ON COMBINING QUOTA-SHARE AND EXCESS OF LOSS

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ABSTRACT

This paper considers reinsurance retention limits in cases where the cedent has a choice between a pure quota-share treaty, a pure excess of loss treaty or a combination of the two. Our primary aim is to find the combination of retention limits which minimizes the skewness coefficient of the insurer's retained risk subject to constraints on the variance and the expected value of his retained risk. The results are given without specifying precisely how the excess of loss reinsurance premium is calculated. It is also shown that, depending to some extent on the constraint on the variance, the solution to the problem is a pure excess of loss treaty if the excess of loss premium is calculated using the expected value or standard deviation principle but that this need not be true if the variance principle is used.

KEYWORDS

Reinsurance, Quota-share, Excess of loss, skewness, coefficient of variation, constrained optimization.

1. INTRODUCTION

This paper considers reinsurance retention limits in cases where the cedent has a choice between a pure quota-share treaty, a pure excess of loss treaty or a combination of the two. Such combinations occur in practice; see, for example, GERATHEWOHL (1980, Vol. 2, p. 371).

We assess the effects on the insurer of a particular combination of reinsurance treaties by considering three moment functions of the insurer's retained risk. These functions are the skewness coefficient and the variance of the insurer's net claims and the insurer's expected net profit. Our primary aim is to find the combination of retention limits which minimizes the skewness coefficient of the insurer's net claims, subject to a maximum value for the variance of the insurer's net claims and a minimum value for the insurer's expected net profit.

In Section 3 we show that the solution to this problem is unchanged if we replace the skewness coefficient by the coefficient of variation of the insurer's net claims.

Constrained optimization as a criterion for determining optimal retention limits has been used before, see BÜHLMANN (1970, pp. 114–119), but not in relation to a combination of types of reinsurance. Combinations of types of reinsurance

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have not often been discussed in the mathematical insurance literature; one notable exception is LEMAIRE, REINHARD and VINCKE (1981). There are some similarities between our paper and theirs, but also some important differences. For example, in our paper we allow the claim number distribution to be more general than the Poisson (for example negative binomial). There is also a difference in the way in which we assume the reinsurance premiums are calculated. We assume the quota-share premium is calculated on a proportional basis with a commission payment to the insurer; we do not specify how the excess of loss reinsurance premium is calculated but make some assumptions about this premium which are shown to be satisfied if it is calculated using the expected value, standard deviation or variance principles.

In Section 2 we describe in detail the two reinsurance treaties and discuss our assumptions relating to the excess of loss reinsurance premium.

In Section 3 we state our problems and give the solution in general form.

In Section 4 we give the solution to our problems assuming the excess of loss reinsurance premium is calculated using the expected value, the standard deviation or the variance principle. It is shown that, provided the constraint involving the variance of the insurer's retained risk is not too restrictive, the optimal solution is a pure excess of loss treaty in the first two cases but this need not to be true in the last case.

In Section 5 we discuss briefly the necessity of the assumptions made concerning the claim number distribution.

In Section 6 we give a numerical example to illustrate our results.

2. THE REINSURANCE ARRANGEMENT AND THE COST OF THE EXCESS OF LOSS REINSURANCE

2.1. *The Reinsurance Arrangement*

Consider a risk for which the aggregate gross (of reinsurance) claims in some fixed time interval are denoted by a random variable Y. We assume Y has a compound distribution, so that

$$Y = \sum_{i=1}^{N} X_i$$

where $\{X_i\}$, with $0 \leq x_0 < X_i < x_1 \leq +\infty$, is a sequence of i.i.d. random variables, with common distribution function F, representing the amounts of the individual claims and N is a random variable, independent of the X_i's, representing the number of claims in the time interval. We shall assume that F is continuous and that the third moments of X_i and N are finite (although this will not always be necessary). Let λ_1, λ_2 and λ_3 denote the mean, variance and third central moment of N. Throughout this and the following two sections, which contain our main results, we shall make the following two assumptions:

$$\lambda_2 - \lambda_1 \geq 0 \tag{2.1.1}$$

$$2\lambda_2^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 \geq 0. \tag{2.1.2}$$

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In Section 5 we shall comment on the necessity of these assumptions for our results but for the present we remark that both assumptions will hold if N has either a Poisson or a negative binomial distribution.

We assume the insurer of the risk arranges a combination of quota-share and excess of loss reinsurance in the following way:

Firstly, the insurer chooses a quote-share retention level which we denote a so that the insurer's aggregate claims, net of quota-share reinsurance, are aY. We assume the cost of the quota-share reinsurance is calculated on a proportional basis with a commission payment. (See CARTER (1979, p. 87).) More precisely, let P denote the insurer's gross (of expenses and reinsurance) premium income in respect of this risk. We assume an amount eP is used to cover the insurer's expenses, irrespective of the level of reinsurance. The premium of the quote-share reinsurance is $(1-a)P$ less a commision payment of $c(1-a)P$.

Secondly, the insurer chooses an excess of loss retention level which we denote M so that the insurer's aggregate claims, net of quota-share and excess of loss reinsurance, can be represented by a random variable $Y(a, M)$, where

$$Y(a, M) = \sum_{i=1}^{N} \min(aX_i, M).$$

We denote by $P(a, M)$ the premium paid to the reinsurer in respect of the excess of loss arrangement and we assume the premiums for the two arrangements are calculated independently of each other. (It could be argued that there should be a connection between the two reinsurance premium calculations since 100% reinsurance should cost the same for the two types of treaty but we do not make this extra assumption). Hence the insurer's net (of expenses and reinsurance costs) premium income is

$$P(c-e) + aP(1-c) - P(a, M).$$

2.2. *The Cost of the Excess of Loss Reinsurance*

Let $C(a, M)$ denote the cost to the insurer of the excess of loss reinsurance arrangement, so that

$$C(a, M) = P(a, M) - E[aY - Y(a, M)] \tag{2.2.1}$$

Throughout this paper we make the following assumptions concerning $C(a, M)$:

$$C(a, M) \in \mathscr{C}_1 \quad \text{for } a, M > 0, \tag{2.2.2}$$

where $\mathscr{C}_i$ is the class of functions with continuous derivatives of order i.

(2.2.3) If $x_1 = +\infty$, $\quad \lim_{M\to\infty} C(a, M) = 0 \quad$ for any $a \in (0, 1]$

If $x_1 < +\infty$, $\quad C(a, M) = 0 \quad$ for any $M \geq ax_1$ and any $a \in (0, 1]$

$$\partial C/\partial M < 0 \quad \text{for } M \in (0, ax_1) \text{ and any } a \in (0, 1] \tag{2.2.4}$$

(2.2.5) $\qquad C(a, M)$ is a convex function of a and M.

Assumptions (2.2.3) are natural. (2.2.4) implies only that the cost of the excess of loss arrangement should decrease as more of the risk is retained by the insurer. (2.2.5) is a little more difficult to interpret but it holds in all our examples in Section 2.3. Roughly speaking, if we regard a as fixed, (2.2.4) and (2.2.5) together imply that as M decreases, the cost of reinsurance increases and the rate of increase of this cost should also increase.

From assumption (2.2.2), (2.2.3) and (2.2.4) we can see that

(2.2.6) $$C(a, M) \geq 0 \quad \text{for any } a \in [0, 1] \text{ and any } M \geq 0$$

with

(2.2.7) $$C(a, M) = 0 \quad \text{if and only if } M \geq a x_1.$$

2.3. *Some Examples*

In this section we discuss very briefly the assumptions of Section 2.2 when $P(a, M)$ is calculated according to some well known principles.

When $P(a, M)$ is calculated according to the expected value principle, standard deviation principle or variance principle (see, for example, GERBER (1979, p. 67) it is not difficult to prove that $C(a, M)$ satisfies (2.2.2), (2.2.3) and (2.2.4). If $P(a, M)$ is calculated according to the expected value principle it can be shown that $C(a, M)$ satisfies (2.2.5). Now suppose that $P(a, M)$ is calculated according to the standard deviation principle so that, after a little calculation,

$$C(a, M) = f\{\lambda_1 G^2(a, M) + (\lambda_2 - \lambda_1) H^2(a, M)\}^{1/2}.$$

Where f is a positive loading factor and

$$G(a, M) = \left[\int_{M/a}^{\infty} (ax - M)^2 \, dF(x)\right]^{1/2},$$

$$H(a, M) = \int_{M/a}^{\infty} (ax - M) \, dF(x).$$

It can be shown that given any two non-negative convex functions $g_1(x)$ and $g_2(x)$ the function $g(x) = [g_1^2(x) + g_2^2(x)]^{1/2}$ is still convex. It can also be shown that $H(a, M)$ is a convex function of (a, M) (this is equivalent to say that $C(a, M)$ is convex if $P(a, M)$ is calculated according to the expected value principle). Since $\lambda_2 - \lambda_1 \geq 0$ by assumption, in order to prove that $C(a, M)$ is convex we only have to prove that $G(a, M)$ is convex. The convexity of $G(a, M)$ follows easily since

$$G(a, M) \in \mathscr{C}_2 \quad \text{for } a > 0, M > 0$$

$$\partial^2 G(a, M)/\partial M^2 \geq 0 \text{ using the Cauchy-Schwarz inequality}$$

and

$$\{\partial^2 G(a, M)/\partial M^2\} \cdot \{\partial^2 G(a, M)/\partial a^2\} - \{\partial^2 G(a, M)/\partial a \, \partial M\}^2 = 0.$$

The fact that $C(a, M)$ satisfies (2.2.5) when $P(a, M)$ is calculated according to the variance principle follows directly from the corresponding result for the standard deviation principle. (Since the square of a non-negative convex function is also convex).

3. THE PROBLEM AND ITS SOLUTION

3.1. *The Problem*

The problem, in broad terms, is to choose retention levels a and M which are, in some sense, optimal for the insurer. We shall assess the effects of reinsurance by considering moment functions of the distribution of the insurer's retained risk. More precisely, let $W(a, M)$ be a random variable denoting the insurer's net (of expense and reinsurance) profit and let $E[W(a, M)]$, $V[Y(a, M)]$, $CV[Y(a, M)]$ and $\gamma(Y(a, M)$ be the expected net profit and the variance, coefficient of variation and skewness coefficient of the insurer's net claims respectively. Our main problem is

PROBLEM 1. *Minimize $\gamma[Y(a, M)]$ over the set Γ.*

Where $\Gamma=\{(a, M): 0 \leq a \leq 1$ and $M \geq 0$ and $E[W(a, M)] \geq B$ and $V(Y(a, M)) \leq D\}$

for some constants B and D. It is assumed that B and D are such that $\Gamma \neq \varnothing$. (Note that we assume $CV[Y(a, M)]$ and $\gamma[Y(a, M)]$ are zero if either $a=0$ or $M=0$, as well as $C[Y(a, M)]$ when $a=0$.)

We shall show that any solution to problem 1 is a solution to problem 2 and vice versa, where problem 2 is

PROBLEM 2. *Minimize $CV[Y(a, M)]$ over the set Γ.*

Note that $V[Y(a, M)]=V[W(a, M)]$ and $\gamma[Y(a, M)]=-\gamma[W(a, M)]$ so that problem 1 can be expressed entirely in terms of the insurer's net profit. This is not the case for problem 2 since here is no simple relationship between $CV[Y(a, M)]$ and $CV[W(a, M)]$.

In order to solve the above problems it will be helpful to consider the following simpler problem:

PROBLEM 3. *Minimize $\gamma[Y(a, M)]$ over the set Γ_1.*

Where $\Gamma_1=\{(a, M): 0 \leq a \leq 1,\ M \geq 0,\ E[W(a, M)] \geq B\}$

or equivalently (as we will see),

PROBLEM 4. *Minimize $CV[Y(a, M)]$ over the set Γ_1,*
i.e., we drop the constraint concerning the variance.

3.2. *The Skewness Coefficient and the Coefficient of Variation of the Total Net Claims*

The statement and proof of the following result assume, for convenience, that $x_0>0$.

Result 1. (i) $\gamma[Y(a, M)]$ and $CV[Y(a, M)]$ are functions of class $\mathscr{C}_1$ for $a, M > 0$.

(ii) Both of them are strictly increasing functions of the single variable M/a for $x_0 < M/a < x_1$ and points such that $0 < M/a \leqslant x_0$ and $M/a \geqslant x_1$ give minimum and maximum values respectively of the two functions over the set $\{(a, M): a, M > 0\}$.

(iii) $\gamma[Y(a_1, M_1)] > \gamma[Y(a_2, M_2)]$ if and only if $CV[Y(a_1, M_1)] > CV[Y(a_2, M_2)]$.

Proof. (i) A little calculation gives the following formulae:

$$V[Y(a, M)] = \lambda_1(\beta_2 - \beta_1^2) + \lambda_2\beta_1^2 \tag{3.2.1}$$

$$CV[Y(a, M)] = \{V[Y(a, M)]\}^{1/2}/(\lambda_1\beta_1) \tag{3.2.2}$$

$$\begin{aligned}\gamma[Y(a, M)] = \{&\lambda_3\beta_1^3 + \lambda_1(\beta_3 - 3\beta_1\beta_2 + 2\beta_1^3) \\ &+ 3\lambda_2\beta_1(\beta_2 - \beta_1^2)\}/\{V[Y(a, M)]\}^{3/2}\end{aligned} \tag{3.2.3}$$

where

$$\beta_k = \int_0^{M/a} a^k x^k \, dF(x) + M^k(1 - F(M/a)).$$

Using integration by parts and the assumptions that $F(0) = 0$ and that F is continuous, we have

$$\beta_k = M^k - ka^k \int_0^{M/a} x^{k-1} F(x) \, dx \tag{3.2.4}$$

from which the proof of (i) follows immediately.

(ii) Let $z = M/a$. Then we can see that

$$\beta_k = M^k \alpha_k \tag{3.2.5}$$

where

$$\alpha_k = \int_0^z (x/z)^k \, dF(x) + 1 - F(z).$$

Substituting (3.2.5) into (3.2.1), (3.2.2) and (3.2.3) we see that $CV[Y(a, M)]$ and $\gamma[Y(a, M)]$ can be expressed as functions of the single variable z. We shall show that $d\gamma[Y(z)]/dz > 0$, for $x_0 < z < x_1$

$$\begin{aligned}\frac{d\gamma}{dz}[Y(z)] = {}& 3(1 - F(z))z^{-5}[\lambda_1(\alpha_2 - \alpha_1^2) + \lambda_2\alpha_1^2]^{-5/2} \\ & \times\left\{\lambda_1^2 \int_0^z (x^2z^2 - x^3z)\,dF(x) + \lambda_1(\lambda_2 - \lambda_1)h(z)\right. \\ & \left. + (2\lambda_2^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3)\alpha_1^2 \int_0^z (xz^3 - x^2z^2)\,dF(x)\right\}\end{aligned}$$

where

$$h(z) = 2\int_0^z x(1-F(x))\,dx \cdot \int_0^z (x^2 - xz)\,dF(x) + \int_0^z (1-F(x))\,dx \cdot \int_0^z (xz^2 - x^3)\,dF(x).$$

It is easily checked that $h(0)=0$ and that $dh(z)/dz \geq 0$ so that $h(z) \geq 0$ for $z \geq 0$.

That $d\gamma/dz$ is strictly positive for $x_0 < z < x_1$ then follows from assumptions (2.1.1) and (2.1.2). The proof that $dCV/dz > 0$ is similar to that given above but is somewhat simpler and does not require assumptions (2.1.1) and (2.1.2). The remaining part of (ii) now follows immediately and (iii) follows from (ii).

REMARKS. The equivalence of problem 1 and 2, and of problems 3 and 4, follows from part (iii) of the above result.

A further implication of the result is that the locus of points (a, M) satisfying the relation $\gamma[Y(a, M)] = \text{constant}$, or $CV[Y(a, M)] = \text{constant}$, is a straight line passing through the origin in the (a, M)-plane, a higher value of the constant giving a line with steeper slope.

3.3. *Isocost Curves*

In this section we consider the locus of points (a, M) satisfying the relation $E[W(a, M)] = B$ which is equivalent to

$$P(c-e) + a(P(1-c) - \lambda_1 E(X)) - C(a, M) = B. \tag{3.3.1}$$

Where B is the constant appearing in the definition of the sets Γ and Γ_1. (See Section 3.1). It can be regarded as the set of points with a fixed reinsurance price, since (3.3.1) is equivalent to

$$(1-a)[P(1-c) - \lambda_1 E(X)] + C(a, M) = P(1-e) - \lambda_1 E(X) - B. \tag{3.3.2}$$

Where the left-hand side represents the total reinsurance cost of the arrangement (a, M).

We make the following assumptions about the parameters involved in our problems:

$$P(1-c) - \lambda_1 E(X) > 0 \tag{3.3.3}$$

$$B < P(1-e) - \lambda_1 E(X) \tag{3.3.4}$$

$$B > P(c-e) \tag{3.3.5}$$

$$B > \max\{E[W(a, M)]: 0 < a \leq 1, 0 \leq M \leq ax_0\}. \tag{3.3.6}$$

Assumption (3.3.3) implies that the cost of the quota-share arrangement $[(1-a)(P(1-c) - \lambda_1 E(X))]$ is positive for $0 \leq a < 1$ and also that the cost of this arrangement decreases with the retention a. Then (3.3.3) together with (2.2.6) and (2.2.7) implies that the total reinsurance cost of the arrangement (a, M) is non-negative, and is zero if and only if $a=1$ and $M \geqslant x_1$. Assumption (3.3.4) is then natural since the right hand side represents the insurer's expected profit after expenses but without any reinsurance. Assumptions (3.3.5) and (3.3.6) imply that points such that $a=0$ or $M \leqslant ax_0$ respectively are not feasible solutions to our problems, i.e., we do not consider solutions where the whole risk is passed to the reinsurer through the quota-share arrangement or where the excess of loss retention is less than the smallest possible claim (net of quota-share reinsurance).

The following result discusses the shape of the isocost curves.

RESULT 2. Let

$$\rho(a, M) = E[W(a, M)] - B \quad \text{for any } a, M > 0$$

$$a_0 = [P(e-c)+B]/[P(1-c) - \lambda_1 E(X)]$$

$A = \{a: 0 < a \leqslant 1$ and there exist at least one M, $M < ax_1$, such that $\rho(a, M) = 0\}$.

Then (i) $A = (a_0, 1]$.

(ii) For each $a \in A$ there is a unique M such that $\rho(a, M) = 0$ i.e., there is a function Φ mapping A into $(0, \infty)$ such that $M = \Phi(a)$ is equivalent to $\rho(a, M) = 0$.

(iii) $\Phi(a) \in \mathscr{C}_1$.

(iv) $\lim_{a \to a_0^+} \Phi(a) = a_0 x_1$.

(v) $\lim_{a \to a_0^+} \Phi'(a) = -\infty$.

(vi) $\Phi(a)$ is convex and is strictly convex if $C(a, M)$ is strictly convex.

PROOF. First note that $\rho(a, M) = 0$ is equivalent to

$$\rho(a, M) = P(c-e) + a(P(1-c) - \lambda_1 E(X)) - C(a, M) - B = 0.$$

(i) Let $\hat{a} \leqslant a_0$. It follows from the definition of a_0 and from (2.2.6) and (2.2.7) that $\rho(\hat{a}, M) < 0$ for any $M < \hat{a}x_1$. Hence $\hat{a} \notin A$. Now let $\hat{a} \in (a_0, 1]$. $\rho(\hat{a}, M)$, considered as a function of M, is continuous since $C(a, M)$ is assumed continuous. Also

$$\lim_{M \to \hat{a}x_1^-} \rho(\hat{a}, M) = P(c-e) + \hat{a}(P(1-c) - \lambda_1 E(X)) - B > 0$$

$$\lim_{M \to \hat{a}x_0^+} \rho(\hat{a}, M) < 0 \quad \text{by (3.3.6).}$$

Hence there is at least one M, $M < \hat{a}x_1$, such that $\rho(\hat{a}, M) = 0$.

(ii) Suppose $\rho(a, M_1) = 0 = \rho(a, M_2)$ for some a, M_1 and M_2. Then $C(a, M_1) = C(a, M_2)$ and hence, using (2.2.4), $M_1 = M_2$.

(iii) This follows from the Implicit Function theorem. See, for example, APOSTOL (1963).

(iv) Let $\{a_n\}$ be a sequence such that $a_n > a_0$, $\lim_{n\to\infty} a_n = a_0$ and $\lim_{n\to\infty} \Phi(a_n) = k \leqslant +\infty$. By continuity we have $\rho(a_0, k) = 0$, which implies that $k \geqslant a_0 x_1$ using the definitions of ρ and a and (2.2.6) and (2.2.7). But using part (i) above, $\Phi(a_n) < a_n x_1$ and (iv) follows.

(v) If $x_1 = +\infty$ this is obvious. If $x_1 < +\infty$ we have only to notice that, using (2.2.3) and (2.2.4), both $\partial C/\partial a$ and $\partial C/\partial M$ are zero at the point $(a_0, a_0 x_1)$. Hence $\partial\rho/\partial M$ is zero and $\partial\rho/\partial a$ is strictly positive at $(a_0, a_0 x_1)$.

(vi) Let $a_1, a_2 \in A$ and $0 \leq \lambda \leq 1$. $\rho(a, M)$ is concave since $C(a, M)$ is convex, so we have

$$\begin{aligned}
&\rho(\lambda a_1 + (1-\lambda)a_2, \Phi(\lambda a_1 + (1-\lambda)a_2)) \\
&\qquad = 0 = \lambda\rho(a_1, \Phi(a_1)) + (1-\lambda)\rho(a_2, \Phi(a_2)) \\
&\qquad \leq \rho(\lambda a_1 + (1-\lambda)a_2, \lambda\Phi(a_1) + (1-\lambda)\Phi(a_2)).
\end{aligned}$$

Using the proof of part (i) above we have

$$\lambda\Phi(a_1) + (1+\lambda)\Phi(a_2) \geq \Phi(\lambda a_1 + (1-\lambda)a_2)$$

It is clear that Φ is strictly convex if the same is true for $C(a, M)$.

3.4. *The Variance as a Function of* (a, M).

We shall find the following result useful when proving our main results in the next section.

RESULT 3
(i) $\partial V[Y(a, M)]/\partial a > 0$ for $x_0 < M/a < x_1$,
(ii) $\partial V[Y(a, M)]/\partial M > 0$ for $x_0 < M/a < x_1$.

PROOF. We have already seen that $V[Y(a, M)] \in \mathscr{C}_1$ for $a, M > 0$ (see proof of result 1(i)). Differentiating (3.2.1) we have:

$$\begin{aligned}
\partial V/\partial a = {} & 2\lambda_1 a \int_0^{M/a} x^2\, dF(x) \\
& + 2(\lambda_2 - \lambda_1) \int_0^{M/a} x\, dF(x) \left[\int_0^{M/a} ax\, dF(x) + M(1 - F(M/a)) \right] \\
\partial V/\partial M = {} & 2\lambda_1 (1 - F(M/a)) \left[MF(M/a) - \int_0^{M/a} ax\, dF(x) \right] \\
& + 2\lambda_2 (1 - F(M/a)) \left[\int_0^{M/a} ax\, dF(x) + M(1 - F(M/a)) \right]
\end{aligned}$$

(ii) follows directly and (i) follows from (2.1.1).

3.5. *The Solution*

In this section we solve our problems in general terms.

RESULT 4. (i) The non-negative constraints are redundant in our problems. (ii) The constraint $E[W(a, M)] \geq B$ is active in the optimum of our problems, i.e., in the optimum of our problems this constraint holds as an equality.

PROOF. (i) Follows directly from assumptions (3.3.5) and (3.3.6).

(ii) We shall prove the result for problem 1, and hence problem 2. The proof for problems 3 and 4 is similar but simpler. Let $(a_1, M_1) \in \Gamma$ be such that $E[W(a_1, M_1)] > B$. From the proof of result 2 we know that there exists $M^* < M_1$ where $a_1 x_0 < M^* < a_1 x_1$ and $E[W(a_1, M^*)] = B$. Using result 3(ii) and result 1(ii) we see that

$$V[Y(a_1, M^*)] \leq V[Y(a_1, M_1)] \leq D$$

and

$$\gamma[Y(a_1, M^*)] < \gamma[Y(a_1, M_1)].$$

Let us now consider the solution to problem 3 (and hence to problem 4). We know that the solution lies on the isocost curve $M = \Phi(a)$ and information about the shape of this curve is contained in result 2. Figure 1 gives three examples of

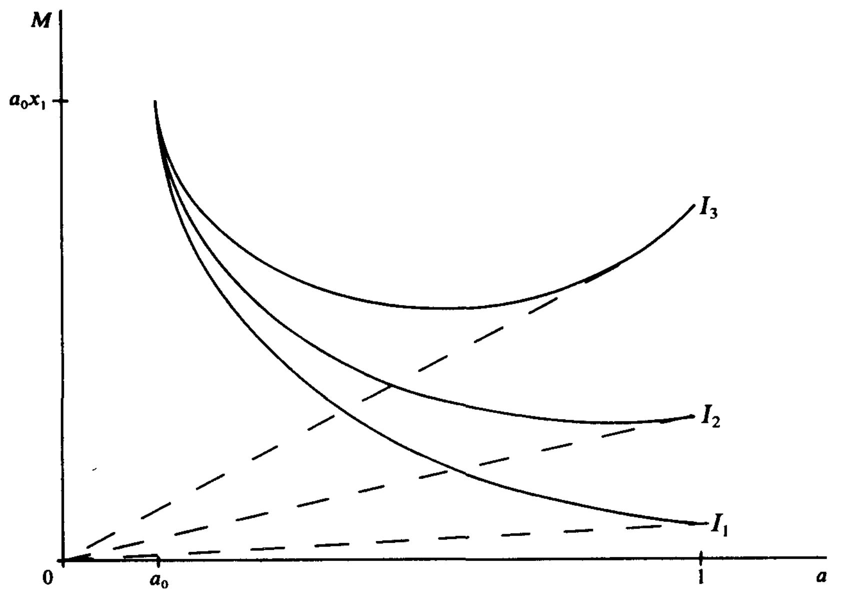

FIGURE 1. Isocost curves in the (a, M)-plane.

isocost curves, labelled I_1, I_2 and I_3. We know that each curve has slope $-\infty$ at the point $(a_0, a_0 x_1)$ and we have assumed x_1 is finite for convenience. We also know that each curve is convex although not necessarily strictly convex. From result 1 we know that straight lines through the origin in fig. 1 represent points of constant skewness, the larger the slope the higher the skewness. Hence it is clear that the solution to problem 3 is the point, or set of points, where the straight line through the origin with the smallest slope intersects the isocost curve. If the isocost curve is decreasing, as in I_1, this point will be $(1, \Phi(1))$, i.e., pure excess of loss reinsurance will be optimal. (Note that LEMAIRE, REINHARD and

VINCKE (1981), by making assumptions about the calculation of the reinsurance premiums different to ours, were able to assume that the isocost curves were decreasing and hence that, in terms of our problem, excess of loss reinsurance was optimal.) Even if the isocost curve is not decreasing, as in I_2, the point $(1, \Phi(1))$ may still be the solution to problem 3. Isocost curve I_3 shows a case where the solution is not $(1, \Phi(1))$.

It is clear that in general the solution to problem 3 will be $(1, \Phi(1))$ unless we can find a point on the isocost curve such that the gradient of the isocost curve at that point equals the slope of the line joining that point to the origin. Such a point may not be unique since the isocost curve may not be strictly convex.

Summarizing we have the following result:

RESULT 5. Let

$$H = \{(a, \Phi(a)) \colon a_0 < a \leq 1, d\Phi(a)/da = \Phi(a)/a\}.$$

Then

(i) if H is empty the solution to problems 3 and 4 is the point $(1, \Phi(1))$.

(ii) if H is not empty, all the points in H are solutions to problems 3 and 4.

REMARKS. (i) We have given a geometrical proof of result 5 but it is possible to give a more formal proof using the Kuhn–Tucker conditions and the facts that $E[W(a, M)]$ is a concave function and $\gamma[Y(a, M)]$, or $CV[Y(a, M)]$, is a quasi-convex function of (a, M). See ARROW and ENTHOVEN (1961).

(ii) Using the definition of Φ the set H can be defined as

$$H = \{(a, M) \colon a \leq 1 \text{ and } E[W(a, M)] = B \text{ and } B + P(e-c) + C(a, M) - a\,\partial C/\partial a - M\,\partial C/\partial M = 0\}. \tag{3.5.1}$$

We are now in a position to solve problems 1 and 2.

RESULT 6. Let

$$a_1 = \inf\{a \colon (a, \Phi(a)) \text{ is a solution to problem 3}\}.$$

Then

(i) if $V[Y(a_1, \Phi(a_1))] \leq D$, $(a_1, \Phi(a_1))$ is a solution of problem 1 and every solution of problem 1 is a solution of problem 3.

(ii) if $V[Y(a_1, \Phi(a_1))] > D$ the solution to problem 1 is $(a^0, \Phi(a^0))$ where

$$a^0 = \sup\{a \colon a \leq 1, E[W(a, M)] = B \text{ and } V[Y(a, M)] = D\}.$$

In this case $a^0 < a_1$.

PROOF. (i) If $(a_1, \Phi(a_1))$ is a solution of problem 3 and $V[Y(a_1, \Phi(a_1))] \leq D$ then clearly $(a_1, \Phi(a_1))$ is a solution of problem 1. If $(a, \Phi(a))$ is another solution to problem 1 we must have $\gamma[Y(a, \Phi(a))] = \gamma[Y(a_1, \Phi(a_1))]$ and so $(a, \Phi(a))$ solves problem 3.

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(ii) Using geometrical arguments and result 3, it is clear that for any a such that $a_1 \leqslant a \leqslant 1$ we have, assuming $V[Y(a_1, \Phi(a_1))] > D$, $\Phi(a_1) \leqslant \Phi(a)$ and $D < V[Y(a_1, \Phi(a_1))] \leqslant V[Y(a, \Phi(a))]$. This shows that $a^0 < a_1$.

On the other hand, $\gamma[Y(a, \Phi(a))]$ is a strictly decreasing function of a for $a_0 < a < a_1$, as is clear when we consider the geometrical proof of result 5.

So for any point $(a, \Phi(a))$ such that $a_0 < a < a^0$ we will have $\gamma[Y(a, \Phi(a))] > \gamma[Y(a^0, \Phi(a^0))]$ and for any point such that $a^0 < a \leqslant 1$ we will have $V[Y(a, \Phi(a))] > D$, otherwise these would be a contradiction to the definition of a^0 or to the mean value theorem.

4. THE SOLUTION IN SOME SPECIAL CASES

In this section we give, briefly, the solution to problems 3 and 4 when the excess of loss reinsurance premium is calculated according to the expected value principle, the standard deviation principle or the variance principle.

RESULT 7. (i) If the excess of loss reinsurance premium is calculated using the expected value principle or standard deviation principle, the solution to Problems 3 and 4 is $(1, \Phi(1))$, i.e., a pure excess of loss arrangement.

(ii) If the excess of loss reinsurance premium is calculated using the variance principle, the solution to problem 3 and 4 is $(\hat{a}, \Phi(\hat{a}))$ where

$$\hat{a} = \min\{2[P(e-c)+B]/[P(1-c)-\lambda_1 E(X)], 1\}.$$

PROOF. (i) The proof is immediate since for both cases

$$B + P(e-c) + C(a, M) - a\frac{\partial C}{\partial a} - M\frac{\partial C}{\partial M} = B + P(e-c) \tag{4.1}$$

which is positive by assumption (3.3.5) and so the set H is always empty (although the relevant isocost curve is not necessarily decreasing). The result follows from result 5(i).

(ii) In this case the left-hand side of (4.1) is equal to

$$B + P(e-c) - C(a, M).$$

Hence the set H is

$$\{(a, M): E[W(a, M)] = B \text{ and } a = 2[B + P(e-c)]/[P(1-c) - \lambda_1 E(X)] \text{ and } a \leqslant 1\}$$

and the result follows from result 5.

5. DISCUSSION

We have assumed throughout Sections 2, 3 and 4 that assumptions (2.1.1) and (2.1.2) hold for the claim number distribution N. It is clear that all our results relating to the coefficient of variation, in particular the solutions to problem 2 and 4, are valid without making assumption (2.1.2), since this assumption was used only in the proof of result 1(ii), and then only in relation to the skewness coefficient.

Assumption (2.1.1) was used in relation to the coefficient of variation to show that $C(a, M)$, and hence the isocost curve $M = \Phi(a)$, is convex when the excess of loss reinsurance premium is calculated according to the standard deviation principle or the variance principle (see Section 2.3). If (2.1.1) does not hold it is not hard to find examples where $P(a, M)$ is calculated according to either the standard deviation principle or the variance principle and where the isocost curve is no longer convex. (One particular example assumes N to be a degenerate random variable always equal to 1, which is equivalent to assuming a combination of quota-share and stop-loss reinsurance). However if (2.1.1) does not hold we can still state result 7(i), relating to the coefficient of variation, since this result is an immediate consequence of result 5(i), and it is easy to see that this result is independent of the convexity of the isocost curves. Assumption (2.1.1) was also used for the proof of result 3(i), which was later applied in the proof of result 6(ii). It is not difficult to see that if (2.1.1) does not hold, but $a_1 = 1$ in result 6, this result is still true. So we can conclude that when $P(a, M)$ is calculated using the expected value or standard deviation principle, the main results relating to the coefficient of variation [i.e., result 7(i) and result 6], hold without the assumption (2.1.1) being fulfilled.

When $P(a, M)$ is calculated according to the variance principle the proofs of Results 5(ii), 6(ii) and 7(ii), relating to the coefficient of variation, are no longer valid without (2.1.1), although it may be possible to prove some of these results without this assumption.

Furthermore (2.1.1) is not a necessary condition for the proof of result 1 relating the skewness coefficient, since this result still holds when N is a degenerate random variable and when the distribution function of the individual claim amounts is absolutely continuous [see LEMAIRE, REINHARD and VINCKE (1981)]. In this particular case all the comments relating to the coefficient of variation apply to the skewness coefficient.

We have already mentioned in Section 2.1 that both (2.1.1) and (2.1.2) hold if N has a Poisson or a Negative Binomial distribution. It is also worth mentioning that (2.1.1), but not (2.1.2), holds for any mixed Poisson distribution and that (2.1.2), but not (2.1.1), holds if N has a binomial distribution or is a degenerate random variable.

6. AN EXAMPLE

In this section we discuss a numerical example that illustrates the results in the previous sections.

We assume the gross aggregate claims are generated by a compound negative binomial distribution with

$$\lambda_1 = 10; \qquad \lambda_2 = 20; \qquad \lambda_3 = 60$$

and

$$F(x)=\begin{cases}0 & \text{if } x\leq 1\\ 1-x^{-4} & \text{if } x>1\end{cases}$$

so that individual claims have a Pareto distribution. We assume

$$P=24;\qquad e=0.35;\qquad B=1.7$$

and the premium loading factor, f, used in the calculation of the excess of loss reinsurance premium is 0.8, 0.45 and 0.4 when the premium calculation used is the expected value principle, the standard deviation principle and the variance principle respectively. Table 1 gives the point (a, M) which is the solution to problems 1 and 2 for various values of c and D. Note that from table 1, we can see that when $c=0.4$ the isocost curves for the three premium calculation principles are not decreasing functions with a.

TABLE 1

Excess of loss premium calculation principle	c	D	(a, M)	$V[W(a, M)]$	$\gamma[Y(a, M)]$	$CV[(a, M)]$
Expected value principle	0.4	33	(1; 1.676)	32.38	0.6763	0.4507
		27	(0.908; 1.57)	27	0.677	0.4511
	0.3	33	(1; 1.676)	32.38	0.6763	0.4507
		27	(0.863; 2.53)	27	0.6886	0.4562
Standard deviaton principle	0.4	33	(1; 1.497)	30.77	0.6743	0.4495
		27	(0.921; 1.48)	27	0.6755	0.4502
	0.3	33	(1; 1.497)	30.77	0.6743	0.4495
		27	(0.846; 18.6)	27	0.7153	0.4609
Variance principle	0.4	33	(0.9375; 1.47)	27.7	0.6751	0.4500
		27	(0.926; 1.46)	27	0.6751	0.4500
	0.3	33	(1; 1.575)	31.54	0.6752	0.4500
		33	(0.854; 3.42)	27	0.6952	0.4580

Let us now consider in more detail the case where $c=.3$ and the standard deviation principle is used for the excess of loss reinsurance premium. Figure 2 shows the variance and the skewness coefficient for seven different isocost curves, starting with $B=32/15$ and decreasing B in steps of 4/30 until we get $B=4/3$. Points with the same subscript correspond to points on the same isocost curve and the smaller the subscript, the higher the value of B. The points $I_1, I_2, \ldots, I_7$ correspond to pure excess of loss treaties and these points together with the points on the solid lines represent solutions of Problem 1 for some value of D. The dotted lines correspond to points that are never solutions to problem 1, although they are on the isocost curved considered. This is because, for example, points between I_5 and I_5'' have both greater skewness and greater variance than I_5.

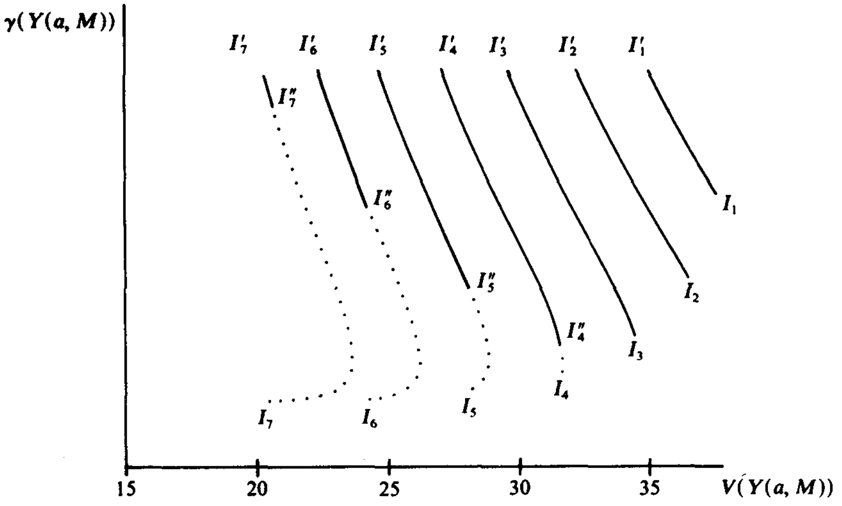

FIGURE 2. Isocost curves in the (V, γ)-plane.

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