ON FINITE GROUPS WHOSE 2-SYLOW SUBGROUPS HAVE CYCLIC SUBGROUPS OF INDEX 2 *

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Introduction

If the finite group G has a 2-Sylow subgroup S of order 2^{a+1}, containing a cyclic subgroup of index 2, then in general S may be one of the following six types [8]:

(i) cyclic;

(ii) Abelian of type $(a, 1)$, $a > 1$;

(iii) dihedral [1];

(iv) generalized quaternion;

(v) $\{\alpha, \beta\}$, $\alpha^{2^a} = \beta^2 = 1$, $\alpha^\beta = \alpha^{2^{a-1}+1}$, $a \geqq 3$;

(vi) $\{\alpha, \beta\}$, $\alpha^{2^a} = \beta^2 = 1$, $\alpha^\beta = \alpha^{2^{a-1}-1}$, $a \geqq 3$.

In cases (i)and (ii), Burnside's theorem shows that G has a normal 2-complement. Case (iii) is of considerable interest, as it occurs with the simple groups $PSL(2, q)$, and has been extensively treated (see the bibliography in [7]). Case (iv) has been dealt with in [5]. In this paper we consider the two remaining cases.

In case (v), G is easily shown to have a normal 2-complement. This is done in § 1. Case (vi) is more interesting (and more difficult). Specific results can be obtained if additional assumptions are made. The main result of the paper is a determination of the structure of G when the centralizer of an involution has an Abelian 2-complement. In particular, it is shown that the only simple groups then occurring are the finite projective group $PSL(3, 3)$ and the Mathieu group M_{11} on 11 letters. These results are obtained in § 4, and two applications are given in § 5.

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[1] The non-cyclic group of order 4 is to be understood as dihedral.

1.

We begin by recalling some useful facts. If S is a p-Sylow subgroup of the finite group G, the *focal group* S^* of S in G is the group generated by all quotients $\sigma'\sigma^{-1}$, where σ and σ' are elements of S conjugate in G. The main property of S^* is the following (see [10]):

S^* is normal in S, and S/S^* is isomorphic to the largest Abelian p-factor group of G.

THEOREM 1. *Let G be a finite group with 2-Sylow subgroup of form*

$$S = \{\alpha, \beta\}, \alpha^{2^a} = \beta^2 = 1, \alpha^\beta = \alpha^{2^{a-1}+1}, a \geqq 3.$$

Then, G has a normal 2-complement.

PROOF. It is easily seen that the elements of S of order 2^b are, for $b > 1$, the elements of form

$$\alpha^{2^{a-b}+n2^{a-b+1}} \quad \text{or} \quad \alpha^{2^{a-b}+n2^{a-b+1}}\beta,$$

where n is an integer. For $b = 1$, we have also the element β. It follows that if σ and σ' are elements of the same order in S then $\sigma'\sigma^{-1}$ is either an even power of α, or the product of such a power with β. Thus the focal group S^* is contained in $\{\alpha^2, \beta\}$, and so is a proper subgroup of S. Thus, G has a non-trivial Abelian 2-factor group, and we can find a normal subgroup H of index 2 in G. H has as 2-Sylow subgroup a subgroup T of index 2 in S. Thus, $T = \{\alpha^2, \gamma\}$, where γ is an element of S. Since α^2 lies in the centre of S, T is Abelian, either cyclic or of type $(a-1,1)$. Since $a-1 > 1$, it follows that the automorphism group of T is a 2-group. Burnside's theorem [8] yields a normal 2-complement for H, and this is also a normal 2-complement for G.

2.

From now on G will always denote a finite group with 2-Sylow subgroup of the form

$$S = \{\alpha, \beta\}, \alpha^{2^a} = \beta^2 = 1, \alpha^\beta = \alpha^{2^{a-1}-1}, a \geqq 3.$$

We put $\tau = \alpha^{2^{a-1}}$, $\pi = \alpha^{2^{a-2}}$, and write $\rho \sim \sigma$ to mean that ρ is conjugate to σ in G, $\rho \nsim \sigma$ for the negation of this statement.

LEMMA 1. *The focal group of S in G is given by*

$$S^* = \begin{cases} \text{(i)} & \{\alpha^2\}, & \textit{if} \ \alpha\beta \nsim \pi \ \textit{and} \ \beta \nsim \tau; \\ \text{(ii)} & \{\alpha^2, \beta\}, & \textit{if} \ \alpha\beta \nsim \pi \ \textit{and} \ \beta \sim \tau; \\ \text{(iii)} & \{\alpha^2, \alpha\beta\}, & \textit{if} \ \alpha\beta \sim \pi \ \textit{and} \ \beta \nsim \tau; \\ \text{(iv)} & S, & \textit{if} \ \alpha\beta \sim \pi \ \textit{and} \ \beta \sim \tau. \end{cases}$$

Proof. The elements of S not in $\{\alpha\}$ are $\beta, \alpha^2\beta, \alpha^4\beta, \cdots$, forming a conjugacy class of S of elements of order 2, and $\alpha\beta, \alpha^3\beta, \alpha^5\beta, \cdots$, forming another conjugacy class of elements of order 4. An even power and an odd power of α cannot be conjugate in G, as they are of different orders. The two elements π and π^{-1} of order 4 in $\{\alpha\}$ are conjugate in S. It is now easy to calculate S, with the results stated.

Theorem 2. *Let G be a finite group with 2-Sylow subgroup of the form*

$$S = \{\alpha, \beta\},\ \alpha^{2^a} = \beta^2 = 1,\ \alpha^\beta = \alpha^{2^{a-1}-1},\ a \geqq 3.$$

Then, one of the following holds:

I. *G has a normal 2-complement.*

II. *G has a normal subgroup of index 2, which has no normal subgroup of index 2 and has dihedral 2-Sylow subgroup.*

III. *G has a normal subgroup of index 2, which has no normal subgroup of index 2 and has 2-Sylow subgroup of generalized quaternion type.*

IV. *G has no normal subgroup of index 2, the involutions of G form a single conjugacy class in G, and the centralizer in G of any involution is a group of type* III.

Proof. Let G/G_2 be the largest Abelian 2-factor group of G. As stated previously, this is isomorphic to S/S^*. G_2 has S^* as 2-Sylow subgroup.

If case (i) of Lemma 1 holds, S^* is cyclic, so that by Burnside's theorem G_2 has a normal 2-complement, which is a normal 2-complement for G.

If cases (ii) or (iii) hold, then $(G : G_2) = 2$, and G_2 can have no normal subgroup of index 2, for otherwise G_2 would have a proper characteristic subgroup K such that G_2/K is a 2-group. Then G/K would be a 2-group of order exceeding 2, and G would have a factor group of order 4, a contradiction. Since $\{\alpha^2, \beta\}$ is dihedral and $\{\alpha^2, \alpha\beta\}$ is of generalized quaternion type, we have the alternatives II, III asserted.

If case (iv) of Lemma 1 holds, we have $G = G_2$, and it remains to verify that the centralizer $C(\tau)$ in G of the involution τ is of type III. $C(\tau)$ contains S since $\tau \in C(S)$. Now, $\alpha\beta$ is conjugate in G to π:

$$\alpha\beta = \pi^\mu, \quad \mu \in G.$$

Since $(\alpha\beta)^2 = \tau = \pi^2$, we have $\tau = \tau^\mu$, i.e. $\mu \in C(\tau)$, and $\alpha\beta$ is conjugate in $C(\tau)$ to π. Since β is not conjugate to τ in $C(\tau)$, case (iii) of Lemma 1 applies to $C(\tau)$, so that $C(\tau)$ is of type III.

3.

3. This section is devoted to giving some examples of finite groups with 2-Sylow subgroup S of the type being discussed, in which the centralizer of the involution τ in the centre of S has an Abelian 2-complement.

(1). S itself.

(2). If $q = r^2$, where r is a power of an odd prime number, we define a group $H(q)$ in the following way: $H(q)$ is the subgroup of the group $P\Gamma L(2, q)$ of all one-dimensional projective semi-linear transformations over $GF(q)$ (cf. [6], where these are called projective collineations) generated by $PSL(2, q)$ and a semi-linear transformation α relative to the automorphism

$$\sigma : x \to x^r$$

of $GF(q)$ of order 2. α is defined by taking a basis and letting α be represented by the semi-linear transformation relative to σ having matrix

$$T = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix},$$

according as $r \equiv 1 \pmod 4$, or $r \equiv -1 \pmod 4$, where b is an element of $GF(q)$ having multiplicative order 2^a, the exact power of 2 dividing $q-1$.

α^2 is the projective linear transformation represented by the matrix

$$T^\sigma T = \begin{pmatrix} b^{r+1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} b^{r-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $r+1$ (respectively $r-1$) is exactly divisible by 2, it follows that $\alpha^2 \in PSL(2, q)$ and that α has order 2^a. Since $PSL(2, q)$ is normal in $P\Gamma L(2, q)$, $H(q)$ is an extension of $PSL(2, q)$ by a group of order 2. Thus $H(q)$ has order $(q-1)q(q+1)$, and has 2-Sylow subgroup S of order 2^{a+1}.

Let β be the involution in $PSL(2, q)$ represented by the matrix

$$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, $\alpha^\beta \alpha$ is represented by the matrix

$$V^{-1} T^\sigma V T = \begin{pmatrix} b & 0 \\ 0 & b^r \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & -b^{r+1} \end{pmatrix}.$$

Since $b^{r-1} = -1$ in the first case, and $b^{r+1} = -1$ in the second case, $\alpha^\beta \alpha$ may be represented by the diagonal matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But, this is equal to $(T^\sigma T)^{2^{a-2}}$, which represents $\alpha^{2^{a-1}}$. Thus,

$$\alpha^\beta = \alpha^{2^{a-1}-1},$$

and so $\{\alpha, \beta\}$ is a 2-Sylow subgroup of $H(q)$, and is of the required type. (We note that $q = r^2 \equiv 1 \pmod 8$, so that $a \geqq 3$.)

The centralizer of $\tau = \alpha^{2^{a-1}}$ in $PSL(2, q)$ has cyclic 2-complement, and this is a 2-complement of the centralizer of τ in $H(q)$, since $(H(q) : PSL(2, q)) = 2$.

(3). $GL(2, 3)$ is a group of order 48 whose 2-Sylow subgroup S is generated by

$$\alpha = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easily checked that $\alpha^8 = \beta^2 = 1$, $\alpha^\beta = \alpha^3$, so that S is of the required type. The centralizer of $\tau = \alpha^4$ is the whole of $GL(2, 3)$, which has a cyclic 2-complement.

(4). $PSL(3, 3) = SL(3,3)$ is a group of order 5616. If τ is an involution in this group, then with respect to a suitable basis τ has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The elements of $C(\tau)$ are then represented by matrices

$$\begin{pmatrix} f^{-1} & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix},$$

where $\begin{pmatrix} b & c \\ d & e \end{pmatrix}$ is an element of $GL(2, 3)$ and f is its determinant. Clearly $C(\tau)$ is isomorphic with $GL(2, 3)$, and so $PSL(3, 3)$ is a group of the required type.

(5) The Mathieu group M_{11} of order 7920 is a quadruply transitive permutation group of degree 11. It may be regarded as a transitive extension of the group $H(9)$ taken as acting on the points of a projective line L over $GF(9)$ [15]. In particular, the 2-Sylow subgroup of M_{11} is a group of order 16 of the required type. If τ is an involution of $H(9)$, τ has two fixed points in L, and so three fixed points a, b, c as an element of M_{11}. The subgroup fixing a, b, c is a quaternion group and so contains only one involution. Hence $C(\tau)$ consists of all permutations of M_{11} permuting a, b, c amongst themselves, and so, by triple transitivity, its order is 48. Thus $C(\tau)$ has a cyclic 2-complement. (In fact, $C(\tau)$ can be shown to be isomorphic with $GL(2, 3)$).

4.

4. We now assume that

(*) *G is a finite group with 2-Sylow subgroup*

$$S = \{\alpha, \beta\}, \quad \alpha^{2^a} = \beta^2 = 1, \quad \alpha^\beta = \alpha^{2^{a-1}-1}, \quad a \geqq 3,$$

such that, for $\tau = \alpha^{2^{a-1}}$, $C(\tau)$ has an Abelian 2-complement.

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Let K be the largest normal odd order subgroup of G. We shall prove that G/K must be isomorphic to one of the groups of § 3.

LEMMA 2. *A finite group H with an Abelian 2-complement is solvable, and every subgroup and quotient group of H has an Abelian 2-complement.*

PROOF. Let S be a 2-Sylow subgroup and C an Abelian 2-complement of H. Since S and C are nilpotent of relatively prime orders and $H = SC$, H is solvable, by a theorem of Wielandt [14]. If L is any subgroup of H, then L is solvable, and so by Hall's extension of Sylow's theorems [9], L has a 2-complement D. Also, D is conjugate to a subgroup of C and so is Abelian. If N is any normal subgroup of G, then $(G : CN)$ is a divisor of $(G : C)$ and so is a power of 2. CN/N is isomorphic to $C/C \cap N$ and so is Abelian of odd order, and is an Abelian 2-complement of G/N.

LEMMA 3. *If G is a group satisfying the condition $(*)$ and K is any normal subgroup of odd order in G, then G/K also satisfies $(*)$.*

PROOF. Let $\bar{\tau} = \tau K$. We need only prove that the centralizer $C(\bar{\tau})$ of $\bar{\tau}$ in G/K has an Abelian 2-complement. If $C(\bar{\tau}) = L/K$, then $\{\tau\}K$ is normal in L, since L centralizes $\tau(\text{mod}\, K)$. Hence, if $\lambda \in L$, $\{\tau\}$ and $\{\tau^\lambda\}$ are 2-Sylow subgroups of $\{\tau\}K$, and so

$$\tau^\lambda = \tau^\mu,$$

for some $\mu \in K$. Thus, $\lambda \in C(\tau)\mu \leqq C(\tau)K$. Since $C(\tau) \leqq L$, we have $L = C(\tau)K$, and so $C(\bar{\tau})$ is isomorphic to $C(\tau)/C(\tau) \cap K$. The result now follows from Lemma 2.

Using this lemma, we can assume also that

$(**)$ *G has no non-trivial normal subgroup of odd order.*

We now consider the cases II, III, IV of Theorem 2 in turn.

Case II. G has a normal subgroup G_2 of index 2 such that G_2 has no normal subgroup of index 2, and has dihedral 2-Sylow subgroup $\{\alpha^2, \beta\}$ of order 2^a. By Lemma 2, the centralizer of τ in G_2 has an Abelian 2-complement. The largest odd order normal subgroup of G_2 is normal in G, and so is trivial, by the assumption $(**)$. By a theorem of Gorenstein and Walter [7], G_2 is isomorphic with the alternating group A_7 of degree 7, or with $PSL(2, q)$, for some odd prime power q. The first case is impossible, as α would induce an automorphism of order 8 in G_2, contradicting the fact that none of the automorphisms of A_7 (which may all be regarded as induced by elements of the symmetric group S_7) is of order 8. We may therefore identify G_2 with $PSL(2, q)$.

The automorphisms of $PSL(2, q)$ are all obtained by conjugation of $SL(2, q)$ by semi-linear transformations (cf. [6]; contragredient transforma-

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tion of $SL(2, q)$ can easily be seen to be equivalent with conjugation by a semi-linear transformation). Let θ be a semi-linear transformation inducing the same automorphism of G_2 as α does. θ^2 induces the same automorphism of G_2 as α^2, which is represented by a linear transformation. Since $PSL(2, q)$ has trivial centralizer in the group $P\Gamma L(2, q)$, it follows that θ^2 is linear. Hence, if σ is the automorphism of $GF(q)$ associated with θ, then $\sigma^2 = 1$.

If $\sigma = 1$, G would be isomorphic with a subgroup of $PGL(2, q)$, and so would have dihedral 2-Sylow subgroup, a contradiction.

Thus, σ is of order 2. Then $q \equiv 1 \pmod 8$, since $q = r^2$, where $GF(r)$ is the fixed field of σ. This implies that 2^a is the exact power of 2 dividing $q-1$. The involutions τ, β can be represented by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If θ has matrix form T, then since θ leaves τ fixed and $\tau^\sigma = \tau$, T must commute (projectively) with τ. It follows that we may take

$$T = \text{(i)} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or (ii)} \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

If case (i) holds, then $\alpha^{2^{a-1}-2} = \beta^\alpha \beta = \beta^\theta \beta = \beta^T \beta$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & b^2 \end{pmatrix}.$$

Since its order is 2^{a-1}, b is an element of $GF(q)$ having multiplicative order 2^a. Now, α^2 is represented by the matrix

$$T^\sigma T = \begin{pmatrix} b^{r+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since this has order 2^{a-1}, $r+1$ is not divisible by 4. Hence $r \equiv 1 \pmod 4$, and we have that G is isomorphic with the group $H(q)$ defined in § 3.

Case (ii) gives the same result in the same way.

Case III. G has a normal subgroup G_2 of index 2 such that G_2 has no normal subgroup of index 2, and has generalized quaternion 2-Sylow subgroup $\{\alpha^2, \alpha\beta\}$ of order 2^a. Again G_2 has no non-trivial normal subgroup of odd order, by the assumption (**). By a theorem of Brauer and Suzuki [5], G_2 has only one involution τ.

$T = \{\tau\}$ is normal in G, and so, by the assumption (*), G has an Abelian 2-complement. G/T has dihedral 2-Sylow subgroup, and, by Lemma 2, the centralizer of an involution in G/T has an Abelian 2-complement. By the result of Gorenstein and Walter, G/T has an odd order normal subgroup N/T such that G/N is isomorphic with $PGL(2, q)$, for some odd q. By Burnside's theorem, N has a normal 2-complement V, which is normal in G. By the assumption (**), V is trivial, so that $N = T$. By Lemma 2, G is solvable, and so $q = 3$. Hence there is an isomorphism

$$\theta : G/T \to PGL(2, 3).$$

Since θ maps G_2/T on $PSL(2, 3)$, and G_2 has only one involution, it follows from a result of Schur [12] that the restriction of θ to G_2/T is induced by an isomorphism of G_2 on $SL(2, 3)$. We can identify G_2 with $SL(2, 3)$, so that θ is the identity map on G_2/T. The element $(\beta T)^\theta$ of $PGL(2,3)$ can be represented by an element $\bar{\beta}$ of $GL(2, 3)$. Now β and $\bar{\beta}$ induce the same automorphism on $G_2 = SL(2, 3)$ since they induce the same automorphism on $G_2/T = PSL(2, 3)$, and no two automorphisms of $SL(2, 3)$ give the same automorphism of $PSL(2, 3)$. Since $\bar{\beta}^2$ induces the same inner automorphism of $SL(2, 3)$ as $\beta^2 = 1$, $\bar{\beta}^2$ lies in the centre $\{\tau\}$ of $SL(2, 3)$. If $\bar{\beta}^2 = \tau$, the 2-Sylow subgroup of $GL(2, 3)$ would be of generalized quaternion type, which is not so. Hence $\bar{\beta}^2 = 1$, and G is isomorphic to $GL(2, 3)$.

Case IV. G has no normal subgroup of index 2, the involutions of G are all conjugate in G, and the centralizer $H = C(\tau)$ is a group of type III. By Lemma 3 and what we have just proved, H has a normal subgroup A of odd order such that

$$H/A \approx GL(2, 3). \tag{1}$$

The Abelian 2-complement of H contains A, and so A is Abelian. We shall show that A is trivial.

The irreducible characters of H with kernel containing A (i.e. the irreducible characters of H/A) are given by Table 1, in which ρ is an element of H whose coset $\bar{\rho}$ with respect to A has order 3, and ω is a square root of -2.

TABLE 1

	1	τ	α^2	ρ	$\rho\tau$	β	α	α^{-1}
φ_0	1	1	1	1	1	1	1	1
φ_1	1	1	1	1	1	-1	-1	-1
φ_2	2	2	2	-1	-1	0	0	0
φ_3	3	3	-1	0	0	-1	1	1
φ_4	3	3	-1	0	0	1	-1	-1
φ_5	4	-4	0	1	-1	0	0	0
φ_6	2	-2	0	-1	1	0	ω	$-\omega$
φ_7	2	-2	0	-1	1	0	$-\omega$	ω

Let D be the set of all roots of τ in G, i.e. the elements σ of G such that τ is a power of σ. D is a subset of H. If in general the coset of an element σ of H with respect to A is denoted by $\bar{\sigma}$, then because A has odd order, σ is a root

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of τ if and only if $\bar{\sigma}$ is a root of $\bar{\tau}$. The classes of H/A consisting of roots of $\bar{\tau}$ are those represented by the cosets of τ, α^2, $\rho\tau$, α and α^{-1}. Now it is easily checked that the module of generalized characters of H/A which vanish on $H-D$ has as a basis the generalized characters

$$\begin{aligned}(2)\qquad \Phi_1 &= \varphi_0+\varphi_2-\varphi_4,\\ \Phi_2 &= \varphi_2-\varphi_6,\\ \Phi_3 &= \varphi_6-\varphi_7,\\ \Phi_4 &= \varphi_1+\varphi_4-\varphi_5,\\ \Phi_5 &= \varphi_1+\varphi_2-\varphi_3.\end{aligned}$$

Denote by $\chi_i(I)$ the sum of the values on all the involutions of G of the (ordinary) irreducible character χ_i of G, and by $\varphi_j(J)$ the sum of the values on all the involutions of H of the character φ_j of H.

LEMMA 4. *Let $\Phi = \sum_j b_j\varphi_j$, $\Phi' = \sum_j b'_j\varphi_j$ be generalized characters of H, which vanish on $H-D$. If the induced generalized characters of G are $\Phi^* = \sum_i c_i\chi_i$, $\Phi'^* = \sum_i c'_i\chi_i$, then*

(i) $\Phi^*(\sigma) = \begin{cases} 0, & \textit{if } \sigma \textit{ is conjugate in } G \textit{ to no element of } D,\\ \Phi(\sigma), & \textit{if } \sigma\in D.\end{cases}$

(ii) $\sum_i c_i c'_i = \sum_j b_j b'_j$.

(iii) $g^{-1}\sum_i \chi_i(I)^2 c_i/\deg\chi_i = h^{-1}\sum_j \varphi_j(J)^2 b_j/\deg\varphi_j$,

where $g = |G|$, $h = |H|$.

These facts are due to Suzuki; an outline of their derivation is given in [13].

Using this lemma and the Frobenius reciprocity law, we easily see that for the generalized characters (2) we have

$$\begin{aligned}\Phi_1^* &= 1_G+\varepsilon(\chi_1-\chi_2),\\ \Phi_2^* &= \varepsilon(\chi_1-\chi_3),\\ \Phi_3^* &= \varepsilon(\chi_3-\chi_4),\\ \Phi_4^* &= \varepsilon\chi_2+\varepsilon_1\chi_5+\varepsilon_2\chi_6,\\ \Phi_5^* &= \varepsilon\chi_1+\varepsilon_1\chi_5+\varepsilon_3\chi_7,\end{aligned}$$

where 1_G is the trivial character of G, $\chi_1, \chi_2, \cdots, \chi_7$ are distinct non-trivial irreducible characters of G, and ε, ε_1, ε_2, ε_3 are all ± 1.

Since all elements of D are 2-singular, Lemma 4(i) implies that Φ_1^* vanishes on all 2-regular elements of G. By Theorem 6 of [4], the same holds for the part of Φ_1^* consisting of characters of the principal 2-block B_1 of G. Since a single character does not vanish on all 2-regular elements, it follows that χ_1 and χ_2 lie in B_1. By considering the other Φ_i^* in turn, we see that $\chi_1, \cdots, \chi_7$ all belong to B_1.

If σ is any 2-regular element of H, then

$$\chi_i(\tau\sigma) = \sum_j d_{ij}\psi_j(\sigma) \qquad (i = 1, \cdots, 7), \tag{3}$$

where the ψ_j are modular irreducible characters of H and the d_{ij} are rational integers, the generalized decomposition numbers [1] of χ_i with respect to τ. By Brauer's Second Main Theorem on blocks [1], $d_{ij} = 0$ unless ψ_j lies in a block b of H such that $B_1 = b^G$. This is so if and only if $b = b_1$, the principal 2-block of $H[2]$. Since A is of odd order, this may be considered as the principal 2-block of H/A.

From Table 1, H/A has only one 2-block, with two modular irreducible characters, the trivial character ψ_1 and the character ψ_2 given by

$$\psi_2(1) = 2, \quad \psi_2(\rho) = -1. \tag{4}$$

The Cartan invariants of b_1 can be calculated to have the values

$$c_{11} = 8, \quad c_{12} = c_{21} = 4, \quad c_{22} = 6. \tag{5}$$

Lemma 4(i) gives relations amongst the values of the χ_i on τ and $\tau\rho$. Using (3) and (4), we can deduce relations amongst the generalized decomposition numbers, and find that they are as in Table 2, in which b, c, d, e are rational integers.

TABLE 2

	1_G	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
ψ_1	1	$b\varepsilon$	$(b+1)\varepsilon$	$b\varepsilon$	$b\varepsilon$	$(d-b)\varepsilon_1$	$(3-d)\varepsilon_2$	$-d\varepsilon_3$
ψ_2	0	$c\varepsilon$	$c\varepsilon$	$(c-2)\varepsilon$	$(c-2)\varepsilon$	$(e-c)\varepsilon_1$	$(2-e)\varepsilon_2$	$-e\varepsilon_3$

By the orthogonality relations on the generalized decomposition numbers [1], we have, using (5), that

$$1+b^2+(b+1)^2+b^2+b^2+(d-b)^2+(3-d)^2+d^2 \leqq 8, \tag{6}$$

$$c^2+c^2+(c-2)^2+(c-2)^2+(e-c)^2+(2-e)^2+e^2 \leqq 6. \tag{7}$$

If d were not 1 or 2, then $(3-d)^2+d^2 \geqq 9$, contradicting (6). Hence, $(3-d)^2+d^2 = 5$, and so

$$3b^2+(b+1)^2+(d-b)^2 \leqq 2.$$

Thus we must have $b = 0$, $d = 1$, and equality holds in (6). If $c \neq 1$, then $c^2+(c-2)^2 \geqq 4$, contradicting (7). Hence $c = 1$, and

$$(e-1)^2+(2-e)^2+e^2 \leqq 2.$$

It follows that $e = 1$, and equality holds in (7). This implies that the

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generalized decomposition numbers with respect to τ of any other irreducible character χ of B_1 are 0, and hence that $\chi(\tau) = 0$. But since χ lies in B_1,

$$g\chi(\tau)/h \deg \chi \equiv g/h \pmod 2$$

by [4]. As g/h is odd, we have a contradiction. Thus $1_G, \chi_1, \chi_2, \cdots, \chi_7$ are the only characters in B_1.

The values of b, c, d, e found give the generalized decomposition numbers on substitution in Table 2. By (3), if $\sigma \in A$, then

$$\begin{aligned} \chi_1(\tau\sigma) &= 2\varepsilon,\ \chi_1(\tau\rho\sigma) = \chi_1(\tau\rho^2\sigma) = -\varepsilon, \\ \chi_6(\tau\sigma) &= 4\varepsilon_2,\ \chi_6(\tau\rho\sigma) = \chi_6(\tau\rho^2\sigma) = \varepsilon_2. \end{aligned} \tag{8}$$

Now let $a = |A|$, $a_1 = |A \cap C(\beta)|$. Since the centralizer of $\bar{\beta}$ in H/A is of order 4, by the structure of $GL(2, 3)$, it follows that the centralizer $C_H(\beta)$ of β in H is a subgroup of SA. Since SA is a split extension of A, we have

$$C_H(\beta) = C_S(\beta)(A \cap C(\beta)),$$

and so $|C_H(\beta)| = 4a_1$. Since every involution in H is conjugate either to τ or to β, we can calculate the values of the $\varphi_i(J)$, and find

$$\begin{aligned} &\varphi_0(J) = 1+12a/a_1, \quad \varphi_1(J) = 1-12a/a_1, \\ &\varphi_2(J) = 2, \quad \varphi_3(J) = 3-12a/a_1, \quad \varphi_4(J) = 3+12a/a_1, \\ &\varphi_5(J) = -4, \quad \varphi_6(J) = \varphi_7(J) = -2. \end{aligned}$$

On applying Lemma 4 to Φ_1, using these values and the values of the $\chi_i(\tau)$ given by (8), and the fact that all involutions are conjugate in G to τ, we obtain the formula

$$g = 2^9 3^2 a^3 f_1(f_1+\varepsilon)/a_1^2(f_1-2\varepsilon)^2, \tag{9}$$

where $f_1 = \deg \chi_1$. The same procedure with Φ_4 gives

$$9/(\varepsilon f_1+1) - 1/(\varepsilon f_1+\varepsilon_2 f_6+1) + 16/\varepsilon_2 f_6 = 2^{10} 3^2 a^3/a_1^2 g,$$

where $f_6 = \deg \chi_6$. Comparison with (9) and simplification leads to the equation

$$f_6^2(2\varepsilon f_1-1)(\varepsilon f_1-8) + 2\varepsilon_2 f_6(\varepsilon f_1+1)(f_1^2-16\varepsilon f_1+4) - 16\varepsilon f_1(\varepsilon f_1+1)^2 = 0.$$

This can be solved to give f_6 in terms of f_1. We obtain

$$\begin{aligned} \varepsilon_2 f_6 &= -2\varepsilon f_1(\varepsilon f_1+1)/(2\varepsilon f_1-1), \quad \text{or} \\ \varepsilon_2 f_6 &= 8(\varepsilon f_1+1)/(\varepsilon f_1-8). \end{aligned} \tag{10}$$

If the first case held, then $2\varepsilon f_1 - 1$ would be a divisor of $\varepsilon f_1 + 1$, which is possible only if $f_1 = 1$, or $f_1 = 2$ and $\varepsilon = 1$. But $f_1 > 1$ since $\chi_1(\tau) = \pm 2$, and we cannot have $f_1 = 2$, $\varepsilon = 1$, by the formula (9). Thus we must have the second case. Hence

$$72 \equiv 8(\varepsilon f_1+1) \equiv 0 \ (\mathrm{mod}\ (\varepsilon f_1-8)).$$

Now, by (9), $f_1 \equiv 2\varepsilon (\mathrm{mod}\ 8)$, for else g would be divisible by 2^5. Also, $f_1 \neq 2\varepsilon$, as we have seen. It follows that we must have

$$\varepsilon = 1, \quad f_1 = 10 \quad \text{or} \quad 26.$$

We consider these two possibilities in turn.

(a) $f_1 = 10$. Then $\varepsilon_2 f_6 = 44$, so that $\varepsilon_2 = 1$, $f_6 = 44$. The degrees of the χ_i may all be found by using Lemma 4(i). We have, putting $f_i = \deg \chi_i$,

$$f_1 = 10,\ f_2 = 11,\ f_3 = 10,\ f_4 = 10,\ f_5 = 55,\ f_6 = 44,\ f_7 = 45.$$

Now χ_1 is characterized as the only irreducible character of degree 10 in the block B_1 which has value 2 on involutions of G. It follows that χ_1 is rational-valued, since a field automorphism transforms χ_1 into a character with the same properties. The kernel of the representation corresponding to χ_1 is of odd order since it does not contain τ, and so is trivial, by the assumption (**). Now, by a theorem of Schur [11], a prime p can occur in the order g with exponent at most

$$[10/(p-1)]+[10/p(p-1)]+[10/p^2(p-1)]+\cdots$$

Thus, g is a divisor of $2^4 3^6 5^2 7 \cdot 11$. Now, by (9),

$$g = 7920a^3/a_1^2 = 2^4 3^2 5 \cdot 11 a^3/a_1^2.$$

Hence a^3/a_1^2 is a divisor of $3^4 5 \cdot 7$ and so also is a.

If σ is an element of prime order p in A, then by rationality of χ_1,

$$\chi_1(\sigma) = \chi_1(\sigma^2) = \cdots = \chi_1(\sigma^{p-1}) = 10-mp,$$

where m is a positive integer. By (8), we have

$$\chi_1(\tau) = \chi_1(\tau\sigma) = \cdots = \chi_1(\tau\sigma^{p-1}) = 2.$$

It follows that if ψ is any irreducible character of $L = \{\tau, \sigma\}$ whose kernel does not contain σ, then by the orthogonality relations ψ occurs in the restriction $\chi_1|L$ with multiplicity $\frac{1}{2}m$. It follows that m is even. Also, since there are $2(p-1)$ such characters ψ, $2(p-1) \leq 10$, and so $p = 3$ or 5. Thus a divides $3^4 5$. For $p = 3$, $m = 2$ or 4, and for $p = 5$, $m = 2$.

Suppose that σ is an element of order 5 in $A_1 = A \cap C(\beta)$. Since β is conjugate to τ, say $\beta^\mu = \tau$, we have that

$$\sigma_1 = \sigma^\mu \in C(\tau) = H.$$

Since elements of H not in A have orders divisible by 2 or 3, $\sigma_1 \in A$. Thus, for any i, $(\beta\sigma^i)^\mu = \tau\sigma_1^i$, and we have

$$\chi_1(\beta\sigma^i) = \chi_1(\tau\sigma_1^i) = 2.$$

Similarly, $\chi_1(\tau\beta\sigma^i) = 2$. Now, if ψ is any irreducible character of $J = \{\tau, \beta, \sigma\}$ whose kernel does not contain σ, then by the orthogonality relations, the multiplicity of ψ in $\chi_1|J$ is $\frac{1}{2}$, which is impossible. Thus 5 is not a divisor of $a_1 = |A_1|$. It follows that 5 does not divide a, since otherwise a^3/a_1^2 would be divisible by 5^3.

Now let σ be an element of order 3 in A_1. Then, as before, $\beta\sigma^i$ is conjugate to $\tau\sigma_1^i$ for all i, where σ_1 is an element of order 3 in H. A conjugate of σ_1 lies either in A or in the coset of ρ or ρ^{-1} with respect to A. It follows from (8) that

$$\chi_1(\beta\sigma) = \chi_1(\beta\sigma^2) = 2 \quad \text{or} \quad -1.$$

Similarly, $\chi_1(\tau\beta\sigma) = \chi_1(\tau\beta\sigma^2) = 2$ or -1. We have seen that $\chi_1(\sigma) = \chi_1(\sigma^2) = 4$ or -2, and

$$\chi_1(\tau) = \chi_1(\beta) = \chi_1(\tau\beta) = \chi_1(\tau\sigma) = \chi_1(\tau\sigma^2) = 2.$$

Suppose that $\chi_1(\sigma) = 4$. Then if ψ is an irreducible character of $J = \{\tau, \beta, \sigma\}$ whose kernel does not contain σ or τ, the multiplicity of ψ in $\chi_1|J$ as calculated by means of the orthogonality relations is $\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$, which is impossible. Hence $\chi_1(\sigma) = -2$.

If A_1 had an elementary Abelian subgroup of order 9, the sum of the values of χ_1 on this subgroup would be -6, which is not a multiple of 9. Thus A_1 must be cyclic. Suppose that A_1 has a subgroup A_2 of order 9. Since the primitive 9-th roots of 1 are algebraically conjugate and χ_1 is rational-valued, the values of χ_1 on the elements of order 9 of A_2 are all equal. Now if ψ is a faithful irreducible character of A_2, the multiplicity of ψ in $\chi_1|A_2$ is calculated to be $\frac{4}{3}$, which is impossible. Hence $a_1 = |A_1| = 1$ or 3.

If $a_1 = 1$, then a^3 divides 3^4, and so $a = 1$ or 3. If $a_1 = 3$, then a^3 divides 3^6, and so $a = 3$ or 9. If $a = 9$, then A is elementary Abelian, since $A = A_1 \times A_1'$, where A_1' is the subgroup of A consisting of elements inverted by conjugation by β.

Suppose that $a \neq 1$. We consider the centralizer $C_H(A)$ of A in H. If $a = 3$, then since $H/C_H(A)$ is isomorphic to a subgroup of the automorphism group of A, $(H : C_H(A)) \leqq 2$. If $a = 9$, then $C_H(A) \supseteqq \{\tau\}A$, and so $H/C_H(A)$ is isomorphic both to a factor group of $PGL(2, 3)$ and to a subgroup of $GL(2, 3)$ (the automorphism group of A). Hence, $(H : C_H(A)) \leqq 6$. Thus, in either case,

$$C_H(A) \supseteqq TA,$$

where $T = \{\alpha^2, \alpha\beta\}$. Now it follows that

$$C(\alpha^2) = \{\alpha\}A.$$

Since this has a normal 2-complement, the principal 2-block of $C(\alpha^2)$ contains only the trivial modular character, and has Cartan invariant 8. Thus each χ_i has at most one non-vanishing generalized decomposition number d_i with respect to α^2, and

$$\chi_i(\alpha^2\sigma) = d_i,$$

for any 2-regular element σ of $C(\alpha^2)$. By applying Lemma 4(i) to the Φ_i, we see that the d_i are given by

$$d_1 = m,\ d_2 = m-3,\ d_3 = d_4 = m-2,\ d_5 = m+n-3,\ d_6 = n,\ d_7 = n+1,$$

where m and n are rational integers. By the orthogonality relations on generalized decomposition numbers,

$$1+m^2+(m-3)^2+2(m-2)^2+(m+n-3)^2+n^2+(n+1)^2 = 8.$$

It easily follows that $m = 2$, $n = 0$

Now consider the group

$$T_1 = T\{\sigma\} = T\times\{\sigma\},$$

where σ is an element of order 3 in A. This has 21 elements of orders 2, 4, 6 or 12, on all of which the value of χ_1 is 2, one element on which χ_1 has value 10, and two elements σ, σ^2 of order 3 on which the value of χ_1 is 4 or -2. If $\chi_1(\sigma) = 4$, the sum of the values of χ_1 on T_1 would be 60, which is not divisible by the order 24 of T_1. Hence $\chi_1(\sigma) = -2$. But now if ψ is an irreducible character of T_1 whose kernel does not contain σ, then the multiplicity of ψ in $\chi_1|T_1$ is computed by means of the orthogonality relations to be $\frac{1}{2}$, which is impossible.

Thus, in this case, $a = 1$.

(b) $f_1 = 26$. In this case, $f_6 = 12$, $\varepsilon_2 = 1$, by (10). Lemma 4(i) gives

$$f_2 = 27,\ f_3 = 26,\ f_4 = 26,\ f_5 = 39,\ f_7 = 13.$$

Since χ_6 is the only irreducible character of degree 12 in the block B_1, χ_6 is rational-valued. The kernel of the corresponding representation is trivial, by the assumption (**). By the theorem of Schur [11], g is a divisor of $2^4 3^8 5^3 7^2 11\cdot 13$. By (9),

$$g = 5616a^3/a_1^2 = 2^4 3^3 13 a^3/a_1^2.$$

Thus a^3/a_1^2 is a divisor of $3^5 5^3 7^2 11$, and so also is a.

If σ is an element of prime order p in A, then as before

$$\chi_6(\sigma) = \chi_6(\sigma^2) = \cdots = \chi_6(\sigma^{p-1}) = 12-mp,$$

where m is a positive integer. By (8),

$$\chi_6(\tau) = \chi_6(\tau\sigma) = \cdots = \chi_6(\tau\sigma^{p-1}) = 4.$$

As in case (a), consideration of the restriction of χ_6 to $L = \{\tau, \sigma\}$ shows that m is even. The multiplicity in $\chi_6|L$ of the non-trivial irreducible character of L whose kernel contains σ is calculated to be $4-\frac{1}{2}m(p-1)$. Hence $\frac{1}{2}m(p-1) \leqq 4$, and so $p = 3$, $m = 2$ or 4; or $p = 5$, $m = 2$; and a^3/a_1^2 divides $3^5 5^3$.

As before, $A_1 = A \cap C(\beta)$ cannot contain an element of order 5, and so a is divisible by 5 to at most the first power.

If σ is an element of order 3 in A_1, then as in case (a) we can calculate the values of χ_6 on all the elements of $\{\tau, \beta, \sigma\}$, with the two possible values 6, 0 for $\chi_6(\sigma) = \chi_6(\sigma^2)$. The value 6 leads to a contradiction. Thus $\chi_6(\sigma) = 0$.

As in case (a), A_1 is of order 1 or 3, and A is the direct product of an elementary Abelian group of order dividing 9 with a cyclic group of order dividing 5. The automorphism group of A is therefore the direct product of a subgroup of $GL(2, 3)$ with a cyclic 2-group. It follows as before that

$$C_H(A) \supseteqq TA,$$

where $T = \{\alpha^2, \alpha\beta\}$. Now the same method as before applied to $T\{\sigma\}$, where σ is an element of order 3 or 5 in A, leads to a contradiction. Thus again $a = 1$.

We have proved therefore that $H \approx GL(2,3)$. Now a theorem of Brauer [3] shows that G is isomorphic either to M_{11} or to $PSL(3, 3)$. (The condition that $G = G'$ in the hypothesis of Brauer's theorem is unnecessary in the present case, as may be seen from Theorem 6 of the appendix. Alternatively, we may note that G' is a group of type IV, and so has the same order as G, by what has been proved.)

These results, together with Lemma 3, immediately imply the following

THEOREM 3. *Let G be a finite group with 2-Sylow subgroup of the form*

$$S = \{\alpha, \beta\}, \alpha^{2^a} = \beta^2 = 1, \alpha^\beta = \alpha^{2^{a-1}-1}, a \geqq 3.$$

If the centralizer of the involution $\alpha^{2^{a-1}}$ has an Abelian 2-complement and K is the largest normal subgroup of odd order in G, then G/K is isomorphic to one of the groups S, $GL(2, 3)$ $PSL(3, 3)$, M_{11}, or $H(q)$ for some q.

COROLLARY. *The only simple groups satisfying the hypotheses of Theorem* 3 *are $PSL(3,3)$ and M_{11}.*

5.

In this section we derive some consequences of the foregoing results.

We denote by J the subgroup of the group $\Gamma L(2, 9)$ of all semi-linear transformations of a two-dimensional vector space over $GF(9)$ generated by $SL(2, 3)$ (regarded as a subgroup of $SL(2, 9)$ taken as a group of matrices) and the transformation γ with matrix

$$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix},$$

semi-linear relative to the non-trivial automorphism of $GF(9)$, where b is a generator of the multiplicative group of $GF(9)$. J is characterized as the extension of $SL(2, 3)$ by an element γ such that $\gamma^2 = \tau$, the involution of $SL(2, 3)$, and γ induces an outer automorphism of $SL(2, 3)$. J has generalized quaternion 2-Sylow subgroup.

THEOREM 4. *Let G be a finite group whose 2-Sylow subgroup S has a cyclic subgroup of index 2. If the centralizer in G of an involution in the centre of S has an Abelian 2-complement and K is the largest normal subgroup of odd order in G, then G/K is isomorphic to one of the groups S, $SL(2, 3)$, J, $GL(2, 3)$, $PSL(3, 3)$, M_{11}, A_7, $PSL(2, q)$, $PGL(2, q)$ or $H(q)$ for some odd q.*

PROOF. The 2-Sylow subgroup of G is as indicated in the introduction. Now Burnside's theorem, the result of Gorenstein and Walter [7], and our theorems give the asserted structure of G in every case except that in which S is of generalized quaternion type. In this case, $\bar{G} = G/K$ has only one involution, by the result of Brauer and Suzuki [5]. By the proof of Lemma 3, the centralizer $\bar{G}$ of this involution has an Abelian 2-complement. If T is the subgroup of order 2 in $\bar{G}$, then if N/T is the largest odd order normal subgroup of $\bar{G}/T$, N has a normal 2-complement V, by Burnside's theorem. V is normal in $\bar{G}$ and hence is trivial, by the maximality of K. Now $\bar{G}/T$ has an Abelian 2-complement, satisfies the conditions of the Gorenstein-Walter theorem, and has no nontrivial normal subgroup of odd order. By Lemma 2, $\bar{G}/T$ is solvable, and thus $\bar{G}/T$ is a 2-group, or isomorphic to $PSL(2, 3)$ or to $PGL(2, 3)$. In the first case, $\bar{G} \approx S$. If $\bar{G}/T \approx PSL(2,3)$, then by the result of Schur [12], $\bar{G} \approx SL(2, 3)$. Now, if $\bar{G}/T \approx PGL(2, 3)$, then the argument used in Case III of § 4 shows that $\bar{G} \approx J$.

THEOREM 5. *Let G be a finite group with a subgroup of order 4 which is its own centralizer in G. If G possesses an involution whose centralizer has an Abelian 2-complement, and K is the largest normal subgroup of odd order in G, then either G/K is isomorphic to one of the groups $PSL(3, 3)$, M_{11}, J, $GL(2, 3)$ $SL(2, 3)$, $H(q)$, $PGL(2, q)$, $PSL(2, q)$ (q odd), or A_7; or else K is a 2-complement for G.*

PROOF. If K is not a 2-complement for G, then by Theorem II of [7] either

(i) the 2-Sylow subgroup of G is of the type considered in Theorem 3, and G has no subgroup of index 2;

(ii) G/K is isomorphic to $SL(2, q)$, $PGL(2, q)$, $PSL(2, q)$ (q odd), or A_7; or

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(iii) G/K has a subgroup G_0/K of index 2 isomorphic to one of the groups named in (ii).

If (i) holds, Theorem 3 shows that G/K is isomorphic to $PSL(3, 3)$ or M_{11}.

If the $SL(2, q)$ case holds in (ii) or (iii), then $q = 3$, by solvability (Lemma 2). It remains to consider the case (iii).

If G_0/K is isomorphic to $SL(2, 3)$, then the 2-Sylow subgroup S of G is an extension of a quaternion group of order 8 by a group of order 2. There are four such extensions. For S to contain a self-centralizing subgroup of order 4, S must be either of generalized quaternion type or of the type considered in Theorem 3. Thus G/K is isomorphic to J or to $GL(2, 3)$, by Theorem 4.

In all other cases of (iii), S is an extension of a dihedral group by a group of order 2. An examination of these extensions shows that S must be either dihedral or of the type considered in Theorem 3. By Theorem 4, G/K is isomorphic to $GL(2, 3)$, $PGL(2, q)$ or $H(q)$ for some q.

Appendix

For completeness we give a proof of the case of Brauer's theorem needed for the proof of Theorem 3.

THEOREM 6. *Let G be a finite group with no subgroup of index 2, such that the centralizer in G of an involution in the centre of a 2-Sylow subgroup of G is isomorphic to $GL(2,3)$. Then G is isomorphic either to M_{11} or to $PSL(3,3)$.*

PROOF. We have Case IV of § 4, with $A = \{1\}$, and retain the notations used and results found there. Since $\Phi_1, \cdots, \Phi_5$ generate the module of generalized characters of H which vanish on $H-D$, any generalized character of H orthogonal to all the Φ_i must vanish on D. In particular, if χ is any irreducible character of G distinct from $1_G, \chi_1, \cdots, \chi_7$, then, by Frobenius reciprocity, the restriction $\chi|H$ is orthogonal to all the Φ_i, so that χ vanishes on D and so on all 2-singular elements of G. By the orthogonality relations on the 2-Sylow subgroup S,

$$\deg \chi \equiv 0 \pmod{16}, \quad \chi \neq 1_G, \chi_1, \cdots, \chi_7. \tag{11}$$

Again, by Frobenius reciprocity, $(\chi_1|H)-\varphi_0+\varphi_3+\varphi_6+\varphi_7$ and $(\chi_6|H)+\varphi_5$ are orthogonal to all the Φ_i. Thus the values of χ_1, χ_6 on D can be found:

$$\begin{aligned} \chi_1(\sigma) &= 1-\varphi_3(\sigma)-\varphi_6(\sigma)-\varphi_7(\sigma), \\ \chi_6(\sigma) &= -\varphi_5(\sigma), \quad \text{for} \quad \sigma \in D. \end{aligned} \tag{12}$$

As in § 4, we have two possibilities.

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(a) $f_1 = 10$. Then, $g = 7920 = 2^4 3^2 5 \cdot 11$. Since the sum of the squares of the degrees of the irreducible characters of G is equal to g, and since

$$g-1-\sum_1^7 f_i^2 = 512,$$

it follows from (11) that there are two more irreducible characters, each of degree 16. Thus there are 10 irreducible characters in all, and G has 10 conjugacy classes. Six of these are represented by 1 and the elements τ, α^2, $\rho\tau$, α and α^{-1} of D. We denote these classes by $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 4 \rangle$, $\langle 6 \rangle$, $\langle 8 \rangle$, $\langle 8 \rangle'$.

Since the order of H is not divisible by 11, the centralizer C_{11} in G of an 11-Sylow subgroup S_{11} is of odd order. Thus the order of the normalizer N_{11} of S_{11} is not divisible by 4. Since $(G:N_{11}) \equiv 1 \pmod{11}$, we must have $|N_{11}| = 55$. We cannot have $C_{11} = N_{11}$, since then G would have 10 classes of elements of order 11. Thus $C_{11} = S_{11}$, and G has two classes $\langle 11 \rangle$, $\langle 11 \rangle'$ of elements of order 11. The remaining two classes must contain elements of order 3 and 5, and we denote these by $\langle 3 \rangle$, $\langle 5 \rangle$. Since there are no elements of order 10, 15 or 55, an element of order 5 generates its own centralizer. Now the orders of all the centralizers of all elements not of order 3 are known, and so the sizes of all the classes may be computed.

The values of χ_1 on $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 4 \rangle$, $\langle 6 \rangle$, $\langle 8 \rangle$ and $\langle 8 \rangle'$ are known, by (12). χ_1 is of 5-defect 0 and so vanishes on $\langle 5 \rangle$. If χ_1 had value 10 on an element of order 11, the kernel of the representation $\mathscr{L}$ corresponding to χ_1 would be of order 11 or 33, so that S_{11} would be normal in G, a contradiction. Hence, since χ_1 is rational, χ_1 has value -1 on $\langle 11 \rangle$, $\langle 11 \rangle'$. By the orthogonality relations, the value of χ_1 on $\langle 3 \rangle$ is 1. All the values of χ_1 have been found, and we have

$$\begin{aligned} \chi_1(\sigma) &= 10,\ \sigma = 1, \\ &= 2,\ \sigma \in \langle 2 \rangle, \langle 4 \rangle, \\ &= -1,\ \sigma \in \langle 6 \rangle, \langle 11 \rangle, \langle 11 \rangle', \\ &= 0,\ \sigma \in \langle 8 \rangle, \langle 8 \rangle', \langle 5 \rangle, \\ &= 1,\ \sigma \in \langle 3 \rangle. \end{aligned} \tag{13}$$

We have seen that a 5-Sylow subgroup S_5 is its own centralizer. Since there is only one class of elements of order 5, we have that the normalizer N_5 of S_5 is a split extension of S_5 by a cyclic group F of order 4. Let $\mathscr{L}^R$ denote the subspace of the representation space of $\mathscr{L}$, the representation corresponding to χ_1, consisting of those vectors left fixed by the subgroup R of G. The dimension $\dim \mathscr{L}^R$ is given by the average value of χ_1 on R. Thus we can compute that $\mathscr{L}^{N_5}$ is a subspace of dimension 2 in the space $\mathscr{L}^F$, which is of dimension 4. Now F is conjugate in G to $\{\alpha^2\}$, and so is contained in a quaternion group Q. $\mathscr{L}^Q$ is a subspace of $\mathscr{L}^F$ of dimension 3.

If $M = \{Q, N_5\}$, then $\mathscr{L}^M = \mathscr{L}^Q \cap \mathscr{L}^{N_5}$, and so dim $\mathscr{L}^M \geqq 1$. Since $\mathscr{L}$ is irreducible, it follows that M is a proper subgroup of G.

Since the number of 5-Sylow subgroups of M is $(M : N_5) \equiv 1 \pmod 5$, we have

$$|M| = 20(5n+1),$$

where n is an odd integer, since $|M|$ is divisible by 8. If $n > 7$, then $(G : M) \leqq 6$, so that G has a transitive permutation representation of degree $\leqq 6$. Since G has no non-trivial irreducible characters of degree less than 10, this representation would be trivial, contradicting the fact that M is proper. If $n = 1$, then M has six 5-Sylow subgroups, and M has a permutation representation $\mathscr{R}$ of degree 6. Then the kernel of $\mathscr{R}$ is the intersection L of all the conjugates of N_5 in M. If a 5-Sylow subgroup S_5 were contained in L, then S_5 would be normal in L and so in M, a contradiction. If L contained an element σ of order 2, then for a non-trivial element μ in S_5, if $\bar{\sigma} = \mathscr{R}(\sigma)$, $\bar{\mu} = \mathscr{R}(\mu)$, $\bar{\sigma}$ transforms $\bar{\mu}$ into $\bar{\mu}^{-1}$, which is distinct from $\bar{\mu}$, contradicting the assumption that $\bar{\sigma} = 1$. Thus L is trivial and $\mathscr{R}$ is faithful. But, this is impossible since M contains a quaternion subgroup, which can have no faithful permutation representation of degree 6. We cannot have $n = 3$ or 5, since then $|M|$ would not divide $|G|$. Hence $n = 7$, $(G : M) = 11$, and G has a transitive permutation representation $\mathscr{P}$ of degree 11. The degrees of the irreducible characters of G being known, it follows that the character of $\mathscr{P}$ is $1_G+\chi_1$, $1_G+\chi_3$ or $1_G+\chi_4$. By (3), we have $\chi_3(\tau) = \chi_4(\tau) = -2$. Thus the character of $\mathscr{P}$ is $1_G+\chi_1$. By (13), only the identity of G is represented by a permutation leaving 4 letters fixed. In particular, $\mathscr{P}$ is faithful. Since $|G| = 11 \cdot 10 \cdot 9 \cdot 8$, $\mathscr{P}(G)$ is quadruply transitive. By a theorem of Jordan (cf. [8], Theorem 5.8.1), G is isomorphic to M_{11}.

(b) $f_1 = 26$. Then, $g = 5616 = 2^4 3^3 13$. Now,

$$g-1-\sum_1^7 f_i^2 = 1024,$$

and so, by (11), G has four more irreducible characters, each of degree 16, and so G has 12 conjugacy classes, six of which, denoted $\langle 1\rangle$, $\langle 2\rangle$, $\langle 4\rangle$, $\langle 6\rangle$, $\langle 8\rangle$, $\langle 8\rangle'$ are represented by 1 and the elements τ, α^2, $\rho\tau$, α and α^{-1} of D. By considering the number of 13-Sylow subgroups of G we see that the normalizer N_{13} and the centralizer C_{13} of a 13-Sylow subgroup S_{13} have orders 39, 13. Thus G has four classes $\langle 13\rangle$, $\langle 13\rangle'$, $\langle 13\rangle''$, $\langle 13\rangle'''$ of elements of order 13. This accounts for all but 728 of the elements of G.

Of the two remaining classes, one is the class $\langle 3\rangle$ of the element ρ of order 3. The other must contain elements of order 3 or 9. Suppose σ is of order 9. Then since σ does not commute with elements of order 2 or 13, the number of conjugates of σ is $2^4 13 = 208$ or $2^4 3 \cdot 13 = 624$. The first case is impossible since it would imply that $|\langle 3\rangle| = 520$, not a divisor of 5616.

Thus σ has 624 conjugates, and generates its own centralizer. In particular the 3-Sylow group S_3 is non-Abelian. Since all elements of order 9 are conjugate, the normalizer $N(\{\sigma\})$ must transform the elements of order 9 in $\{\sigma\}$ transitively. Hence $|N(\{\sigma\})| = 2 \cdot 3^3$, and there are $2^3 13 = 104$ cyclic subgroups of order 9 in G.

Since $|\langle 3 \rangle| = 104$, there are 52 subgroups of order 3 in G, all conjugate. Each cyclic subgroup of order 9 contains exactly one such subgroup. Hence each subgroup of order 3 must be contained in exactly two cyclic subgroups of order 9. But, by the structure of S_3 ([8], § 4.4), the centre of S_3 is contained in three cyclic subgroups of order 9, a contradiction.

Hence G has no elements of order 9, and the remaining class $\langle 3 \rangle'$ contains elements of order 3. Now let σ, σ' be non-conjugate elements of order 3, σ an element of the centre of S_3. The order $|C(\sigma)|$ is not divisible by 4, since otherwise $C(\sigma)$ would have an Abelian subgroup of order 12, contradicting the fact that the centralizer of an involution contains no such subgroup. Since $|C(\sigma)|$ is also not divisible by 13 but is divisible by 3^3, the number of conjugates of σ is $2^3 13$ or $2^4 13$. The second case is impossible as it gives a size 520 for the class of σ'. Hence σ has $2^3 13 = 104$ conjugates, and σ' has 624 conjugates. $|C(\sigma)| = 2 \cdot 3^3$, and $|C(\sigma')| = 3^2$. Thus σ is conjugate to ρ, so that $\sigma \in \langle 3 \rangle$, $\sigma' \in \langle 3 \rangle'$. Also S_3 is non-Abelian of exponent 3.

The values of χ_6 on $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 4 \rangle$, $\langle 6 \rangle$, $\langle 8 \rangle$, $\langle 8 \rangle'$ are known, by (12). If χ_6 had value 12 on an element of order 13, the kernel of the representation corresponding to χ_6 would be of order 13, $3 \cdot 13$, $3^2 \cdot 13$ or $3^3 13$. In any case it would have a normal 13-Sylow subgroup or normal 3-Sylow subgroup which would be normal in G, a contradiction. Hence, since χ_6 is rational, its value on $\langle 13 \rangle$, $\langle 13 \rangle'$, $\langle 13 \rangle''$, $\langle 13 \rangle'''$ is -1. By the orthogonality relations, the values of χ_6 on $\langle 3 \rangle$, $\langle 3 \rangle'$ are found. We have

$$
\begin{aligned}
(14) \qquad \chi_6(\sigma) &= 12, \sigma = 1 \\
&= 4, \ \sigma \in \langle 2 \rangle \\
&= 0, \sigma \in \langle 4 \rangle, \langle 8 \rangle, \langle 8 \rangle', \langle 3 \rangle' \\
&= 1, \sigma \in \langle 6 \rangle \\
&= -1, \sigma \in \langle 13 \rangle, \langle 13 \rangle', \langle 13 \rangle'', \langle 13 \rangle''' \\
&= 3, \sigma \in \langle 3 \rangle.
\end{aligned}
$$

Let S_3 be a 3-Sylow subgroup of G whose centre contains (and so is generated by) the element ρ. Since $|C(\rho)| = 2 \cdot 3^3$, and ρ is conjugate to ρ^{-1}, the normalizer $N(\{\rho\}) = C^*(\rho)$ has order $2^2 3^3$. S_3 is characteristic in $C(\rho)$, which is normal in $C^*(\rho)$. Hence S_3 is normal in $C^*(\rho)$. $C^*(\rho)$ contains the involution τ, and also, by the structure of $C(\tau) \approx GL(2, 3)$, an involution μ which transforms ρ into its inverse and commutes with τ. $\{\tau, \mu\}$ acts as a group of automorphisms of the elementary Abelian group $S_3/\{\rho\}$, which therefore has a subgroup $U/\{\rho\}$ of order 3, invariant under $\{\tau, \mu\}$. Now $\{\tau, \mu\}$

acts as a group of automorphisms of the elementary Abelian group U. Since neither τ nor μ centralizes U, we may assume that

$$U = \{\rho, \lambda\}, \rho^\tau = \rho, \rho^\mu = \rho^{-1}, \lambda^\tau = \lambda^{-1}, \lambda^\mu = \lambda.$$

Thus we know the structure of the subgroup

$$M = \{\tau, \mu, \rho, \lambda\}$$

of order 36. M contains 15 involutions, 12 elements of order 6, and 8 elements of order 3. Let n be the number of elements of M belonging to $\langle 3 \rangle$. Then the sum of the values of χ_6 on M is $84+3n$. Since this must be divisible by the order 36 of M, $n = 8$. Thus all elements of order 3 in M belong to $\langle 3 \rangle$, and the average value of χ_6 on M is 3, i.e. dim $\mathscr{L}^M = 3$, where $\mathscr{L}$ is the representation corresponding to χ_6, and $\mathscr{L}^M$ the subspace of the representation space consisting of vectors left fixed by M. dim $\mathscr{L}^{C(\tau)}$ is computed to be 2. Since $T = \{\tau, \mu, \rho\}$ is a subgroup of both M and $C(\tau)$, $\mathscr{L}^M$ and $\mathscr{L}^{C(\tau)}$ are subspaces of $\mathscr{L}^T$, which has dimension 4. Thus $\mathscr{L}^M \cap \mathscr{L}^{C(\tau)}$ has dimension at least 1, and $L = \{M, C(\tau)\}$ is a proper subgroup of G.

Clearly $|L|$ is divisible by $2^4 3^2$, and so $(G : L)$ is a divisor of $3 \cdot 13$. Let $\mathscr{P}$ be the transitive permutation representation of G on the right cosets of L. If $(G : L) = 3$, G would have a nontrivial irreducible character of degree $\leqq 2$, which is not so. If $(G : L) = 39$, then $C(\tau)$ is a subgroup of L of index 3. The intersection of the conjugates of $C(\tau)$ in L is a subgroup of index 3 or 6 in L, and so is either $C(\tau)$ or its unique subgroup K of index 2. Since τ generates the centre of both $C(\tau)$ and K, it follows that $\{\tau\}$ is normal in L, so that $L \subseteqq C(\tau)$, a contradiction. Hence, $(G : L) = 13$.

L is not normal in G, since otherwise G would have 13 characters of degree 1. Hence L is its own normalizer, and we may regard $\mathscr{P}$ as a permutation representation of G on the conjugates $L_1, \cdots, L_{13}$ of L, which we call *lines*. The character of $\mathscr{P}$ must be $1+\chi_6$, since all non-trivial irreducible characters of G apart from χ_6 have degree exceeding 12. Thus $\mathscr{P}$ is doubly transitive, and faithful, by (14). We identify $\mathscr{P}(G)$ with G.

By (14), if $\sigma \in \langle 3 \rangle$, then $1+\chi_6(\sigma) = 4$, so that σ fixes exactly 4 lines. We define a *point* to be such a set of 4 lines. By double transitivity, any two lines belong to at least one point.

If L_1, L_2 are two lines, let σ_1, σ_2 be elements of $\langle 3 \rangle$ each fixing both L_1, L_2, i.e. lying in $L_1 \cap L_2$. $L_1 \cap L_2$ contains no elements of order 4, since these each fix only one line. Hence the 2-Sylow subgroup of $L_1 \cap L_2$ is of elementary Abelian type, and so of order at most 4 since G contains no elementary Abelian subgroup of order 8. $L_1 \cap L_2$ contains no elements of $\langle 3 \rangle$ since these fix no lines. In particular $L_1 \cap L_2$ does not contain a 3-Sylow subgroup of G.

Suppose $L_1 \cap L_2$ has non-normal 3-Sylow subgroup V. If $|V| = 9$, then $|L_1 \cap L_2| = 36$. V is its own normalizer in $L_1 \cap L_2$, and so, by Burn-

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side's theorem, $L_1 \cap L_2$ has normal 2-Sylow subgroup W. The centralizer C of W in $L_1 \cap L_2$ must have order 12. We may assume $\sigma_1 \in C$. If L_1, L_2 belong to more than one point, we may assume that there is a line L_3 fixed by σ_1 but not fixed by σ_2. $L_1 \cap L_2 \cap L_3$ is a proper subgroup of $L_1 \cap L_2$. If L_4 is the fourth line fixed by σ_1, then the three involutions in $L_1 \cap L_2$ commute with σ_1 and so permute L_1, L_2, L_3, L_4 amongst themselves. Since they each fix L_1 and L_2, at least one of them fixes all four lines. Hence $|L_1 \cap L_2 \cap L_3|$ is even, and so is 6 or 12, since $L_1 \cap L_2$ has no subgroup of order 18. If $|L_1 \cap L_2 \cap L_3| = 6$, then computation shows that the average value of $1+\chi_6$ on $L_1 \cap L_2 \cap L_3$, which is the number of transitive constituents of $L_1 \cap L_2 \cap L_3$, is 5. L_1, L_2 and L_3 form three of these constituents. This leaves two constituents whose sizes are divisors of 6 whose sum is 10. But there are no such numbers. Hence $|L_1 \cap L_2 \cap L_3| = 12$. Now the number of constituents is found to be 4, again giving a contradiction.

Now take $|V| = 3$, so that $|L_1 \cap L_2| = 12$ and $L_1 \cap L_2$ is isomorphic to the alternating group A_4. As before, if L_1, L_2 belong to more than one point we can take a line L_3 fixed by σ_1 but not by σ_2. $L_1 \cap L_2 \cap L_3$ is a proper subgroup of $L_1 \cap L_2$ and so is of order 3. If L_4 is the fourth line fixed by σ_1, then $C(\sigma_1)$ permutes L_1, L_2, L_3, L_4 amongst themselves. Since $\sigma_1 \in \langle 3 \rangle$, $C(\sigma_1)$ contains a 3-Sylow subgroup of G. As 4! is divisible by 3 to the first power only, there is a subgroup of order 3^2 fixing L_1, L_2, L_3, L_4, and this is a contradiction.

If $L_1 \cap L_2$ has normal 3-Sylow subgroup, then σ_1 and σ_2 commute, and so if the lines left fixed by σ_1 are L_1, L_2, L_3, L_4 then σ_2 permutes these amongst themselves and leaves L_1, L_2 fixed. Since σ_2 has order 3, σ_2 leaves L_1, L_2, L_3, L_4 all fixed. This completes the proof that the two lines L_1, L_2 belong to exactly one point.

If L is a line, each point of L lies on four lines, three of which are distinct from L. There are 12 lines distinct from L, each of which meets L in exactly one point. Hence L has four points. Thus there are four points on each line and four lines on each point, so that the number of points is 13, the number of lines. Now by [8], Theorem 20.8.1, we have a projective plane $\mathscr{G}$ which being of order 3 is Desarguesian. Clearly G is a group of collineations of $\mathscr{G}$. Since $PSL(3, 3)$, the full collineation group of $\mathscr{G}$, has order 5616, the order of G, we have $G \approx PSL(3, 3)$.

This finishes the proof of Theorem 6.

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