.1.2}$$

where $(t)_m=t(t+1)\cdots(t+m-1)$, $(t)_0=1$. Let $Ef(x)=f(x+1)$ be the shift operator. Then we can rewrite (3.1.1) as

$$\begin{aligned}&\sum_{j=0}^{\omega}\frac{(-1)^j}{(\omega-j)!\,(j+n)!\,(2j+1)!}\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}E^{p-j}x^{2j+2n}\\&=\sum_{j=0}^{\omega}\frac{(-1)^j}{(\omega-j)!\,(j+n)!\,(2j+1)!}(E^{1/2}-E^{-1/2})^{2j}x^{2j+2n}.\end{aligned} \tag{3.1.3}$$

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Using Taylor theorem $E = e^D$ where D is the differentiation operator (see [GKP]) we have:

$$(E^{1/2} - E^{-1/2})^{2j} = (e^{D/2} - e^{-D/2})^{2j} = P(D)^{2j} D^{2j},$$

where

$$P(D) = \frac{2\sinh D/2}{D} = \frac{e^{D/2} - e^{-D/2}}{D} = 1 + \frac{D^2}{24} + O(D^4). \tag{3.1.4}$$

Substituting this into the sum and applying D^{2j} to x^{2j+2n} we get:

$$\begin{aligned}
&\sum_{j=0}^{\omega} \frac{(-1)^j (2j+2n)!\, P(D)^{2j} x^{2n}}{(\omega - j)!\,(j+n)!\,(2j+1)!\,(2n)!} \\
&\quad = \frac{1}{\omega!\, n!}\, {}_2F_1\big(n + 1/2, -\omega; 3/2; P(D)^2\big)\, x^{2n} \\
&\quad = \frac{1}{\omega!\, n!}\, {}_2F_1\big(1 - n, \omega + 3/2; 3/2; P(D)^2\big) \big(1 - P(D)^2\big)^{\omega - n + 1} x^{2n}.
\end{aligned} \tag{3.1.5}$$

The first equality is obtained by representing the sum as a hypergeometric series and the second equality follows from the Euler transformation (see [GKP]):

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b}\, {}_2F_1(c - a, c - b; c; z). \tag{3.1.6}$$

Note that on both sides we have in fact polynomials in D since $-\omega \le 0$ and $1 - n \le 0$ and therefore both hypergeometric series are finite (otherwise they would not be well defined).

On the other hand, due to (3.1.4) we have

$$\big(1 - P(D)^2\big)^{\omega - n + 1} = O(D^{2\omega - 2n + 2}),$$

and hence

$$\big(1 - P(D)^2\big)^{\omega - n + 1} x^{2n} = 0$$

for $\omega \ge 2n$. This completes the proof of the S^1-identity. ■

Note that for $\omega = 2n - 1$ the identity (1.2.1) does not hold (see [Z]) and hence $2n$ is "sharp" as was mentioned in Section 1.1.

3.2 Proof of Theorem 1.2.2

As in the previous section, we symmetrize the inner sumation indices and prove that

$$\sum_{j=0}^{\omega} \frac{(-1)^n \Gamma(\omega + 5/2)}{(\omega - j)!\,(j+n)!\,(2j+3)} \sum_{l=-j-1}^{j+1} (-1)^l \frac{l^2 (l^2 - 1)^{j+n}}{(j+l+1)!\,(j-l+1)!} = -\frac{\sqrt{\pi}}{4 \cdot n!},$$

for $n \ge 1, \omega \ge 2n$.

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We transform the inner sum:

$$\frac{1}{(2j+2)!}\sum_{l=-j-1}^{j+1}\binom{2j+2}{j-l+1}(l^2-1)^{j+n}l^2$$
$$=\frac{(-1)^{j+1}}{(2j+2)!}\sum_{p=0}^{2j+2}(-1)^p(p-j-1)!\binom{2j+2}{p}\big((p-j-1)^2-1\big)^{j+n}.$$

Let us substitute this to the initial expression changing the summation index $j\to j+1$. Denote $\omega'=\omega+1$, $n'=n-1$. We have

(3.2.1)
$$\sum_{j=0}^{\omega'}\frac{(-1)^{n'+1}\Gamma(\omega'+3/2)(-1)^j}{(\omega'-j)!\,(j+n')!\,(2j+1)!}\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}(p-j)^2\big((p-j)^2-1\big)^{j+n'}.$$

Let us open the last bracket. We get:

$$\sum_{r=0}^{j+n'}(-1)^r\binom{n'+j}{r}\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}(p-j)^{2j+2n'-2r+2}.$$

Note that (see (1.13) in [Go])

$$\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}(p-j)^s=0 \tag{3.2.2}$$

for $s<2j$ and

$$\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}(p-j)^{2j}=(2j)!\,. \tag{3.2.3}$$

Therefore non-zero contribution comes only from $2j+2n'-2r+2\geq 2j$, *i.e.*, $r\leq n'+1$. This implies that (3.2.1) can be rewritten as

(3.2.4)
$$(-1)^{n'+1}\Gamma(\omega'+3/2)\sum_{r=0}^{n'+1}(-1)^r\binom{n'+1}{r}$$
$$\cdot\sum_{j=1}^{\omega'}\frac{(-1)^j}{(\omega'-j)!\,(j+t-1)!\,(2j+1)!}\sum_{p=0}^{2j}(-1)^p\binom{2j}{p}(p-j)^{2j+2t},$$

where $t = n' - r + 1$. Consider the last two sums:

$$\sum_{j=1}^{\omega'} \frac{(-1)^j}{(\omega' - j)!\,(j + t - 1)!\,(2j + 1)!} \sum_{p=0}^{2j} (-1)^p \binom{2j}{p} (p - j)^{2j+2t}. \tag{3.2.5}$$

Let us show that (3.2.5) vanishes for $\omega' \geq 2t+1$ which is always the case since $\omega \geq 2n$ and $t \leq n' + 1 = n$). We use Lemma 5.1.1 (see Section 5.1) taking $s = 1$ in (5.1.3). Applying the same arguments as in the proof of Theorem 1.2.1 we get that (3.2.5) vanishes for $r < n' + 1$. Therefore the only non-zero contribution to (3.2.4) comes from $r = n' + 1$. Taking this into account and substituting (3.2.3) into (3.2.4) we finally obtain:

$$\frac{\Gamma(\omega' + 3/2)}{(n' + 1)!\,a'!} \sum_{j=1}^{\omega'} \frac{(-1)^j j}{2j + 1} \binom{\omega'}{j} = -\frac{\Gamma(\omega' + 3/2)\sqrt{\pi}}{4(n' + 1)!\,\Gamma(\omega' + 3/2)} = -\frac{\sqrt{\pi}}{4 \cdot n!}$$

which completes the proof of Theorem 1.2.2. ∎

4 Proofs of Theorems 1.3.1 and 1.4.1

4.1 Proof of Theorem 1.3.1

Denote $z = k + \alpha$. The inner sum in (1.1.3) is equal to:

$$\begin{aligned}
&\sum_{z=\alpha}^{j+\alpha} (-1)^{z+\alpha} \frac{(z + \alpha - 1)!\,2z(z^2 - \alpha^2)^{j+n}}{(z - \alpha)!\,(2\alpha)!} \binom{2j + 2\alpha}{j + \alpha + z} \\
&\quad = 2 \cdot \sum_{z=\alpha}^{j+\alpha} \frac{(-1)^{z+\alpha}}{(2\alpha)!} \prod_{\beta=0}^{\alpha-1} (z^2 - \beta^2)(z^2 - \alpha^2)^{j+n} \binom{2j + 2\alpha}{j + \alpha + z} \\
&\quad = \sum_{z=-j-\alpha}^{j+\alpha} \frac{(-1)^{z+\alpha}}{(2\alpha)!} \prod_{\beta=0}^{\alpha-1} (z^2 - \beta^2) \binom{2j + 2\alpha}{j + \alpha + z} (z^2 - \alpha^2)^{j+n}.
\end{aligned}$$

Denote $l = z + j + \alpha$. Then the last sum can be rewritten as

$$\frac{(-1)^j}{(2j + 2\alpha)!} \sum_{l=0}^{2j+2\alpha} (-1)^l \prod_{\beta=0}^{\alpha-1} \bigl((l - j - \alpha)^2 - \beta^2\bigr) \binom{2j + 2\alpha}{l} \bigl((l - j - \alpha)^2 - \alpha^2\bigr)^{j+n}. \tag{4.1.1}$$

Let $\omega' = \omega + \alpha$, $n' = n - \alpha$ and let $j := j + \alpha$ be the new summation index. Due to (4.1.1) we can represent (1.1.3) as:

$$\begin{aligned}
a_{n,2\alpha+1} = {} & \frac{2(-1)^{n'}\Gamma(\omega' + \frac{3}{2})}{(2\alpha)!} \sum_{s=1}^{\alpha} K_s^{\alpha} \cdot \sum_{j=0}^{\omega'} \frac{(-1)^j}{(\omega' - j)!\,(j + n' - r)!\,(2j + 1)!} \\
& \cdot \sum_{r=0}^{j+n'} \frac{(-1)^r \alpha^{2r}}{r!} \sum_{l=0}^{2j} (-1)^l \binom{2j}{l} (l - j)^{2j+2n'-2r+2s},
\end{aligned} \tag{4.1.2}$$

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where K_s^α are defined by (1.3.3). Note that if $2j + 2n' - 2r + 2s < 2j$ the last sum vanishes due to (3.2.2). Therefore if $r \le n' + s$ we can rewrite (4.1.2) as

$$
\begin{aligned}
a_{n,2\alpha+1} = {} & \frac{2(-1)^{n'}\Gamma(\omega' + \frac{3}{2})}{(2\alpha)!} \sum_{s=1}^{\alpha} K_s^\alpha \sum_{r=0}^{n'+s} \frac{(-1)^r \alpha^{2r}}{r!} \\
& \cdot \sum_{j=0}^{\omega'} \frac{(-1)^j}{(\omega' - j)!\,(j + n' - r)!\,(2j+1)!} \\
& \cdot \sum_{l=0}^{2j} (-1)^l \binom{2j}{l} (l - j)^{2j+2n'-2r+2s},
\end{aligned}
\tag{4.1.3}
$$

using the fact that $(j + n' - r)! = 0$ for $r > j + n'$. Let us note that Lemma 5.1.1 implies that the last two sums in (4.1.3) vanish if $r < n' + s$ and $\omega \ge 2n$. Indeed, this follows from (5.1.3) for $t = n' - r + s$ in the same way as vanishing of (3.2.5) in the proof of Theorem 1.2.2. Therefore the only non-zero contribution again comes only from $r = n' + s$ when the inner sum is equal to $(2j)!$ by (3.2.3). Hence we obtain:

$$
a_{n,2\alpha+1} = \frac{2\Gamma(\omega' + \frac{3}{2})}{(2\alpha)!} \sum_{s=1}^{\alpha} K_s^\alpha \frac{(-1)^s \alpha^{2n'+2s}}{(n'+s)!} \sum_{j=0}^{\omega'} \frac{(-1)^j}{(\omega' - j)!\,(j-s)!\,(2j+1)}.
\tag{4.1.4}
$$

Note that

$$
\begin{aligned}
& \sum_{j=0}^{\omega'} \frac{(-1)^j}{(\omega' - j)!\,(j-s)!\,(2j+1)} \\
& \quad = \frac{(-1)^s}{(\omega' - s)!} \sum_{j=0}^{\omega' - s} (-1)^j \binom{\omega' - s}{j} \frac{1}{2j + 2s + 1} \\
& \quad = \frac{(-1)^s}{(\omega' - s)!} \int_0^1 \left(\sum_{j=0}^{\omega' - s} (-1)^j \binom{\omega' - s}{j} x^{2j+2s} \right) dx \\
& \quad = \frac{(-1)^s}{(\omega' - s)!} \int_0^1 x^{2s} (1 - x^2)^{\omega' - s}\, dx = \frac{(-1)^s \Gamma(s + \frac{1}{2})}{2\Gamma(\omega' + 3/2)},
\end{aligned}
$$

where the last equality follows from [GR]. Substituting this into (4.1.4) after certain cancellations we obtain (1.3.2). In particular, taking $\alpha = 2$ and $\alpha = 3$ we get (1.3.4). The proof of Theorem 1.3.1 is complete. ∎

4.2 Proof of Theorem 1.4.1

The first steps of the proof are similar to that of Theorem 1.3.1. Let $n' = n - \nu + 1$, $\omega' = \omega + \nu - 1$ and let $j := j + \nu - 1$ be the new summation index. Similarly to

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(4.1.3) we obtain the following formula from (1.1.3):

$$
\begin{aligned}
a_{n,2\nu} = {} & \frac{2(-1)^{n'+1}(\omega'+1)!}{(2\nu-1)!} \sum_{s=0}^{\nu-1} K_s^{\nu} \sum_{r=0}^{n'+s} \frac{(-1)^r(\nu-\frac{1}{2})^{2r}}{r!} \\
& \cdot \sum_{j=0}^{\omega'} \frac{(-1)^j}{(\omega'-j)!\,(j+n'-r)!\,(2j+2)!} \\
& \cdot \sum_{l=0}^{2j+1} (-1)^l \binom{2j+1}{l} \Big(l-j-\frac{1}{2}\Big)^{2j+2n'-2r+2s+1}.
\end{aligned}
\tag{4.2.1}
$$

However, from this moment the situation is quite different. If in the proof of Theorem 1.3.1 only one term corresponding to $r = n' + s$ gave a non-zero contribution, now all terms with $n' - r + s \geq 0$ contribute to the sum. Indeed, repeating the arguments of Theorem 1.2.2 we get that the last two sums in (4.2.1) are equal to:

$$
\sum_{j=0}^{\omega'} \frac{(-1)^j(2j+2n'-2r+2s+1)!\,P^{2j+1}}{(\omega'-j)!\,(j+n'-r)!\,(2j+2)!\,(2n'-2r+2s)!} x^{2n'-2r+2s}\Big|_{x=0}
\tag{4.2.2}
$$

where P is given by (3.1.4). Setting $t = n' - r + s$ in (5.2.3) in Lemma 5.2.1 (see Section 5.2) we get that if $\omega \geq 2n$, (4.2.2) is equal to

$$
\frac{(n'-r)_s}{2(n'+s-r)!\,(\omega'+1)!} P^{-1}(x^{2n'-2r+2s})\Big|_{x=0}.
\tag{4.2.3}
$$

Let us compute $P^{-1}(x^{2t})|_{x=0}$. We have

$$
P^{-1} = \frac{D}{e^{-D/2} - e^{D/2}} = \sum_{i=0}^{\infty} P_{2i} D^{2i}
$$

and

$$
P^{-1}(x^{2t})|_{x=0} = (2t)!\,P_{2t}.
$$

Computing P_{2t} we get Bernoulli numbers. Indeed,

$$
(2t)!\,P_{2t} = -2\left(\frac{B_{2t}}{2^{2t}} - \frac{B_{2t}}{2}\right).
\tag{4.2.4}
$$

Indeed, by a well-known formula (see [GKP])

$$
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!},
$$

and on the other hand

$$
\frac{z/2}{e^{z/2} - 1} - \frac{1}{2}\frac{z}{e^z - 1} = -\frac{1}{2} P^{-1}(z)
$$

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which implies (4.2.4). Let us substitute (4.2.4) into (4.2.3) and further into (4.2.1). After certain combinatorial transformations we obtain (1.4.2). In particular, we take t as the new summation index, $t = 0, 1, \ldots, n - \nu + 1 + s$. Note that if $n < \nu$ then $t \le s$ and hence $(t - s)_s = (t - s)(t - s + 1) \cdots (t - 1) = 0$ unless $t = 0$ when $(-s)_s = (-1)^s s!$. This explains why the second sum disappears in (1.4.2) for $n < \nu$.

It is easy to check that taking $\nu = 1$ we get (1.4.4). The proof is complete. ■

4.3 Proof of Theorem 1.1.2 for $2n \le \omega < 3n$

Take the arguments in the proofs of Theorems 1.3.1 and 1.4.1 in the reverse order. Starting with (1.3.2) in odd dimensions and (1.4.2) in even dimensions we arrive to (1.1.3). Note that the proofs of Theorems 1.3.1 and 1.4.1 are valid for $\omega \ge 2n$ (*cf.* Theorems 1.2.1 and 1.2.2) and hence formula (1.1.3) holds under the same condition. This completes the proof of Theorem 1.1.2. ■

5 Auxiliary Combinatorial Lemmas

5.1 Odd Dimensions

Lemma 5.1.1 *Let $\omega' \ge 2t + s$, $s \ge 0$, $t \ge 1$. Then*

(5.1.2)
$$\sum_{j=0}^{\omega'} \frac{(-1)^j (2j+2t)!\, z^j}{(\omega' - j)!\,(j + t - s)!\,(2j+1)!} = \sum_{k=0}^{s} Q_{k,s}(z)\, {}_2F_1\Big(-\omega' + k, \frac{1}{2} + t + k; \frac{3}{2} + k; z\Big),$$

where $Q_{k,s}(z)$ are some polynomials in z. Moreover,

$$\sum_{j=0}^{\omega'} \frac{(-1)^j (2j+2t)!}{(\omega - j)!\,(j + t - s)!\,(2j+1)!} P(D)^{2j} x^{2t} = 0, \tag{5.1.3}$$

where $P(D)$ is defined by (3.1.4).

Proof Denote the sum at the left hand side by $\sigma_s(z)$. Let us proceed by induction. For $s = 0$ the statement follows from (3.1.5). Suppose we proved it for all $s \le s_0$. Let us prove it for $s_0 + 1$. It is easy to see that

$$\sigma_{s_0+1}(z) = (t - s_0)\sigma_{s_0}(z) + z \frac{d\sigma_{s_0}}{dz}. \tag{5.1.4}$$

By the induction hypothesis and the rule for differentiation of a hypergeometric function (see [Er]) we obtain:

$$\frac{d\sigma_{\sigma_0}}{dz} = \sum_{k=0}^{s_0} Q'_{k,s_0}(z)\, {}_2F_1\left(-\omega' + k, \frac{1}{2} + t + k; \frac{3}{2} + k; z\right)$$
$$+ \sum_{k=0}^{s_0} Q_{k,s_0}(z) \frac{(k - \omega')(\frac{1}{2} + t + k)}{k + \frac{3}{2}}\, {}_2F_1\left(-\omega' + k + 1, \frac{3}{2} + t + k; \frac{5}{2} + k; z\right).$$

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Substituting this to (5.1.4) implies (5.1.2).

Let us prove (5.1.3). We use (5.1.2) and apply arguments of the previous section starting with (3.1.5) to each term of the sum $\sigma_s(z)$. Note that each hypergeometric function in the right-hand side of (5.1.2) is in fact a finite series since $-\omega' + k < 0$ for all $k = 0, 1, \ldots, s$. Due to (3.1.6) we have the following condition for vanishing of the left hand side in (5.1.3):

$$3/2 + k - 1/2 - t - k + \omega' - k = \omega' - k - t + 1 > t, \tag{5.1.5}$$

that is $\omega' \geq 2t + k$. But we have supposed that $\omega' \geq 2t + k$ and since $k \leq s$ we get (5.1.5). The last thing we have to verify is that using the Euler transformation (3.1.6) we always get a finite hypergeometric series. This is indeed so since

$$3/2 + k - 1/2 - t - k = 1 - t \leq 0$$

due to the condition $t \geq 1$. This completes the proof of the Lemma. ∎

5.2 Even Dimensions

Lemma 5.2.1 *Let $\omega' \geq 2t + s$, $s \geq 0$, $t \geq 0$. Then*

$$\begin{aligned} &\sum_{j=0}^{\omega'} \frac{(-1)^j (2j+2t+1)!\, z^{j+1}}{(\omega'-j)!\,(j+t-s)!\,(2j+2)!} \\ &\qquad = \frac{(2t)!\,(t-s)_s}{2(\omega'+1)!\,t!} + \sum_{k=0}^{s} Q_{k,s}(z)\,{}_2F_1\left(-1-\omega'+k, \frac{1}{2}+t+k; \frac{1}{2}+k; z\right), \end{aligned} \tag{5.2.2}$$

where $Q_{k,s}(z)$ are some polynomials in z. Moreover,

$$\sum_{j=0}^{\omega'} \frac{(-1)^j (2j+2t+1)!\, P^{2j+1}}{(\omega'-j)!\,(j+t-s)!\,(2j+2)!} x^{2t}\Big|_{x=0} = \frac{(2t)!\,(t-s)_s}{2(\omega'+1)!\,t!} P^{-1}(x^{2t})\Big|_{x=0}, \tag{5.2.3}$$

where $P(D)$ is defined by (3.1.4).

Proof Again, we proceed by induction over s. For $s = 0$ this can be checked by a direct computation (*e.g.* using [W]). Denoting the left-hand side of (5.2.2) by $\zeta_s(z)$ similarly to (5.1.4) we have

$$\zeta_{s_0+1}(z) = (t - s_0 - 1)\zeta_{s_0}(z) + z\frac{d\zeta_{s_0}}{dz}. \tag{5.2.4}$$

As in the proof of Lemma 5.1.1 this implies the induction step and proves (5.2.2). The relation (5.2.3) follows from (5.2.2) in a similar way as (5.1.3) follows from (5.1.2). ∎

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https://doi.org/10.4153/CJM-2002-040-4 Published online by Cambridge University Press