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EMBEDDINGS OF FREE TOPOLOGICAL VECTOR SPACES

ARKADY LEIDERMAN and SIDNEY A. MORRIS✉

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Abstract

It is proved that the free topological vector space $\mathbb{V}([0,1])$ contains an isomorphic copy of the free topological vector space $\mathbb{V}([0,1]^n)$ for every finite-dimensional cube $[0,1]^n$, thereby answering an open question in the literature. We show that this result cannot be extended from the closed unit interval $[0,1]$ to general metrisable spaces. Indeed, we prove that the free topological vector space $\mathbb{V}(X)$ does not even have a vector subspace isomorphic as a topological vector space to $\mathbb{V}(X \oplus X)$, where X is a Cook continuum, which is a one-dimensional compact metric space. This is also shown to be the case for a rigid Bernstein set, which is a zero-dimensional subspace of the real line.

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1. Introduction and preliminary results

In 1976 Nickolas [25] proved that the free topological group on the closed unit interval $[0,1]$ has a topologically isomorphic copy of the free topological group on the n-dimensional cube $[0,1]^n$ as a closed subgroup, for every finite n. The analogous problem for free abelian topological groups remained open for two decades until it was shown to be true by Leiderman *et al.* [18] using the powerful Kolmogorov superposition theorem which answered Hilbert's 13th problem. The analogous problem for free topological vector spaces was the subject of the open question in [8]. Theorem 2.4, which is one of the main results of our paper, answers this question positively: for any finite-dimensional metrisable compact space X, the free topological vector space $\mathbb{V}(X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}[0,1]$. The proof depends on a 2011 result of Levin [19] which is also closely related to the Kolmogorov superposition theorem. Observe that the closed unit interval $[0,1]$ in Theorem 2.4 cannot be replaced by any zero-dimensional metrisable compact space, in view of Theorem 2.7.

All vector spaces considered in this paper are vector spaces over the field of real numbers. While we do consider non-Hausdorff topological groups and non-Hausdorff topological vector spaces, our focus will always be on those which are Hausdorff.

Definition 1.1 [22, 23]. A class $\mathfrak{V}$ of not necessarily Hausdorff topological groups is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, not necessarily Hausdorff quotient groups, and arbitrary products with the Tychonoff product topology.

The class of all topological groups and the class of all abelian topological groups are both varieties of topological groups as is the class of all boolean topological groups, that is, those topological groups G with the property that $x^2 = 1$, the identity, for all $x \in G$.

Definition 1.2 [6, 24]. A class $\mathfrak{V}$ of not necessarily Hausdorff topological vector spaces is said to be a *variety of topological vector spaces* if it is closed under the operations of taking subspaces, not necessarily Hausdorff quotient spaces, and arbitrary products with the Tychonoff product topology.

The following varieties are the object of extensive research: the variety of all topological vector spaces, the variety of all locally convex spaces and the variety of all locally convex spaces with the weak topology.

Definition 1.3 [4]. If C is a class of topological groups (or a class of topological vector spaces), then the variety generated by C is the smallest variety of topological groups (or topological vector spaces) that contains C.

Important examples include the variety of all locally convex spaces with the weak topology, which is the variety of topological vector spaces generated by $\mathbb{R}$, and the variety of locally convex spaces generated by each of the classical Banach spaces, ℓ_p, c_0, and so on.

Definition 1.4. If $\mathfrak{V}$ is a variety of topological groups (or a variety of topological vector spaces), then the *free topological group (or free topological vector space) of* $\mathfrak{V}$ on a Tychonoff space X is a topological group (or topological vector space) $F(X, \mathfrak{V})$ in $\mathfrak{V}$ containing the space X such that every continuous mapping f from X to a $G \in \mathfrak{V}$ gives rise to a unique continuous homomorphism (linear operator) $\bar{f} : F(X, \mathfrak{V}) \to G$ with $\bar{f}$ agreeing with f on X.

Definition 1.5 (See [6, 7, 28, 29]).

(i) If $\mathfrak{V}$ is the variety of all topological vector spaces, $F(X, \mathfrak{V})$ is said to be the *free topological vector space* on X and is denoted by $\mathbb{V}(X)$.
(ii) If $\mathfrak{V}$ is the variety of all locally convex spaces, then $\mathfrak{V}$ is said to be the *free locally convex space* on X and is denoted by $L(X)$.
(iii) If $\mathfrak{V}$ is the variety of all locally convex spaces with the weak topology, then $F(X, \mathfrak{V})$ is denoted by $L_p(X)$.
(iv) If $\mathfrak{V}$ is the variety of all topological groups, then $F(X, \mathfrak{V})$ is the *free topological group* on X and is denoted by $F(X)$.
(v) If $\mathfrak{V}$ is the variety of all abelian topological groups, then $F(X, \mathfrak{V})$ is the *free abelian topological group* on X and is denoted by $A(X)$.

(vi) If $\mathfrak{V}$ is the variety of all boolean topological groups, then $F(X, \mathfrak{V})$ is the *free boolean topological group* on X and is denoted by $B(X)$.

Remark 1.6. By the adjoint functor theorem [20] and Theorem 2.6 of [22], for any Tychonoff space X and variety $\mathfrak{V}$ which has a member containing X as a subspace, $F(X, \mathfrak{V})$ exists and is essentially unique (that is, unique up to isomorphism). Note that any variety which contains $\mathbb{R}^X$ has such a member. Further, $\mathbb{V}(X)$, $L(X)$, $L_p(X)$, $F(X)$, $A(X)$ and $B(X)$ are Hausdorff (see [7, 28, 29]).

If X is compact, then clearly it is a closed subset of $L_p(X)$, $L(X)$, $\mathbb{V}(X)$, $F(X)$, $A(X)$ and $B(X)$. Indeed by the Stone–Čech compactification argument in [10], for any Tychonoff space X, the space X is a closed subset of each of these. (For example, if X is any Tychonoff space and $\phi : X \to \beta X$ is the canonical one-to-one continuous map into its Stone–Čech compactification, there is a continuous linear operator Φ from $\mathbb{V}(X)$ to $\mathbb{V}(\beta X)$ which extends ϕ. As βX is compact, it is a closed subspace of $\mathbb{V}(\beta X)$ and then $\Phi^{-1}(\beta X) = X$ implies that X is a closed subspace of $\mathbb{V}(X)$.)

Definition 1.7. For a Tychonoff space X, let $C_p(X)$ denote the topological vector space of all continuous real-valued functions defined on X equipped with the pointwise convergence topology (or weak topology), that is, the topology it inherits as a subspace of the product space $\mathbb{R}^X$.

Notation 1.8. Let A be a subset of a vector space E and $n \in \mathbb{N}$. The subset $\mathrm{sp}_n(A)$ of E is defined by

$$\mathrm{sp}_n(A) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in [-n, n], x_i \in A \text{ for all } i = 1, \ldots, n\};$$

and the subset $\mathrm{SP}_n(A)$ of E is defined by

$$\mathrm{SP}_n(A) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \ x_i \in A \text{ for all } \lambda_i \in \mathbb{R} \text{ and } i = 1, \ldots, n\}.$$

Clearly, $\mathrm{sp}_n(A) \subseteq \mathrm{SP}_n(A)$. The *span* of A in E is the vector subspace $\mathrm{sp}(A)$ of E given by $\mathrm{sp}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{sp}_n(A) = \bigcup_{n\in\mathbb{N}} \mathrm{SP}_n(A)$.

Remark 1.9. By definition, the underlying vector space of $\mathbb{V}(X)$, $L(X)$ and $L_p(X)$ is the vector space with the Hamel basis X. Further, the topology $\tau(\mathbb{V}(X))$ of $\mathbb{V}(X)$ is finer than the topology $\tau(L(X))$ of $L(X)$ which is, in turn, finer than the topology $\tau(L_p(X))$ of $L_p(X)$. Thus, if X is compact, $\mathrm{sp}_n(X)$ is a compact subspace of $\mathbb{V}(X)$ and so $\mathrm{sp}_n(X)$ has precisely the same topology as a subspace of each of $\mathbb{V}(X)$, $L(X)$ and $L_p(X)$. Further, by the Stone–Čech compactification argument used in Remark 1.6, we see that, for any Tychonoff space X, $\mathrm{sp}_n(X)$ is a closed subspace of $\mathbb{V}(X)$, $L(X)$ and $L_p(X)$. Indeed, if $\mathfrak{V}$ is any variety of locally convex spaces, $\mathrm{sp}_n(X)$ is a closed subspace of the free topological vector space $F(X, \mathfrak{V})$ of $\mathfrak{V}$.

Definition 1.10. A Hausdorff topological space X is said to be a k_ω-*space* if X is the union of an increasing sequence $X_1, X_2, \ldots, X_n, \ldots$ of compact subspaces, with the property that a subset $A \subset X$ is closed in X if (and only if) $A \cap X_n$ is a closed subspace of X_n for each $n \in \mathbb{N}$. Further, in such a case, X is said to have k_ω-decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$.

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Theorem 1.11 [7]. *If X is a k_ω-space with k_ω-decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$, then $\mathbb{V}(X)$ is a k_ω-space with k_ω-decomposition $\mathbb{V}(X) = \bigcup_{n\in\mathbb{N}} \mathrm{sp}_n(X_n)$.*

Corollary 1.12. *Let Y be a k_ω-space and let X be a closed subspace of Y. Then $\mathrm{sp}(X)$, with the topology induced on it as a subspace of $\mathbb{V}(Y)$, is isomorphic as a topological vector space to $\mathbb{V}(X)$.*

Corollary 1.13. *Let X be a k_ω-space with k_ω-decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$. If K is a compact subspace of $\mathbb{V}(X)$, then $K \subseteq \mathrm{sp}_n(X_n)$ for some $n \in \mathbb{N}$.*

Proposition 1.14. *If X is any Tychonoff space, then $\mathrm{SP}_n(X)$ is a closed subspace of $\mathbb{V}(X)$ and also a closed subspace of $L(X)$. Indeed, if $\mathfrak{V}$ is any variety of locally convex spaces, then $\mathrm{SP}_n(X)$ is a closed subspace of the free topological vector space $F(X, \mathfrak{V})$ of $\mathfrak{V}$.*

Proof. Proposition 0.5.16 of [2] says that $\mathrm{SP}_n(X)$ is closed in $L_p(X)$. Therefore, it is closed in every finer topology on the underlying vector space. Thus $\mathrm{SP}_n(X)$ is closed in $\mathbb{V}(X)$ and in $L(X)$. Indeed, as $\mathfrak{V}(\mathbb{R})$ is the smallest nontrivial variety of locally convex spaces [6], $\mathrm{SP}_n(X)$ is a closed subspace of $F(X, \mathfrak{V})$. □

Remark 1.15. Clearly, it is important to describe explicitly, for any Tychonoff space X, the topology of $\mathrm{sp}_n(X)$ in $\mathbb{V}(X)$. Joiner [12] gave a useful description of the topology of the words of (reduced) length n in the free topological group and the free abelian topological group on a Tychonoff space X. A simpler proof of this result was given by Hardy, Morris and Thompson in [10] using Stone–Čech compactifications. Unknown to Joiner, Hardy, Morris and Thompson, A.V. Arhangel'skii proved a slightly different result much earlier but it appeared in a somewhat obscure rotary-printed book in Russian [1] and has not been translated into English. For further commentary, see [28].

It is natural to seek a similar result for free topological vector spaces. We address firstly the compact case.

Proposition 1.16. *Let X be a compact Hausdorff space. Consider an element $w \in \mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X) \subseteq \mathbb{V}(X)$ for $n \in \mathbb{N}$, where $w = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, each $\lambda_i \neq 0$, each $x_i \in X$ and $x_i \neq x_j$ for $i \neq j \in \{1, \ldots, n\}$. Put $M = |\lambda_1| + \cdots + |\lambda_n|$. A base of open neighbourhoods of w in $\mathrm{SP}_n(X)$ is the family of all subsets of the form $V_1U_1 + V_2U_2 + \cdots + V_nU_n$, where the sets U_i are pairwise disjoint open neighbourhoods of x_i in X, and $0 \notin V_i$ is an open neighbourhood of λ_i in $[-M, M]$ for $i = 1, \ldots, n$.*

Proof. Let U be any open neighbourhood in $\mathrm{SP}_n(X)$ of w. Then $U = U' \cap \mathrm{SP}_n(X)$ for U' an open neighbourhood of w in $\mathbb{V}(X)$. Since $\mathbb{V}(X)$ is a topological vector space, there exist an open neighbourhood $0 \notin V_i$ of λ_i in $[-M, M]$ and U'_i an open neighbourhood of x_i in $\mathbb{V}(X)$, for $i = 1, \ldots, n$, with $V_1U'_1 + V_2U'_2 + \cdots V_nU'_n \subseteq U'$. Put $U_i = U'_i \cap X$ for $i = 1, \ldots, n$. Then $V_1U_1 + V_2U_2 + \cdots + V_nU_n \subseteq U' \cap \mathrm{SP}_n(X) = U$.

To complete the proof we show that every set of the form $V_1U_1 + \cdots + V_nU_n$, where U_i is an open neighbourhood of x_i in X, the U_i are pairwise disjoint and $0 \notin V_i$ is an

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open neighbourhood of λ_i in $[-M, M]$, for $i = 1, \dots, n$, is an open neighbourhood in $\mathrm{SP}_n(X)$ of w.

Define the compact set $Z = [-M, M] \times X \times [-M, M] \times X \times \cdots \times [-M, M] \times X$ (that is, the product of n copies of $[-M, M] \times X$) and the surjective continuous map $\psi : Z \to \mathrm{SP}_n(X)$ by $\psi(\gamma_1, y_1, \gamma_2, y_2, \dots, \gamma_n, y_n) = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n$.

Let P be the set of all permutations of the finite set of integers $1, 2, \dots, n$. Put

$$A = \bigcup_{p \in P} V_{p(1)} \times U_{p(1)} \times \cdots \times V_{p(n)} \times U_{p(n)}.$$

Then A is an open subset of the compact space Z and A is saturated with respect to the mapping ψ. As ψ is surjective and $\psi(A) = V_1 U_1 + \cdots + V_n U_n$, it follows that $V_1 U_1 + \cdots + V_n U_n$ is an open subset of $\mathrm{SP}_n(X)$, as required. □

Next we generalise Proposition 1.16 from the compact case to a Tychonoff space X by applying the standard Stone–Čech compactification argument of [10].

Theorem 1.17. *Let X be a Tychonoff space and let $w \in \mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X) \subseteq \mathbb{V}(X)$, for $n \in \mathbb{N}$, where $w = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, each $\lambda_i \neq 0$, each $x_i \in X$ and $x_i \neq x_j$ for $i \neq j \in \{1, \dots, n\}$. Put $M = |\lambda_1| + \cdots + |\lambda_n|$. A base $\mathcal{B}$ of open neighbourhoods of w in $\mathrm{SP}_n(X)$ is the family of all subsets of the form $V_1 U_1 + V_2 U_2 + \cdots + V_n U_n$, where the sets U_i are pairwise disjoint open neighbourhoods of x_i in X, and $0 \notin V_i$ is an open neighbourhood of λ_i in $[-M, M]$ for $i = 1, \dots, n$.*

Proof. Recall that we defined the space Z as the product of n copies of $[-M, M] \times X$ and that ψ is the surjective continuous map $\psi : Z \to \mathrm{SP}_n(X)$ which is defined by $\psi(\gamma_1, y_1, \gamma_2, y_2, \dots, \gamma_n, y_n) = \gamma_1 y_1 + \gamma_2 y_2 + \cdots + \gamma_n y_n$.

Let βX be the Stone–Čech compactification of the Tychonoff space X, so that there exists an embedding $\beta : X \to \beta X$. As $\mathbb{V}(X)$ is a free topological vector space, the map β extends to a continuous linear operator $\overline{\beta} : \mathbb{V}(X) \to \mathbb{V}(\beta X)$.

Define the compact space

$$Y = [-M, M] \times \beta X \times [-M, M] \times \beta X \times \cdots \times [-M, M] \times \beta X$$

(that is, the product of n copies of $[-M, M] \times \beta X$), and let i be the natural embedding of the topological space Z into the compact space Y. Consider $\mathrm{SP}_n(\beta X) \subseteq \mathbb{V}(\beta X)$ and $\mathrm{SP}_n(X) \subseteq \mathbb{V}(X)$. Let $j : \mathrm{SP}_n(X) \to \mathrm{SP}_n(\beta X)$ be the restriction of $\overline{\beta}$ to $\mathrm{SP}_n(X)$. So we have the commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\phi"] & \mathrm{SP}_n(\beta X) \\ Z \arrow[u, "i"] \arrow[r, "\psi"'] & \mathrm{SP}_n(X) \arrow[u, "j"'] \end{tikzcd}$$

By Proposition 1.16, the family of all $V_1 U'_1 + V_2 U'_2 + \cdots + V_n U'_n$, where all U'_i are open pairwise disjoint neighbourhoods of x_i in βX and all $0 \notin V_i$ are

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open neighbourhoods of λ_i in $[-M, M]$, for $i = 1, \dots, n$, forms a basis of open neighbourhoods in $\mathrm{SP}_n(\beta X)$ of $w = \lambda_1 x_1 + \cdots + \lambda_n x_n$ in $\mathrm{SP}_n(\beta X)$. As

$$j^{-1}(V_1 U_1' + V_2 U_2' + \cdots + V_n U_n') = V_1 \beta^{-1}(U_1') + V_2 \beta^{-1}(U_2') + \cdots + V_n \beta^{-1}(U_n'),$$

the required result immediately follows. □

Remark 1.18. It follows from Remark 1.9 that Proposition 1.16 remains true if $\mathbb{V}(X)$ is replaced by $L(X)$ or $L_p(X)$. The proof of Theorem 1.17 is easily modified to show that it also remains true if $\mathbb{V}(X)$ is replaced by $L(X)$ or $L_p(X)$.

Remark 1.19. Arhangel'skii proved an analogous result to Theorem 1.17 for $L_p(X)$. (See Proposition 0.5.17 of [2].) From Arhangel'skii's result for $L_p(X)$, one can derive an alternative proof of Theorem 1.17.

Corollary 1.20. *In the notation of Theorem 1.17, consider the projection maps* $\phi_i : V_1 U_1 + V_2 U_2 + \cdots + V_n U_n \cap (\mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X)) \to U_i \subseteq X$, *where the maps* ϕ_i *are defined by* $\phi_i(\lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n) = z_i$ *and each* $z_i \in U_i$. *Then the* ϕ_i *are continuous mappings.*

Definition 1.21. A topological space is said to be a *continuum* if it is a compact metrisable connected space.

All continua considered in this paper are *nondegenerate*, meaning that they contain at least two points.

Definition 1.22 [5, 11]. A continuum X is said to be a *Cook continuum* if, for every subcontinuum $K \subseteq X$, the only continuous maps of K into X are the identity map and the constant maps.

Proposition 1.23 [16, Section 47, III.1]. *Every nonempty open subset of a continuum* X *contains a nondegenerate subcontinuum.*

Corollary 1.24. *Every nonempty open subset of a Cook continuum* X *contains a nondegenerate subset* M *such that* M *is a Cook continuum.*

A Cook continuum is a one-dimensional topological space. We shall also consider another special topological space known as a *rigid Bernstein set* B in the real line $\mathbb{R}$. The set B does not contain any interval and therefore, it is zero-dimensional. A detailed description of the properties of B and its construction can be found in [21, Example 6.13.1] (see also [14, 27]). According to Pol, the standard method of construction of B was originated by Kuratowski in 1925. The following strong rigidity property of B is, in fact, a consequence of [14, Corollary 3.3].

Proposition 1.25 [15]. *If* G *is an uncountable* G_δ*-subset of the rigid Bernstein set* B, *then, for each finite collection of continuous functions* $f_j : G \to B$, $j \in \{1, 2, \dots, n\}$, *there exists an uncountable* G_δ*-subset* G' *of* G *such that the restriction* $f_j \restriction G'$ *is either the identity or constant, for every* $j \in \{1, 2, \dots, n\}$.

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2. Embedding into the free topological vector space on [0, 1]

We begin with a lemma which follows easily from Theorem 1.11.

LEMMA 2.1. *Let Y be a compact Hausdorff space and let X be a compact subspace of $\mathbb{V}(Y)$ such that:*

(i) *the members of X are linearly independent; and*
(ii) $\mathrm{sp}(X)$ *is a closed vector subspace of* $\mathbb{V}(Y)$.

Then $\mathrm{sp}(X)$*, with the topology induced as a subspace of* $\mathbb{V}(Y)$*, is isomorphic as a topological vector space to* $\mathbb{V}(X)$.

PROPOSITION 2.2. *Let X and Y be compact Hausdorff spaces. Consider the following three properties.*

(i) *$L_p(X)$ is isomorphic as a topological vector space to a closed vector subspace of $L_p(Y)$ and there exists an $n \in \mathbb{N}$ with $X \subseteq \mathrm{sp}_n(Y)$.*
(ii) *$L(X)$ is isomorphic as a topological vector space to a closed vector subspace of $L(Y)$ and there exists an $n \in \mathbb{N}$ with $X \subseteq \mathrm{sp}_n(Y)$.*
(iii) *$\mathbb{V}(X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(Y)$.*

Then (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii).

PROOF. Without the additional assumption that there exists an $n \in \mathbb{N}$ with $X \subseteq \mathrm{sp}_n(Y)$, the equivalence of (i) and (ii) has been proved in [17] and [18].

Now assume that (i) is true. We note that, by Theorem 1.11, $\mathrm{sp}_n(Y)$ is a compact subspace of $\mathbb{V}(Y)$. Therefore, by Remark 1.9, $\mathrm{sp}_n(Y)$ has precisely the same compact topology as a subspace of each of $\mathbb{V}(Y)$, $L(Y)$ and $L_p(Y)$. So, by (i), X has the same compact topology as a subspace of $\mathrm{sp}_n(Y)$ in both spaces $L_p(Y)$ and $\mathbb{V}(Y)$. Further, by (ii), $\mathrm{sp}(X)$ is a closed subspace of $L_p(Y)$ and so $\mathrm{sp}(X)$ is a closed subspace of $\mathbb{V}(Y)$ as $\mathbb{V}(Y)$ has a finer topology than $L_p(Y)$. Lemma 2.1 now implies that (iii) is true. □

Recall that $\dim X$ stands for the covering dimension of a Tychonoff space X.

PROPOSITION 2.3. *Let X and Y be any Tychonoff spaces. If the free topological vector spaces $\mathbb{V}(X)$ and $\mathbb{V}(Y)$ are isomorphic, then $\dim X = \dim Y$.*

PROOF. For every Tychonoff space X, the topological vector spaces $\mathbb{V}(X)$ and $L(X)$ have the same continuous linear functionals [7]. Therefore, the dual space to $\mathbb{V}(X)$ equipped with the weak topology is isomorphic to $C_p(X)$. The function spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, and hence $\dim X = \dim Y$ by the celebrated Pestov result [26]. □

We denote by $\mathbb{I}$ the closed unit interval $[0, 1]$. Below we resolve positively the open problem posed in [8]. This result strikingly contrasts with Proposition 2.3.

THEOREM 2.4. *Let X be a finite-dimensional metrisable compact space. Then the free topological vector space $\mathbb{V}(X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(\mathbb{I})$.*

PROOF. According to Levin's theorem [19], for every n-dimensional metrisable compact space X there exists a linear continuous and **open** mapping T from $C_p(\mathbb{I})$ onto $C_p(X)$. As $L_p(X)$ is the dual space of $C_p(X)$, the dual mapping T^* is an isomorphic linear embedding of $L_p(X)$ into $L_p(\mathbb{I})$.

Since T is an open mapping, T^* is a closed mapping. So, as has been observed in [17], we can identify $L_p(X)$ with a **closed** linear subspace of $L_p(\mathbb{I})$. Furthermore, an analysis of the mapping T constructed in [19] shows that $T^*(X)$ is contained in the subspace

$$M_k(\mathbb{I}) = \{\lambda_1 t_1 + \cdots + \lambda_k t_k\} \subset L_p(\mathbb{I}),$$

where each $\lambda_i \in [-1, 1]$, $t_i \in \mathbb{I}$ and $k = k(n)$ depends only on the dimension n.

Therefore, condition (i) of Proposition 2.2 is satisfied with $Y = \mathbb{I}$. Thus, by Proposition 2.2, condition (iii) is satisfied, that is, $\mathbb{V}(X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(\mathbb{I})$. □

We immediately obtain the following result of [8].

COROLLARY 2.5. *Let $\mathbb{S}^k$ be the k-dimensional unit sphere. Then the free topological vector space $\mathbb{V}(\mathbb{S}^k)$ is isomorphic as a topological vector space to a closed vector subspace of $V(\mathbb{I}^k)$.*

COROLLARY 2.6. *Let $X = \bigoplus_{i\in\mathbb{N}} X_i$ be the free union of a countable family of the finite-dimensional metrisable compact spaces X_i. Then the free topological vector space $\mathbb{V}(X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(\mathbb{I})$.*

PROOF. For each $i \in \mathbb{N}$, let $\mathbb{I}_i$ be a homeomorphic copy of $\mathbb{I}$. Then, by Theorem 2.4, there is a topological vector space embedding of $\mathbb{V}(X_i)$ into $\mathbb{V}(\mathbb{I}_i)$. Therefore, $\mathbb{V}(\bigoplus_{i\in\mathbb{N}} X_i)$ embeds as a topological vector space into $\mathbb{V}(\bigoplus_{i\in\mathbb{N}} \mathbb{I}_i)$, which then embeds into $\mathbb{V}(\mathbb{R})$, which, in turn, embeds into $\mathbb{V}(\mathbb{I})$, by [8]. □

THEOREM 2.7 [8]. *Let X and Y be compact metrisable spaces and suppose that $\mathbb{V}(X)$ is isomorphic as a topological vector space to a vector subspace of $\mathbb{V}(Y)$. If Y is zero-dimensional, then X is zero-dimensional as well.*

Thus, the segment $\mathbb{I}$ in Theorem 2.4 cannot be replaced by any zero-dimensional metrisable compact space. Note also that, for $L(X)$, a more general result is known: if X and Y are compact spaces such that $L(X)$ is isomorphic as a topological vector space to a vector subspace of $L(Y)$, then $\dim Y = 0$ implies that $\dim X = 0$ [13].

Recall the following theorem which was proved in [18].

THEOREM 2.8. *For a Tychonoff space X, the following are equivalent.*

(i) *$A(X)$ embeds isomorphically as a topological subgroup into $A(\mathbb{I})$.*
(ii) *$F(X)$ embeds isomorphically as a topological subgroup into $F(\mathbb{I})$.*

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(iii) *X is a k_ω-space such that every compact subspace of X is finite-dimensional and metrisable.*

The statements of Theorems 2.4 and 2.8 suggest the following very interesting problem.

PROBLEM 2.9. Let X be a Tychonoff space. Is it true that $\mathbb{V}(X)$ embeds isomorphically as a topological vector space into $V(\mathbb{I})$ if and only if X is a k_ω-space such that every compact subspace of X is finite-dimensional and metrisable?

The necessity in Problem 2.9 is known.

REMARK 2.10. Let X be the one-point compactification of the disjoint union of the Euclidean cubes $\mathbb{I}^n$ for $n \in \mathbb{N}$. Then $L(X)$ embeds as a topological vector space into $L(\mathbb{I})$ (see [18, Remark 4.6] and [9]), but $\mathbb{V}(X)$ does not embed as a topological vector space into $\mathbb{V}(\mathbb{I})$ because X is not contained in $\mathrm{sp}_n(\mathbb{I})$ for any $n \in \mathbb{N}$.

EXAMPLE 2.11. Here we will discuss the conditions of Proposition 2.2. Denote by $I' = I'' = [0, 1]$. Define a mapping $T : C_p(I') \to C_p(I'')$ by

$$Tf(x) = 2f(x) - f\Big(\frac{x+1}{2}\Big).$$

Clearly, T is linear and continuous. It is proved in [17] that T is also a surjective and nonopen mapping. Therefore, the dual mapping T^* isomorphically embeds $L_p(I'')$ into $L_p(I')$ as a nonclosed vector subspace. For convenience, we identify $T^*(L_p(I''))$ with $L_p(I'')$ and $T^*(I'')$ with I''. Note that the second condition of item (i) in Proposition 2.2 is evidently fulfilled: $I'' \subseteq \mathrm{sp}_2(I')$. However, we show that $\mathrm{sp}(I'')$ with the topology induced from $\mathbb{V}(I')$ is not isomorphic to $\mathbb{V}(I'')$, which is opposite to the conclusion of Proposition 2.2. It is proved in [17] that, for every $m \in \mathbb{N}$, $x \in [0, 1]$, $f \in C(I')$, the following formula holds:

$$\sum_{n=0}^{m} \frac{1}{2^n} Tf\Big(\frac{x+2^n-1}{2^n}\Big) = 2f(x) - \frac{1}{2^m} f\Big(\frac{x+2^{m+1}-1}{2^{m+1}}\Big).$$

It means that $\mathrm{sp}_2(I') \cap \mathrm{sp}(I'')$ is not closed in $\mathbb{V}(I')$, and hence $\mathrm{sp}(I'')$ is not closed in $\mathbb{V}(I')$. Therefore, $\mathrm{sp}(I'')$ with the topology induced from $\mathbb{V}(I')$ is not isomorphic to $\mathbb{V}(I'')$ because $\mathbb{V}(I'')$ is a complete topological vector space, so $\mathbb{V}(I'')$ is closed in any topological vector space containing it.

It is natural now to ask the following questions.

PROBLEM 2.12. Let X and Y be metrisable compact spaces. Consider the following conditions.

(i) The free abelian topological group, $A(Y)$, on Y is isomorphic as a topological group to a closed subgroup of $A(X)$.
(ii) The free boolean topological group, $B(Y)$, on Y is isomorphic as a topological group to a closed subgroup of $B(X)$.

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(iii) The free topological vector space, $\mathbb{V}(Y)$, on Y is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(X)$.

Does (i) $\Longrightarrow$ (ii)? Does (ii) $\Longrightarrow$ (i)? Does (i) $\Longrightarrow$ (iii)? Does (iii) $\Longrightarrow$ (i)?

To the best of our knowledge, Problem 2.12 is open even for Tychonoff spaces X and Y.

PROBLEM 2.13. Let X be a finite-dimensional metrisable compact space. Is $B(X)$ isomorphic as a topological group to a subgroup of $B(\mathbb{I})$?

3. Embedding of $\mathbb{V}(X \oplus X)$ into $\mathbb{V}(X)$

Denote by $X \times X$ ($X \oplus X$) the square (the free topological sum) of two copies of a topological space X, respectively. The following proposition is an immediate consequence of Theorem 2.4 and Corollary 1.12.

PROPOSITION 3.1. *For every finite-dimensional metrisable compact space X containing a homeomorphic copy of the closed unit interval $\mathbb{I}$, the free topological vector space $\mathbb{V}(X \times X)$ is isomorphic as a topological vector space to a closed vector subspace of $\mathbb{V}(X)$.*

PROPOSITION 3.2. *For every infinite zero-dimensional metrisable locally compact space X, the free topological vector spaces $\mathbb{V}(X)$ and $\mathbb{V}(X \oplus X)$ are isomorphic as topological vector spaces.*

PROOF. It is easy to deduce from the main result of [3] that $F(X)$ and $F(X \oplus X)$ (and, therefore, also $A(X)$ and $A(X \oplus X)$) are isomorphic as topological groups for every infinite zero-dimensional metrisable locally compact space X.

Denote $Y = X \oplus X$. It follows that $A(X)$ and $A(Y)$ are isomorphic as topological groups. Let $\alpha : A(X) \to A(Y)$ be a witnessing isomorphism. Note that $A(X)$ and $A(Y)$ are topological subgroups of $\mathbb{V}(X)$ and $\mathbb{V}(Y)$, respectively. Then the restriction $\alpha{\restriction_X} : X \to A(Y) \subset \mathbb{V}(Y)$ is a continuous mapping, and thus it can be lifted to a continuous linear operator $T : \mathbb{V}(X) \to \mathbb{V}(Y)$. It is easy to see that T is surjective since α is surjective. Analogously, the inverse isomorphism $\alpha^{-1} : A(Y) \to A(X)$ gives rise to a continuous linear operator $S : \mathbb{V}(Y) \to \mathbb{V}(X)$. The mappings T^{-1} and S coincide on Y and so T^{-1} and S coincide on $\mathbb{V}(Y)$, which means that T is a topological isomorphism between $\mathbb{V}(X)$ and $\mathbb{V}(Y)$. □

It is perhaps surprising, then, that there exists a one-dimensional metrisable compact space X such that $A(X \oplus X)$ does not embed isomorphically into $A(X)$. In view of Proposition 3.2 a metrisable compact space X having such a property cannot be zero-dimensional. Nevertheless, a zero-dimensional space X which is a subspace of the reals $\mathbb{R}$, and such that $A(X \oplus X)$ does not embed isomorphically into $A(X)$, does exist.

These results have been proved in [15]. We shall now proceed to prove, in Theorem 3.4, the analogous result for free topological vector spaces. The proof is very similar to the free abelian topological group case of [15], but a little more complex. We include the details for completeness. However, we also point out clearly for the

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first time that the proof gives rather more than the bland statement that $V(X \oplus X)$ does not embed into $\mathbb{V}(X)$; it shows that $X \oplus X$ cannot be embedded homeomorphically in $\mathbb{V}(X)$ in such a way that it is a Hamel basis for the vector subspace it spans. This allows the result to be extended to $L(X)$ and $L_p(X)$.

Lemma 3.3. *Let $\alpha : M \to \mathbb{V}(X)$ be a homeomorphic embedding, where:*

(a) *$M \subseteq X$ and both X and M are Cook continua; or*

(b) *X is a rigid Bernstein set and M is an uncountable G_δ-subset of X.*

Then in case (a) there exists a Cook continuum $M_1 \subseteq M$ and in case (b) there exists an uncountable G_δ-subset M_1 of M, such that the following hold.

(i) *For some $n \in \mathbb{N}$, $\alpha(M_1) \subseteq \mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X)$.*

(ii) *$\alpha(M_1) \subseteq V_1U_1 + V_2U_2 + \cdots + V_nU_n$, where the latter set $V_1U_1 + \cdots + V_nU_n$ is a fixed member of the basis $\mathcal{B}$, as in the notation of Theorem 1.17.*

(iii) *For precisely one $i \in \{1, 2, \ldots, n\}$ and* **all** *$x \in M_1$, we have $x_i(x) = x$, where $\alpha(x) = \lambda_1(x)x_1(x) + \lambda_2(x)x_2(x) + \cdots + \lambda_n(x)x_n(x) \in V_1U_1 + V_2U_2 + \cdots + V_nU_n$. Without loss of generality one can assume that this is the case for $i = 1$.*

(iv) *For each $i \in \{2, \ldots, n\}$, there exists an element $z_i \in X$ such that, for* **all** *$x \in M_1$, we have $x_i(x) = z_i$.*

(v) *Denote by T the vector subspace $\mathrm{sp}\{z_2, z_3, \ldots, z_n\} \subset \mathbb{V}(X)$, where all z_i are fixed by the previous item (iv). Then, for* **all** *$x \in M_1$, there is a real number $\lambda(x)$ and an element $t(x) \in T$ such that $\alpha(x) = \lambda(x)x + t(x)$.*

Proof. Case (a). By Corollary 1.13, as $\alpha(M)$ is compact, we can choose the smallest natural number n such that $\alpha(M) \subset \mathrm{SP}_n(X)$. Define the set $W_n \subset M$ as $W_n = \alpha^{-1}(\alpha(M) \cap (\mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X)))$. Since $\alpha(M) \cap (\mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X))$ is an open subspace of the connected compact space $\alpha(M)$, the set W_n is uncountable and open in M. Further, consider the open cover of W_n by the sets $W_n \cap \alpha^{-1}(\alpha(M) \cap U)$, where $U \in \mathcal{B}$, as in the notation of Theorem 1.17. As W_n is a separable metrisable space, this cover has a countable subcover. Since W_n is uncountable, at least one member G of the subcover must also be uncountable. So G is a nontrivial open subset of the Cook continuum M.

By Corollary 1.24, there exists a subspace M_1 of $G \subseteq M$ such that M_1 is a Cook continuum. Clearly, (i) and (ii) are true, namely, $\alpha(M_1) \subset U \cap (\mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X))$ and $U = V_1U_1 + V_2U_2 + \cdots + V_nU_n \in \mathcal{B}$.

Let $\phi_j : V_1U_1 + V_2U_2 + \cdots + V_nU_n \to X$, for each $j \in \{1, 2, \ldots, n\}$, be defined by the following rule: if $x = \lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n \in V_1U_1 + V_2U_2 + \cdots + V_nU_n$, then $\phi_j(x) = x_j$. Then every composition map $\phi_j \circ \alpha \restriction_{M_1} : M_1 \to X$ is a continuous map of the Cook continuum $M_1 \subseteq X$ into X, where $\alpha \restriction_{M_1}$ is the restriction of α to M_1.

Since M_1 is a Cook continuum, each $\phi_j \circ \alpha \restriction_{M_1}$ is either the identity map or a constant map. Further, as α is a homeomorphism, so too $\alpha \restriction_{M_1}$ is a homeomorphism of M_1 onto its image. Thus, not all $\phi_j \circ \alpha \restriction_{M_1}$ can be constant maps. As the open neighbourhoods U_i are assumed to be disjoint, we conclude that $\phi_i \circ \alpha \restriction_{M_1}$ is the identity map, that is, $x_i(x) = x$ for all $x \in M_1$, for precisely one $i \in \{1, 2, \ldots, n\}$. Without loss of generality, one can assume that $i = 1$. So $x_1(x) = x$ for all $x \in M_1$. This proves (iii).

Now, for each $j \in \{2, \ldots, n\}$, $\phi_j \circ \alpha \restriction_{M_1} (M_1) = \{z_j\}$ for some fixed $z_j \in X$. This proves (iv). Finally, $\alpha(x) = \lambda_1(x)x + \lambda_2(x)z_2 + \cdots + \lambda_n(x)z_n = \lambda(x)x + t(x)$, where $\lambda(x) = \lambda_1(x)$, $t(x) \in T$. Thus, we obtain (v), which completes the proof of the Lemma in case (a).

Case (b). For each $n \in \mathbb{N}$, consider $W_n = \alpha^{-1}(\alpha(M) \cap (\mathrm{SP}_n(X) \setminus \mathrm{SP}_{n-1}(X)))$. The set W_n can be expressed as the difference of two closed subsets in a separable metrisable set. Therefore, every W_n is a G_δ-subset of M.

Since M is uncountable, we can choose n such that W_n is uncountable. Exactly as in case (a) we find an open uncountable $G \subseteq W_n$ such that $\alpha(G)$ is contained in some $U \in \mathcal{B}$.

Now the finite collection of continuous functions $\phi_j \circ \alpha \restriction_G : G \to X$, $j \in \{1, 2, \ldots, n\}$ is correctly defined. Therefore, by Proposition 1.25, there exists an uncountable G_δ-subset M_1 of $G \subseteq M$ such that every $\phi_j \circ \alpha \restriction_{M_1}$ is either the identity map or a constant map. It is easy to see that (i) and (ii) are true, and we complete the proof of (iii) exactly as in case (a). Items (iv) and (v) also can be proved without any change in the arguments, which finishes the proof of the Lemma in case (b). □

Theorem 3.4. *Let X be either a Cook continuum or a rigid Bernstein set. Then the free topological vector space $\mathbb{V}(X \oplus X)$ is not isomorphic as a topological vector space to a vector subspace of $\mathbb{V}(X)$. Indeed, the topological space $X \oplus X$ cannot be homeomorphically embedded in $\mathbb{V}(X)$ such that it is a Hamel basis for the vector space it spans.*

Proof. Consider case (a): X is a Cook continuum; case (b): X is a rigid Bernstein set. In both cases let Y' and Y'' be two copies of the space X and denote by $h' : X \to Y'$ and $h'' : X \to Y''$ the witnessing homeomorphisms. Assume that $Z = Y' \cup Y''$ is a subspace of $\mathbb{V}(X)$ with $Y' \cap Y'' = \emptyset$: that is, Z is homeomorphic to the free sum $X \oplus X$. We shall prove that Z is not a Hamel basis for the vector subspace of $\mathbb{V}(X)$ that it generates.

Firstly, we apply Lemma 3.3 with $M = X$. So there are:

- a natural number n and a Cook continuum $M_1 \subseteq M$ (in case (a)); and
- an uncountable G_δ-subset M_1 of M (in case (b));

such that, for **all** $x \in M_1$, $h'(x) = \lambda(x)x + t(x)$, where $\lambda(x)$ is a real number, and $t(x) \in T$. Note that T is a fixed $n - 1$-dimensional vector subspace of $\mathbb{V}(X)$.

Next we apply Lemma 3.3, with $M = M_1$. So there are

- a natural number k and a Cook continuum $M_2 \subseteq M_1$ (case (a));
- an uncountable G_δ-subset M_2 of M_1 (case (b))

such that, for **all** $x \in M_2$, $h''(x) = \gamma(x)x + q(x)$, where $\gamma(x)$ is a real number and $q(x) \in Q$. Here Q is a fixed $(k - 1)$-dimensional vector subspace of $\mathbb{V}(X)$. Denote also by S the vector subspace $T + Q$ of $\mathbb{V}(X)$. Then $\dim S \leq n + k - 2$.

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Let m be any point in M_2. Put $w' = h'(m) \in \mathrm{sp}(Y')$, $w'' = h''(m) \in \mathrm{sp}(Y'')$.

Then $\gamma(m)w' - \lambda(m)w'' = \gamma(m)\lambda(m)m + \gamma(m)t(m) - \lambda(m)\gamma(m)m - \lambda(m)q(m) = \gamma(m)t(m) - \lambda(m)q(m)$. Thus, $s = \gamma(m)w' - \lambda(m)w'' \in S$. Now we take at least $n + k - 1$ distinct points $m_1, m_2, \ldots, m_D$ in M_2. Put $w'_i = h'(m_i) \in \mathrm{sp}(Y')$, $w''_i = h''(m_i) \in \mathrm{sp}(Y'')$ for each $i = 1, 2, \ldots, D$. Then we obtain D distinct points $s_i = \gamma(m_i)w'_i - \lambda(m_i)w''_i$ in the vector subspace S. Since $D \geq \dim S + 1$, we conclude that the elements $\{s_i : i = 1, 2, \ldots, D\}$ are not linearly independent.

Therefore, the elements $w'_1, w''_1, \ldots, w'_D, w''_D$ are not linearly independent as well. But all $w'_1, w''_1, \ldots, w'_D, w''_D$ are distinct points in $Z = Y' \cup Y''$. Hence Z is not a Hamel basis for the vector subspace that it spans. So $\mathrm{sp}(Y' \cup Y'')$ is not isomorphic as a topological vector space to $\mathbb{V}(Y' \oplus Y'')$. □

We note that Theorem 3.4 says more than $\mathbb{V}(X \oplus X)$ is not isomorphic as a topological vector space to a vector subspace of $\mathbb{V}(X)$. It says that **any** topological vector space which has the topological space $X \oplus X$ as a Hamel basis is not isomorphic as a topological vector space to a vector subspace of $\mathbb{V}(X)$.

As a corollary to Theorem 3.4 and its proof we obtain the following strengthening of the result from [15].

Theorem 3.5. *Let X be either a Cook continuum or a rigid Bernstein set. Then $L(X \oplus X)$ is not isomorphic as a topological vector space to a vector subspace of $L(X)$ and $L_p(X \oplus X)$ is not isomorphic as a topological vector space to a vector subspace of $L_p(X)$. Indeed the topological space $X \oplus X$ cannot be homeomorphically embedded in $L(X)$ or $L_p(X)$ in such a way that it is a Hamel basis for the vector space it spans.*

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ARKADY LEIDERMAN, Department of Mathematics,
Ben-Gurion University of the Negev, Beer Sheva, P.O.B. 653, Israel
e-mail: arkady@math.bgu.ac.il

SIDNEY A. MORRIS, Centre for Informatics and Applied Optimization,
Federation University, Australia, P.O.B. 663, Ballarat,
Victoria, 3353, Australia
and
Department of Mathematics and Statistics,
La Trobe University, Melbourne, Victoria, 3086, Australia
e-mail: morris.sidney@gmail.com