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A STRUCTURED DESCRIPTION OF THE GENUS SPECTRUM OF ABELIAN p-GROUPS

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Abstract. The genus spectrum of a finite group G is the set of all g such that G acts faithfully on a compact Riemann surface of genus g. It is an open problem to find a general description of the genus spectrum of the groups in interesting classes, such as the Abelian p-groups. Motivated by earlier work of Talu for odd primes, we develop a general combinatorial method, for arbitrary primes, to obtain a structured description of the so-called reduced genus spectrum of Abelian p-groups, including the reduced minimum genus. In particular, we determine the complete genus spectrum for a large subclass, namely, those having 'large' defining invariants. With our method we construct infinitely many counterexamples to a conjecture of Talu, which states that an Abelian p-group is recoverable from its genus spectrum. Finally, we give a series of examples of our method, in the course of which we prove, for example, that almost all elementary Abelian p-groups are uniquely determined by their minimum genus, and that almost all Abelian p-groups of exponent p^2 are uniquely determined by their minimum genus and Kulkarni invariant.

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1. Introduction.

1.1. Genus spectrum. Given a compact Riemann surface X of genus $g \geq 0$, a finite group G is said to act on X, if G can be embedded into the group $\mathrm{Aut}(X)$ of biholomorphic maps on X. While $\mathrm{Aut}(X)$ is infinite as long as $g \leq 1$, by the Hurwitz theorem **[5]** we have $|\mathrm{Aut}(X)| \leq 84 \cdot (g-1)$ as soon as $g \geq 2$. Thus, in the latter case, there are only finitely many groups G, up to isomorphism, acting on X.

But conversely, given a finite group G there always is an infinite set $\mathrm{sp}(G)$ of integers $g \geq 0$, called the *(genus) spectrum* of G, such that there is a Riemann surface X of genus g being acted on by G; in this case, g is called a genus of G. Note that we are in particular including the cases $g \leq 1$. In **[10]**, the problem of determining $\mathrm{sp}(G)$ is called the *Hurwitz problem* associated with G, and the problem of finding the *minimum* *genus* $\min \operatorname{sp}(G)$ of G, also called its *strong symmetric genus*, has stimulated particular interest. For more details we refer the reader to [**1**, **12**], and the references given there.

To attack the Hurwitz problem, let $\Delta(G) := \frac{|G|}{\exp(G)}$, where $\exp(G)$ denotes the exponent of G, that is the least common multiple of the orders of its elements. Then, let the *reduced (genus) spectrum* of G be defined by

$$\operatorname{sp}_0(G) := \left\{ \frac{g-1}{\Delta(G)} \in \mathbb{Z} \ : \ g \in \operatorname{sp}(G) \right\},$$

where the number $\frac{g-1}{\Delta(G)}$ is called the reduced genus associated with g. It follows from [**6**], together with a special consideration of the case $g = 0$, that

$$\operatorname{sp}_0(G) \subseteq \mathbb{S} := \frac{1}{\epsilon(G)} \cdot (\{-1\} \cup \mathbb{N}_0)$$

is a co-finite subset, where $\epsilon(G)$ divides $\gcd(2, |G|)$ and can be determined from the structure of G, as is recalled in (2.2). A word of caution is in order here: In [**6**] the notion of reduced genus is defined differently, by taking $\epsilon(G)$ into account as well, while our choice leads to fewer case distinctions.

The *reduced minimum genus* of G, that is the reduced genus associated with the minimum genus of G, equals $\mu_0(G) := \min \operatorname{sp}_0(G)$. Moreover, following [**7**], the *reduced stable upper genus* $\sigma_0(G)$ of G is the smallest element of $\mathbb{S}$ such that all elements of $\mathbb{S} \setminus \operatorname{sp}_0(G)$ are less than $\sigma_0(G)$; the genus $\sigma(G)$ associated with $\sigma_0(G)$ is called the *stable upper genus* of G. The elements of $\mathbb{S} \setminus \operatorname{sp}_0(G)$ exceeding $\mu_0(G)$ are called the *reduced spectral gap* of G; the associated genera form the *spectral gap* of G. Hence, solving the Hurwitz problem for G amounts to determining $\mu_0(G)$ and $\sigma_0(G)$ and the reduced spectral gap of G.

1.2. Our approach to Abelian p-groups. We now restrict ourselves to p-groups, where p is a prime. Not too much is known about the genus spectrum of groups within this class, even if we only look at interesting subclasses, for example, those given by bounding a certain invariant, such as rank, exponent, nilpotency class, or co-class; see [**12**].

This still holds if we restrict further to the class of Abelian p-groups, which are the groups we are interested in from now on, their general form being

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e},$$

where $e \geq 1$, $r_i \geq 0$ for $1 \leq i \leq e-1$, and $r_e \geq 1$. We point out that, in particular, contrary to [**9**, **13**], we are allowing for arbitrary primes $p \geq 2$ throughout.

We give an outline of the paper: In Section 2 we recall a few facts about Riemann surfaces and their automorphism groups.

Section 3 is devoted to prepare the combinatorial tools featuring prominently later on: Given a prime p, and a non-increasing sequence $\underline{a} := (a_1, \ldots, a_e)$ of non-negative integers, the associated *p-mainline integer* is defined as $\wp(\underline{a}) := \sum_{i=1}^{e} a_i p^{e-i}$. Moreover, given any non-increasing sequence $\underline{s} := (s_1, \ldots, s_e)$ of non-negative integers, let $\mathcal{P}(\underline{s})$ be the set of all p-mainline integers $\wp(\underline{a})$, where $\underline{a}$ is bounded below component-wise by $\underline{s}$. The connection to Abelian p-groups with defining invariants $(r_1, \ldots, r_e)$ is given by letting the sequence $\underline{s}$ be defined as

$$s_i := 1 + \sum_{j=i}^{e} r_j \quad \text{for} \quad 1 \leq i \leq e.$$

We are interested in the structure of $\mathcal{P}(\underline{s})$, whose minimum obviously equals $\wp(\underline{s})$. It can be shown that $\mathcal{P}(\underline{s})$ is a co-finite subset of the non-negative integers, and thus the combinatorial problems arising are to determine the smallest m such that all integers from m on actually are elements of $\mathcal{P}(\underline{s})$, and to describe the gap consisting of the non-mainline integers between $\wp(\underline{s})$ and m.

Having these preliminaries in place we turn our attention to Abelian p-groups and their genus spectrum: Our starting point in Section 4 is Talu's approach [**13**] towards a general description of the genus spectrum of Abelian p-groups, in the case where p is odd. Building on these ideas, we describe the smooth epimorphisms, in the sense of (2.1), onto a given Abelian p-group, where p is arbitrary. The resulting general necessary and sufficient arithmetic condition for their existence, which we still refer to as *Talu's theorem*, is given in Theorems 3 and 4; in proving the latter, we in particular close a gap in the proof of [**13**, Theorem 3.3]. In Section 5, this is translated into a combinatorial description of the domain of the reduced genus map, yielding a structured description of the reduced spectrum of G, which is presented in Section 5.3, and leading to a method of computing the reduced minimum genus $\mu_0(G)$ of G culminating in Theorem 5, which says that $\mu_0(G)$ is given as the minimum of at most $e + 1$ numbers, given explicitly in terms of the defining invariants $(r_1, \ldots, r_e)$. In particular, in Section 5.6, we obtain an independent proof and an improved version of Maclachlan's method [**8**, Theorem 4] for the special case of Abelian p-groups.

Having these combinatorial tools in place, in Section 6, we turn to Abelian p-groups with 'large' invariants, by assuming that

$$r_i \geq p - 1 \quad \text{for} \quad 1 \leq i \leq e - 1, \quad \text{and} \quad r_e \geq \max\{p - 2, 1\}.$$

In these cases, we are able to determine both the reduced minimum genus $\mu_0(G)$ as well as the reduced stable upper genus $\sigma_0(G)$ in terms of the defining invariants $(r_1, \ldots, r_e)$ of G. More precisely, our main result says the following.

Main Theorem. *Let G have 'large' invariants as specified above. Then, the reduced minimum and stable upper genus of G is given as*

$$\mu_0(G) = \sigma_0(G) = \frac{1}{2} \cdot \left(-1 - p^e + \sum_{i=1}^{e} (p^e - p^{e-i}) \cdot r_i \right).$$

In particular, the reduced spectral gap is empty.

We point out that, in contrast, the main focus of [**13**] is on Abelian p-groups having 'small' invariants, fulfilling $s_i \leq (e - i + 1) \cdot (p - 1)$, for $1 \leq i \leq e$.

In Section 7, we turn to an aspect of the general question of how much information about a group is encoded into its spectrum, at best whether its isomorphism type can be recovered from it. Since in view of the examples in [**9**] this cannot possibly hold without restricting the class of groups considered, the class of Abelian p-groups seems to be a good candidate to look at. More specifically, *Talu's conjecture* [**13**] says that, whenever p is odd, the spectrum of a non-trivial Abelian p-group already determines the group up to isomorphism. Moreover, although this cannot possibly hold in full generality for $p = 2$, in view of the examples considered below, one might expect that it still holds true up to finitely many finite sets of exceptions.

But, as a consequence of Main Theorem 1, we are able to produce infinitely many counterexamples to Talu's conjecture (both for p odd and $p = 2$), that is pairs of non-isomorphic Abelian p-groups having the same spectrum. We present two distinct kinds of counterexamples, consisting of groups having the same order and exponent, and of groups where these invariants are different, in Section 7.1 and Sections 7.2–7.4, respectively. This even shows that there cannot be an absolute bound on the cardinality of a set of Abelian p-groups sharing one and the same spectrum, even if we restrict ourselves to groups having the same order and exponent.

In order to show the effectiveness of the combinatorial methods developed, in Sections 8 and 9, we work out various explicit examples, where in particular we get new proofs of a number of earlier results scattered throughout the literature:

In Theorem 7, we determine the Abelian p-groups of non-positive reduced minimum genus, where we recover the Abelian p-groups amongst the well-known finite groups acting on surfaces of genus $g \leq 1$, see [**11**, Appendix] or [**2**, Section 6.7]. In particular, the non-cyclic Abelian groups of order at most 9, which have to be treated as exceptions in [**8**, Theorem 4], reappear here naturally. In Theorem 8, we deal with cyclic p-groups, whose smallest genus ≥ 2 we determine. In particular, we recover the results in [**4**] and [**6**, Proposition 3.3], and we show that such a group is uniquely determined by its smallest genus ≥ 2, with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$. Similarly, in Theorem 9, we consider Abelian p-groups of rank 2, for which we also determine the smallest genus ≥ 2. In particular, we improve the bound in [**6**, Proposition 3.4], and we show that such a group is uniquely determined by its smallest genus ≥ 2, with the exception of the groups $\{\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z}_4^2\}$. For Abelian p-groups of cyclic deficiency 1, where p is odd, we recover part of [**9**, Theorem 5.4] and [**9**, Corollary 5.5]. Finally, in Sections 8.1 and 8.2, we completely determine the spectrum of the Abelian 2-groups of order dividing 16, and of the Abelian 3-groups of order dividing 27, respectively.

In Theorem 10 and Proposition 7, we determine the reduced minimum genus of elementary Abelian p-groups, and we show that such a group is uniquely determined by its minimum genus, with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_2^2\}$; for p odd this is claimed in [**9**, Corollary 7.3]. Similarly, in Theorem 11 and Proposition 8, we determine the reduced minimum genus of Abelian p-groups of exponent p^2, and we show that such a group is uniquely determined by its minimum genus and its Kulkarni invariant, with the exception of the groups $\{\mathbb{Z}_4^2, \mathbb{Z}_2 \oplus \mathbb{Z}_4\}$; for p odd this is claimed in [**13**, Theorem 3.8].

In particular, these results imply the following theorem related to Talu's conjecture:

THEOREM. *Talu's conjecture holds within the following subclasses of the class of non-trivial Abelian p-groups (including the case $p = 2$):*

(a) the class of cyclic p-groups (with the exception $\{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$),
(b) the class of Abelian p-groups of rank 2 (with the exception $\{\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8\}$),
(c) the class of elementary Abelian p-groups (with the exception $\{\mathbb{Z}_2, \mathbb{Z}_2^2\}$),
(d) the class of Abelian p-groups of exponent p^2 (without exception).

2. Groups acting on Riemann surfaces. We assume the reader is familiar with the basic theory of Riemann surfaces, as is exhibited, for example, in [**1**, **2**], so that here we are just content with recalling a few facts. The connection between geometry and group theory is given by the following well-known theorem. We point out that it is often only used for $g \geq 2$, in which case the 'groups with signature' occurring are the Fuchsian groups, but it actually holds for all $g \geq 0$; see, for example, [**1**, Section 1], [**2**, Chapter 6] and [**11**].

THEOREM 1. *A finite group G acts on a compact Riemann surface X, if and only if there is $\Gamma \leq \mathrm{Aut}(U)$, where U is a simply-connected Riemann surface and Γ is a group with finite signature in the sense of Section 2.1, and a smooth epimorphism $\phi : \Gamma \longrightarrow G$, such that X is isomorphic to the orbit space $U/\ker(\phi)$.*

2.1. Smooth epimorphisms. We keep the notation of Theorem 1. A group Γ is said to be a *group with (finite) signature* if it has a distinguished generating set

$$\{a_k, b_k \;:\; 1 \leq k \leq h\} \quad \cup \quad \{c_j \;:\; 1 \leq j \leq s\}$$

for some $h, s \in \mathbb{N}_0$, subject to the order relations

$$c_j^{n_j} = 1, \quad \text{where} \quad n_j \in \mathbb{N} \setminus \{1\}$$

for $1 \leq j \leq s$, and the 'long' relation

$$\prod_{k=1}^{h} [a_k, b_k] \cdot \prod_{j=1}^{s} c_j = 1,$$

where $[a, b] := a^{-1}b^{-1}ab$ denotes the commutator of a and b. More generally, there might also be order relations of the form '$c^{\infty} = 1$', that is no order relation for the generator c at all; but, since we are requiring X to be compact, and hence the orbit space X/G to be compact as well, these cases do not occur here; see [**11**, Appendix].

An epimorphism $\phi : \Gamma \longrightarrow G$ with torsion-free kernel is called *smooth*. This is equivalent to the condition that $\phi(c_j) \in G$ has order n_j, for all $1 \leq j \leq s$. In this case, the $(s+1)$-tuple $(n_1, \ldots, n_s; h)$ is called a *signature* of G, with *periods* $n_1, \ldots, n_s \geq 2$ and *orbit genus* $h \geq 0$. The orbit space X/G has genus h, and the branched covering $X \longrightarrow X/G$ gives rise to the *Riemann–Hurwitz equation*

$$g - 1 = |G| \cdot \left(h - 1 + \frac{1}{2} \cdot \sum_{i=1}^{s} \left(1 - \frac{1}{n_i} \right) \right).$$

2.2. Kulkarni's Theorem. To describe the structure of the genus spectrum of a finite group G, in [**6**], a group theoretic invariant $N(G) \in \mathbb{N}$, now called the *Kulkarni invariant* of G, is introduced, such that

$$\mathrm{sp}(G) \setminus \{0\} \subseteq 1 + N(G) \cdot \mathbb{N}_0,$$

and $\mathrm{sp}(G) \setminus \{0\}$ is a co-finite subset of $1 + N(G) \cdot \mathbb{N}_0$. Moreover, we have

$$N(G) = \frac{1}{\epsilon(G)} \cdot \frac{|G|}{\exp(G)},$$

where $\epsilon = \epsilon(G) \in \{1, 2\}$ is determined by the structure of G as follows.

If $|G|$ is odd, then $\epsilon := 1$; if $|G|$ is even, letting $\tilde{G}$ be a Sylow 2-subgroup of G, then $\epsilon := 1$ provided the subset $\{a \in \tilde{G}; |a| < \exp(\tilde{G})\} \subseteq G$ forms a subgroup of $\tilde{G}$ of index 2, otherwise $\epsilon := 2$.

This yields the description of the non-negative part of the reduced spectrum $\mathrm{sp}_0(G)$ as stated earlier. As for its negative part, the well-known description of finite group actions on compact Riemann surfaces of genus $g = 0$, see [**11**, Appendix] or [**2**, Section 6.7], says that in this case G is cyclic, dihedral, alternating or symmetric of isomorphism type in $\{\mathbb{Z}_n, \mathrm{Dih}_{2n}, \mathrm{Alt}_4, \mathrm{Sym}_4, \mathrm{Alt}_5\}$; hence, we indeed get $\Delta(G) = \epsilon(G)$.

2.3. The case of p-groups. We turn to the case of interest for us: Let G be a p-group of order p^n and exponent p^e, where $e \leq n \in \mathbb{N}_0$.

If $\phi : \Gamma \longrightarrow G$ is a smooth epimorphism, then all the periods are of the form p^i, where $0 \leq i \leq e$. Hence, we may abbreviate any signature $(n_1, \ldots, n_s; h)$ of G by the $(e+1)$-tuple $(x_1, \ldots, x_e; h)$, being called the associated *p-datum*, where

$$x_i := |\{1 \leq j \leq s; n_j = p^i\}| \in \mathbb{N}_0.$$

The set $D(G)$ of all p-data of G, being afforded by smooth epimorphisms, is called the *data spectrum* of G. Then, the Riemann–Hurwitz equation gives rise to the *genus map* $g : D(G) \longrightarrow \mathrm{sp}(G)$ defined by

$$g(x_1, \ldots, x_e; h) := 1 + p^{n-e} \cdot \left((h-1) \cdot p^e + \frac{1}{2} \cdot \sum_{i=1}^{e} x_i(p^e - p^{e-i})\right).$$

Letting the *cyclic deficiency* of G be defined as

$$\delta = \delta(G) := \log_p(\Delta(G)) = n - e \in \mathbb{N}_0,$$

in view of Kulkarni's theorem (2.2), we have $N(G) = \frac{1}{\epsilon(G)} \cdot p^{\delta(G)}$. Then, the *reduced genus map*

$$g_0 : D(G) \longrightarrow \mathrm{sp}_0(G) \subseteq \frac{1}{\epsilon(G)} \cdot (\{-1\} \cup \mathbb{N}_0) \subseteq \frac{1}{2} \cdot (\{-1\} \cup \mathbb{N}_0),$$

given by associating the reduced genus $\frac{g-1}{p^\delta} \in \mathrm{sp}_0(G)$ with any $g \in \mathrm{sp}(G)$, reads

$$g_0(x_1, \ldots, x_e; h) = (h-1) \cdot p^e + \frac{1}{2} \cdot \sum_{i=1}^{e} x_i(p^e - p^{e-i}).$$

3. Mainline integers. In this section, we consider sequences of non-negative integers from a certain purely combinatorial viewpoint. We develop a little piece of general theory, as far as will be needed in Sections 5 and 6.

3.1. Integer sequences. Given finite sequences $\underline{a} = (a_1, \ldots, a_e) \in \mathbb{N}_0^e$ and $\underline{b} = (b_1, \ldots, b_e) \in \mathbb{N}_0^e$ of non-negative integers, of length $e \geq 1$, we write $\underline{a} \leq \underline{b}$, and say that $\underline{b}$ *dominates* $\underline{a}$, if $a_i \leq b_i$ for all $1 \leq i \leq e$. We will be mainly concerned with the set of *non-increasing* sequences

$$\mathcal{N} = \mathcal{N}(e) := \{\underline{a} = (a_1, \ldots, a_e) \in \mathbb{N}_0^e \ : \ a_1 \geq \cdots \geq a_e\}.$$

We introduce a few combinatorial notions concerning integer sequences: To this end, we fix $p \in \mathbb{N}$; later on p will be a prime, but here is no need to assume this.

(i) For an arbitrary sequence $\underline{a} = (a_1, \ldots, a_e) \in \mathbb{N}_0^e$, let

$$\wp(\underline{a}) = \wp(a_1, \ldots, a_e) := \sum_{i=1}^{e} a_i p^{e-i} \in \mathbb{N}_0.$$

Then, the *(p-)mainline integers* associated with $\underline{a}$ are defined as

$$\mathcal{P}(\underline{a}) = \mathcal{P}(a_1, \ldots, a_e) := \{\wp(\underline{b}) \in \mathbb{N}_0 \ : \ \underline{b} \in \mathcal{N}, \ \underline{a} \leq \underline{b}\}.$$

Note that we allow for arbitrary $\underline{a}$ to start with, while the sequences $\underline{b}$ used in the definition of $\mathcal{P}(\underline{a})$ are required to be non-increasing. It will turn out that there always is a non-increasing sequence affording a given set of mainline integers.
The *hull sequence* $\underline{\tilde{a}} = (\tilde{a}_1, \ldots, \tilde{a}_e) \in \mathcal{N}$ of $\underline{a}$ is defined recursively by letting $\tilde{a}_e := a_e$ and

$$\tilde{a}_i := \max\{\tilde{a}_{i+1}, a_i\} \quad \text{for} \quad e-1 \geq i \geq 1;$$

note that this definition is actually independent of the chosen integer p. Hence, we have $\underline{a} \leq \underline{\tilde{a}}$, where $\underline{a} = \underline{\tilde{a}}$, if and only if $\underline{a} \in \mathcal{N}$;

(ii) Given a non-increasing sequence $\underline{a} = (a_1, \ldots, a_e) \in \mathcal{N}$, its *p-enveloping sequence* $\underline{\hat{a}} = (\hat{a}_1, \ldots, \hat{a}_e) \in \mathcal{N}$ is defined recursively by $\hat{a}_e := a_e$ and

$$\hat{a}_i := \max\{\hat{a}_{i+1} + (p-1), a_i\} \quad \text{for} \quad e-1 \geq i \geq 1;$$

hence, we have $\underline{a} = \underline{\tilde{a}} \leq \underline{\hat{a}}$, where $\underline{a} = \underline{\hat{a}}$ if $p = 1$.
Moreover, whenever $e \geq 2$ let

$$||\underline{a}|| = ||(a_1, \ldots, a_e)|| := \min\{a_i - a_{i+1} \ : \ 1 \leq i \leq e-1\},$$

and let $||\underline{a}|| := \infty$ for $e = 1$; note that, despite notation, $||\cdot||$ is not a norm in sense of metric spaces. In particular, we have $\underline{a} = \underline{\hat{a}}$ if and only if $||\underline{a}|| \geq p-1$.

PROPOSITION 1. *Given $\underline{a} \in \mathbb{N}_0^e$, then we have $\mathcal{P}(\underline{a}) = \mathcal{P}(\underline{\tilde{a}})$.*

Proof. Let $\underline{b} = (b_1, \ldots, b_e) \in \mathcal{N}$. If $\underline{\tilde{a}} \leq \underline{b}$, then from $\underline{a} \leq \underline{\tilde{a}}$ we also get $\underline{a} \leq \underline{b}$. Conversely, if $\underline{a} \leq \underline{b}$, then we have $\tilde{a}_e = a_e \leq b_e$, and recursively for $e-1 \geq i \geq 1$ we get $\tilde{a}_{i+1} \leq b_{i+1} \leq b_i$ and $a_i \leq b_i$, hence $\tilde{a}_i \leq b_i$; this implies that $\underline{\tilde{a}} \leq \underline{b}$. □

PROPOSITION 2. *Given $\underline{a} \in \mathcal{N}$, the set $\mathbb{N}_0 \setminus \mathcal{P}(\underline{a})$ is finite.*

Proof. We consider the p-enveloping sequence $\underline{\hat{a}} = (\hat{a}_1, \ldots, \hat{a}_e) \in \mathcal{N}$ of $\underline{a}$, and we show that any $m \geq \wp(\underline{\hat{a}})$ is a mainline integer: To this end, write $m - \wp(\underline{\hat{a}})$ in a partial p-adic expansion as $m - \wp(\underline{\hat{a}}) = \sum_{i=1}^{e} b_i p^{e-i}$, where $b_i \geq 0$ such that $b_2, \ldots, b_e \leq p-1$, but b_1 might be arbitrarily large. Then, we have $m = \sum_{i=1}^{e} (\hat{a}_i + b_i) p^{e-i}$. Since for $1 \leq i \leq e-1$ we have $\hat{a}_i - \hat{a}_{i+1} \geq p-1 \geq b_{i+1} - b_i$, thus $\hat{a}_i + b_i \geq \hat{a}_{i+1} + b_{i+1}$, this implies that $m \in \mathcal{P}(\underline{a})$. □

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3.2. Combinatorial problems. The general aim now is to investigate the structure of $\mathcal{P}(\underline{a})$, for a given sequence $\underline{a} \in \mathbb{N}_0^e$: By Proposition 1, we have

$$\mu(\underline{a}) := \min \mathcal{P}(\underline{a}) = \min \mathcal{P}(\underline{\tilde{a}}) = \wp(\underline{\tilde{a}}),$$

where $\underline{\tilde{a}} \in \mathcal{N}$ is the associated hull sequence. Moreover, by Proposition 2, the set $\mathcal{P}(\underline{a}) = \mathcal{P}(\underline{\tilde{a}})$ is a co-finite subset of $\mathbb{N}_0$. In consequence, the problems associated with $\underline{a}$ are to determine the smallest integer $\sigma(\underline{a}) \in \mathbb{N}_0$ such that all $m \geq \sigma(\underline{a})$ are elements of $\mathcal{P}(\underline{a})$, and to determine the gap $\{\mu(\underline{a}) + 1, \ldots, \sigma(\underline{a}) - 1\} \setminus \mathcal{P}(\underline{a})$.

Note that by the proof of Proposition 2, we have $\mu(\underline{a}) \leq \sigma(\underline{a}) \leq \wp(\underline{\hat{a}})$, where $\underline{\hat{a}}$ is the associated p-enveloping sequence. Hence, in particular we have shown the following.

THEOREM 2. *Given $\underline{a} \in \mathcal{N}$ such that $||\underline{a}|| \geq p - 1$, then we have $\mu(\underline{a}) = \sigma(\underline{a}) = \wp(\underline{a})$, that is the associated mainline integers are given as $\mathcal{P}(\underline{a}) = \mathbb{N}_0 + \wp(\underline{a})$.*

4. Talu's theorem revisited. In this section, we develop our approach to describe the smooth epimorphisms onto a given Abelian p-group. We first prepare the setting.

4.1. Abelianisations. Let Γ be a group with signature, given by the p-datum $(x_1, \ldots, x_f; h)$, where $h \geq 0, f \geq 0$ and $x_f > 0$; note that we are allowing for the case $f = 0$, where the p-datum becomes $(-; h)$. Thus, Γ is generated by the set

$$\{a_k, b_k \ : \ 1 \leq k \leq h\} \quad \cup \quad \{c_{ij} \ : \ 1 \leq i \leq f, \ 1 \leq j \leq x_i\},$$

subject to the order relations

$$c_{ij}^{p^i} = 1, \quad \text{for} \quad 1 \leq i \leq f \quad \text{and} \quad 1 \leq j \leq x_i,$$

and the long relation

$$\prod_{k=1}^{h} [a_k, b_k] \cdot \prod_{i=1}^{f} \prod_{j=1}^{x_i} c_{ij} = 1.$$

Let $0 \leq f' \leq f$ be defined as follows:

$$f' := \begin{cases} 0, & \text{if } \sum_{i=1}^{f} x_i \leq 1, \\ \max\{1 \leq d \leq f \ : \ \sum_{i=d}^{f} x_i \geq 2\}, & \text{if } \sum_{i=1}^{f} x_i \geq 2. \end{cases}$$

In other words, we have $f' = 0$ if and only if the p-datum is $(-; h)$ or $(0, \ldots, 0, 1; h)$, while otherwise we have $f' = f$ if and only if $x_f \geq 2$, and if $x_f = 1$ then $1 \leq f' < f$ is largest such that $x_{f'} > 0$.

It follows from the above presentation that the Abelianisation $H := \Gamma/[\Gamma, \Gamma]$ of Γ, where $[\Gamma, \Gamma]$ denotes the derived subgroup of Γ, can be written as

$$H \cong \begin{cases} \mathbb{Z}^{2h}, & \text{if } f' = 0, \\ \mathbb{Z}^{2h} \oplus \mathbb{Z}_p^{x_1} \oplus \mathbb{Z}_{p^2}^{x_2} \oplus \cdots \oplus \mathbb{Z}_{p^f}^{x_f - 1}, & \text{if } f' = f, \\ \mathbb{Z}^{2h} \oplus \mathbb{Z}_p^{x_1} \oplus \mathbb{Z}_{p^2}^{x_2} \oplus \cdots \oplus \mathbb{Z}_{p^{f'}}^{x_{f'}}, & \text{if } 1 \leq f' < f. \end{cases}$$

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Indeed, identifying the elements of Γ with their images under the natural map $\Gamma \longrightarrow H$, we conclude that H is generated by the set

$$\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{f-1} \cup \mathcal{C}_f,$$

reflecting its decomposition as a direct sum of cyclic subgroups, where

$$\begin{aligned} \mathcal{C}_0 &:= \{a_k, b_k \in H \;:\; 1 \leq k \leq h\}, \\ \mathcal{C}_i &:= \{c_{ij} \in H \;:\; 1 \leq j \leq x_i\}, \qquad \text{for} \quad 1 \leq i \leq f-1, \\ \mathcal{C}_f &:= \{c_{fj} \in H \;:\; 1 \leq j \leq x_f - 1\}. \end{aligned}$$

4.2. Abelian groups. Let G be a non-trivial Abelian p-group given by

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e},$$

where $e \geq 1$, and $r_i \geq 0$ for $1 \leq i \leq e-1$, and $r_e \geq 1$. Moreover, let

$$\{g_{ij} \;:\; 1 \leq i \leq e,\ 1 \leq j \leq r_i\}$$

be a generating set reflecting the decomposition as a direct sum of cyclic subgroups.

Proceeding similarly as above, let $0 \leq e' \leq e$ be defined as follows:

$$e' := \begin{cases} 0, & \text{if } \sum_{i=1}^{e} r_i \leq 1, \\ \max\{1 \leq d \leq e \;:\; \sum_{i=d}^{e} r_i \geq 2\}, & \text{if } \sum_{i=1}^{e} r_i \geq 2. \end{cases}$$

Thus, we have $e' = 0$ if and only if $G \cong \mathbb{Z}_{p^e}$ is cyclic, while otherwise we have $e' = e$ if and only if $r_e \geq 2$, and if $r_e = 1$ then $1 \leq e' < e$ is largest such that $r_{e'} > 0$.

In particular, letting $\Omega_{e-1}(G) := \{g \in G \;:\; g^{p^{e-1}} = 1\}$, we observe that $\Omega_{e-1}(G)$ is a subgroup of G of index p if and only if $e' < e$. Hence, using the notation of Kulkarni's theorem 2.2, we have $\epsilon(G) = 2$ if and only if $p = 2$ and $e' = e$.

For the remainder of this section, we keep the notation fixed in Sections 4.1 and 4.2. Now, since any group homomorphism from Γ to an Abelian group factors through H, from Section 2.1, we get the following:

Proposition 3. *There is a smooth epimorphism $\phi : \Gamma \longrightarrow G$ if and only if there is an epimorphism $\varphi : H := \Gamma/[\Gamma, \Gamma] \longrightarrow G$ such that $\varphi(c_{ij})$ has order p^i, for $1 \leq i \leq f$ and $1 \leq j \leq x_i$, and $\prod_{j=1}^{x_f-1} \varphi(c_{fj})$ has order p^f.*

We also call such an epimorphism $\varphi : H \longrightarrow G$ smooth. Having this in place, we are prepared to state a necessary and sufficient arithmetic condition for the existence of a smooth epimorphism $\phi : \Gamma \longrightarrow G$. By Proposition 3, this amounts to giving such a condition for a smooth epimorphism $\varphi : H \longrightarrow G$, which is done in (3) and (4) for necessity and sufficiency, respectively. We call this collection of statements *Talu's theorem*, for the following reasons.

We pursue a strategy similar to the one employed in [**13**, Lemma 3.2] and [**13**, Theorem 3.3], where the statements of Theorems 3 and 4 are proven for the case p odd. Here, we are developing a general approach, which covers the case $p = 2$ as well, and with which we recover the results in [**13**] in a more unified manner. In particular, we

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close a gap in the proof of [**13**, Theorem 3.3], where the element there playing a role similar to the element 'g' in our proof of Theorem 4 is incorrectly stated.

THEOREM 3. *If there exists a smooth epimorphism $\varphi : H \longrightarrow G$, then we have $f' = f \leq e$, and the following inequalities are fulfilled:*

$$2h + \sum_{j=i}^{f} x_j \geq 1 + \sum_{j=i}^{e} r_j, \quad \textit{for} \quad 1 \leq i \leq f, \quad \textit{and} \quad 2h \geq \sum_{j=f+1}^{e} r_j.$$

Moreover, if $p = 2$ and $e' < f$, then x_f is even.

Proof. For $0 \leq i \leq e$ let $\Omega_i(G) = \{g \in G \ : \ g^{p^i} = 1\}$ and $\mho_i(G) = \{g^{p^i} \in G \ : \ g \in G\}$ be the characteristic subgroups of G consisting of all elements of order dividing p^i, and of all p^i-th powers, respectively. In particular, $\Omega_1(G)$ is an $\mathbb{F}_p$-vector space, where $\mathbb{F}_p$ denotes the field with p elements.

Now, the existence of the smooth epimorphism $\varphi : H \longrightarrow G$ implies $f' = f \leq e$. We have $\mho_e(H) \leq \ker(\varphi)$, thus letting

$$\tilde{H} := H/\mho_e(H) \cong \mathbb{Z}_p^{x_1} \oplus \mathbb{Z}_{p^2}^{x_2} \oplus \cdots \oplus \mathbb{Z}_{p^f}^{x_f - 1} \oplus \mathbb{Z}_{p^e}^{2h}$$

yields an epimorphism $\tilde{\varphi} : \tilde{H} \longrightarrow G$. Hence, dualising we get a monomorphism $\tilde{\varphi}^* : G^* := \operatorname{Hom}(G, \mathbb{C}^*) \longrightarrow \operatorname{Hom}(\tilde{H}, \mathbb{C}^*) = \tilde{H}^*$, that is $G \cong G^*$ is isomorphic to a subgroup of $\tilde{H}^* \cong \tilde{H}$. Thus, $\Omega_i(G)$ and $\mho_i(G)$ can be identified with subgroups of $\Omega_i(\tilde{H})$ and $\mho_i(\tilde{H})$, respectively, and hence we have

$$\dim_{\mathbb{F}_p}(\Omega_1(\mho_i(G))) \leq \dim_{\mathbb{F}_p}(\Omega_1(\mho_i(\tilde{H}))).$$

Now, for $0 \leq i \leq e - 1$ we have

$$\Omega_1(\mho_i(G)) \cong \mathbb{Z}_p^{r_{i+1}} \oplus \mathbb{Z}_p^{r_{i+2}} \oplus \cdots \oplus \mathbb{Z}_p^{r_e},$$

which yields

$$\dim_{\mathbb{F}_p}(\Omega_1(\mho_i(G))) = \sum_{j=i+1}^{e} r_j.$$

Similarly, for $0 \leq i \leq f - 1$ we have

$$\Omega_1(\mho_i(\tilde{H})) \cong \mathbb{Z}_p^{x_{i+1}} \oplus \mathbb{Z}_p^{x_{i+2}} \oplus \cdots \oplus \mathbb{Z}_p^{x_{f-1}} \oplus \mathbb{Z}_p^{x_f - 1} \oplus \mathbb{Z}_p^{2h},$$

yielding

$$\dim_{\mathbb{F}_p}(\Omega_1(\mho_i(\tilde{H}))) = 2h - 1 + \sum_{j=i+1}^{f} x_j,$$

while for $f \leq i \leq e - 1$ we get

$$\dim_{\mathbb{F}_p}(\Omega_1(\mho_i(\tilde{H}))) = 2h.$$

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Finally, let $p = 2$ and $e' < f \leq e$. Then, G has the form

$$G \cong \mathbb{Z}_2^{r_1} \oplus \mathbb{Z}_4^{r_2} \oplus \cdots \oplus \mathbb{Z}_{2^{e'}}^{r_{e'}} \oplus \mathbb{Z}_{2^e},$$

and thus

$$\Omega_f(G)/\Omega_{f-1}(G) \cong \mathbb{Z}_{2^{e-f}}/\mathbb{Z}_{2^{e-f+1}} \cong \mathbb{Z}_2.$$

Now, we observe that $\varphi(c_{fj}) \in \Omega_f(G) \setminus \Omega_{f-1}(G)$, for $1 \leq j \leq x_f - 1$, where $\prod_{j=1}^{x_f-1} \varphi(c_{fj}) \notin \Omega_{f-1}(G)$ as well, implying that $x_f - 1$ is odd. □

THEOREM 4. *Let $f' = f \leq e$, where in case $p = 2$ and $e' < f$ we additionally assume that x_f is even, such that*

$$2h + \sum_{j=i}^{f} x_j \geq 1 + \sum_{j=i}^{e} r_j, \quad \textit{for} \quad 1 \leq i \leq f, \quad \textit{and} \quad 2h \geq \sum_{j=f+1}^{e} r_j.$$

Then, there exists a smooth epimorphism $\varphi : H \longrightarrow G$.

Proof. By the inequalities assumed, we have

$$|\mathcal{C}_0 \cup \mathcal{C}_f \cup \mathcal{C}_{f-1} \cup \cdots \cup \mathcal{C}_i| \geq \sum_{j=i}^{e} r_j, \quad \text{for} \quad 1 \leq i \leq f, \quad \text{and} \quad |\mathcal{C}_0| \geq \sum_{j=f+1}^{e} r_j,$$

where the latter sum is empty if $e = f$. Thus, we may choose a subset $\mathcal{D}_{f+1} \subseteq \mathcal{C}_0$ of cardinality $\sum_{j=f+1}^{e} r_j$. Subsequently, for $f \geq i \geq 1$, we may recursively choose, disjointly from $\mathcal{D}_{f+1}$, pairwise disjoint sets

$$\mathcal{D}_i = \{d_{i,1}, \ldots, d_{i,r_i}\} \subseteq \mathcal{C}_0 \cup \mathcal{C}_f \cup \mathcal{C}_{f-1} \cup \cdots \cup \mathcal{C}_i$$

of cardinality r_i. Let

$$\mathcal{C}_i' := \mathcal{C}_i \setminus \left(\bigcup_{j=1}^{i} \mathcal{D}_j \right) \quad \text{for} \quad 1 \leq i \leq f, \quad \text{and} \quad \mathcal{C}_0' := \mathcal{C}_0 \setminus \left(\bigcup_{j=1}^{f+1} \mathcal{D}_j \right).$$

We are going to define a homomorphism $\varphi : H \longrightarrow G$ by specifying the image of $\mathcal{C}$:

The direct summand $\langle \mathcal{D}_{f+1} \rangle$ of H is a free Abelian group of rank $\sum_{j=f+1}^{e} r_j$, hence choosing $\varphi(c)$ appropriately, for $c \in \mathcal{D}_{f+1} \subseteq \mathcal{C}_0$, the direct summand

$$G' := \langle g_{ij} \ : \ f + 1 \leq i \leq e, \ 1 \leq j \leq r_i \rangle \cong \mathbb{Z}_{p^{f+1}}^{r_{f+1}} \oplus \mathbb{Z}_{p^{f+2}}^{r_{f+2}} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$$

of G becomes an epimorphic image of $\langle \mathcal{D}_{f+1} \rangle$. Thus, letting $\varphi(c) := 1$ for $c \in \mathcal{C}_0'$, we are done in the case $f = 0$. Hence, we may assume that $f' = f > 0$; thus, we have $x_f \geq 2$ and $\mathcal{C}_f \neq \emptyset$, where we may assume that $\mathcal{C}_f \cap \mathcal{D}_f \neq \emptyset$ whenever $r_f > 0$.

Now, for $d_{ij} \in \mathcal{C}_0 \cap \mathcal{D}_i$, where $1 \leq i \leq f$, we let $\varphi(d_{ij}) := g_{ij}$. Moreover, for $d_{ij} \in \mathcal{C}_k \cap \mathcal{D}_i$, where $1 \leq i \leq k < f \leq e$, we let $\varphi(d_{ij}) := g_{ij} \cdot g_{e,r_e}^{p^{e-k}}$, while for $c \in \mathcal{C}_k'$ we let $\varphi(c) := g_{e,r_e}^{p^{e-k}}$. To specify $\varphi(c)$ for $c \in \mathcal{C}_f$ we need some flexibility:

For $d_{ij} \in \mathcal{C}_f \cap \mathcal{D}_i$, where $1 \leq i \leq f$, we let $\varphi(d_{ij}) = g_{ij} \cdot c'$, for some $c' \in G$, while for $c \in \mathcal{C}_f'$ we just write $\varphi(c) = c'$. Then, we have to show that the elements c' can be

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chosen suitably to give rise to an epimorphism such that all $\varphi(c)$, where $c \in \mathcal{C}_f$, as well as $g := \prod_{c\in\mathcal{C}_f} \varphi(c)$ have order p^f.

In particular, $\varphi(c)$ will have order p^f, if $c \in \mathcal{C}_f \setminus \mathcal{D}_f$ and $c' \in G$ is chosen to have order p^f, or if $c \in \mathcal{C}_f \cap \mathcal{D}_f$ and $c' \in G'$ is chosen to have order dividing p^f. Moreover, φ will be an epimorphism whenever $f < e$ and we choose $c' \in G'$ for all $c \in \mathcal{C}_f \cap (\bigcup_{i=1}^f \mathcal{D}_i)$. The order condition on g will be checked by showing that the image of g under a suitable projection of G onto one of its direct summands already has order p^f. We now distinguish various cases as follows:

(i) Let $f < e' \le e$. Then, pick $c_0 \in \mathcal{C}_f$, and let $c_0' := g_{e',1}^{p^{e'-f}}$, while for $c_0 \neq c \in \mathcal{C}_f$ let $c' := g_{e,r_e}^{p^{e-f}}$; note that for $e' = e$ we have $r_e \ge 2$. Then, projecting g onto $\langle g_{e',1}\rangle$ yields c_0', which has order p^f.

(ii) Let $f = e' \le e$. Then, since $r_f = r_{e'} > 0$, we may assume that $d_{e',1} \in \mathcal{C}_f \cap \mathcal{D}_f$. For $c \in \mathcal{C}_f \setminus \mathcal{D}_f$ let $c' := g_{e,r_e}^{p^{e-f}}$, while for $c \in \mathcal{C}_f \cap \mathcal{D}_f$ let $c' := 1$; note that for $f = e' = e$ we have $r_e \ge 2$, and $d_{e,r_e} \in \mathcal{C}_0 \cup \mathcal{C}_f$ implies that φ is an epimorphism. Projecting g onto $\langle g_{e',1}\rangle$ yields $g_{e',1}$, which has order p^f.

(iii) Let $e' < f < e$. Then, for $c \in \mathcal{C}_f$ let $c' := (g_{e,1}^{p^{e-f}})^{a_c}$, where a_c is chosen coprime to p. Projecting g onto $\langle g_{e,1}\rangle$ yields $(g_{e,1}^{p^{e-f}})^a$, where $a := \sum_{c\in\mathcal{C}_f} a_c$. The latter element has order p^f if and only if a is coprime to p. If p is odd, this can be achieved by picking any $c \in \mathcal{C}_f$ and replacing a_c by $a_c + 1$ or $a_c - 1$, if necessary. If $p = 2$, then a_c is odd for all $c \in \mathcal{C}_f$, which, since $|\mathcal{C}_f| = x_f - 1$ is odd, implies that a is odd;

(iv) Let $e' < f = e$. Then, since $r_f = r_e = 1$, we may assume that $\mathcal{C}_f \cap \mathcal{D}_f = \{d_{e,1}\}$. For $c \in \mathcal{C}_f$ let $c' := g_{e,1}^{a_c}$, where a_c is chosen co-prime to p for $c \neq d_{e,1}$, while for $c = d_{e,1}$ we choose a_c such that $1 + a_c$ is co-prime to p. This implies that $\varphi(d_{e,1})$ has order p^f and that φ is an epimorphism. Projecting g onto $\langle g_{e,1}\rangle$ yields $g_{e,1}^a$, where $a := 1 + \sum_{c\in\mathcal{C}_f} a_c$. The latter element has order p^f if and only if a is co-prime to p. If p is odd, this can be achieved by picking $c \in \mathcal{C}_f$ and replacing a_c by $a_c + 1$ or $a_c - 1$, if necessary. If $p = 2$, then a_c is odd for all $d_{e,1} \neq c \in \mathcal{C}_f$, and $1 + a_c$ is odd for $c = d_{e,1}$, which, since $|\mathcal{C}_f| = x_f - 1$ is odd, implies that a is odd.

$\square$

5. Transforming to mainline integers. In this section, we show how mainline integers, as introduced in Section 3, can be used for the problem of determining the reduced genus spectrum of Abelian p-groups. In order to be able to reformulate the results of Section 4, we have to introduce quite a bit of notation, which we do in a series of steps.

5.1. Translating to non-increasing sequences. We define $\alpha : \mathbb{N}_0^{e+1} \longrightarrow \mathbb{N}_0^{e+1}$ by

$$\alpha(x_1, \ldots, x_e; x_0) := \left(\sum_{i=1}^{e} x_i + 2x_0, \sum_{i=2}^{e} x_i + 2x_0, \ldots, x_e + 2x_0, 2x_0\right),$$

which is injective and has image, using the notation from Section 3.1,

$$\mathrm{im}(\alpha) = \mathcal{N}'(e+1) := \{(a_1, \ldots, a_{e+1}) \in \mathcal{N}(e+1) \;:\; a_{e+1} \in 2\mathbb{N}_0\}.$$

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The inverse map $\alpha^{-1} : \mathcal{N}'(e+1) \longrightarrow \mathbb{N}_0^{e+1}$ is given by

$$\alpha^{-1}(a_1, \ldots, a_{e+1}) := \left(a_1 - a_2, \ldots, a_e - a_{e+1}; \frac{a_{e+1}}{2}\right).$$

5.2. Translating the reduced genus map. We continue to assume that G is a non-trivial Abelian p-group of exponent p^e. Letting $D(G) \subset \mathbb{N}_0^{e+1}$ be the data spectrum of G as introduced in Section 2.3, let

$$A(G) := \alpha(D(G)) \subset \mathbb{N}_0^{e+1}.$$

Then, the reduced genus map $g_0 : D(G) \longrightarrow \frac{1}{2} \cdot (\{-1\} \cup \mathbb{N}_0)$, given by

$$g_0(x_1, \ldots, x_e; h) = -p^e + \left(h + \frac{1}{2} \cdot \sum_{i=1}^{e} x_i\right) \cdot p^e - \frac{1}{2} \cdot \sum_{i=1}^{e} x_i p^{e-i},$$

can be rephrased as $\gamma = g_0 \circ \alpha^{-1} : A(G) \longrightarrow \frac{1}{2} \cdot (\{-1\} \cup \mathbb{N}_0)$, where explicitly

$$\gamma(a_1, \ldots, a_{e+1}) = -p^e + \frac{a_{e+1}}{2} + \frac{p-1}{2} \cdot \wp(a_1, \ldots, a_e),$$

where still $\wp(a_1, \ldots, a_e) := \sum_{i=1}^{e} a_i p^{e-i}$.

Moreover, as will become clear below, elements of the form $(x_1, \ldots, x_i, 0, \ldots, 0; h) \in D(G)$, for some $0 \leq i \leq e$, are of particular importance. These translate into elements of the form $(a_1, \ldots, a_i, 2a, \ldots, 2a) \in \mathcal{N}'(e+1)$. For the latter, we have

$$\gamma(a_1, \ldots, a_i, 2a, \ldots, 2a) = -p^e + a + \frac{p-1}{2} \cdot \wp(a_1, \ldots, a_i, 2a, \ldots, 2a),$$

where the argument of $\wp$ is a sequence of length e, and yields

$$\wp(a_1, \ldots, a_i, 2a, \ldots, 2a) = p^{e-i} \cdot \sum_{j=1}^{i} a_j p^{i-j} + 2a \cdot \sum_{j=0}^{e-i-1} p^j.$$

From that, we get

$$\gamma(a_1, \ldots, a_i, 2a, \ldots, 2a) = -p^e + p^{e-i} \cdot \left(a + \frac{p-1}{2} \cdot \wp(a_1, \ldots, a_i)\right).$$

In particular, for $i = 0$ we get $\gamma(2a, \ldots, 2a) = (a-1) \cdot p^e$, while for $i = e$ we recover $\gamma(a_1, \ldots, a_e, 2a) = -p^e + a + \frac{p-1}{2} \cdot \wp(a_1, \ldots, a_e)$. Note that we have $\gamma(a_1, \ldots, a_i, 2a, \ldots, 2a) \in \mathbb{Z}$, unless $p = 2$ and $i = e$ and a_e odd, in which case we have $\gamma(a_1, \ldots, a_e, 2a) \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

5.3. Translating Talu's theorem. Let again $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$, where $e \geq 1$, $r_i \geq 0$ for $1 \leq i \leq e-1$, and $r_e \geq 1$. Moreover, for $1 \leq i \leq e+1$ we fix

$$s_i := 1 + \sum_{j=i}^{e} r_j.$$

Hence, we have $\underline{s} := (s_1, \ldots, s_{e+1}) \in \mathcal{N}(e+1)$ such that $s_e \geq 2$ and $s_{e+1} = 1$. Now rephrasing Theorems 3 and 4 yields the following structured description of $A(G)$:

(i) For p odd, we have

$$A(G) = A_0 \cup A_1 \cup \cdots \cup A_e,$$

where for $0 \leq i \leq e$ we let, setting $a_0 := \infty$,

$$\begin{aligned} A_i \quad := \quad & \{\underline{a} \in \mathcal{N}'(e+1) \ : \ (a_1, \ldots, a_i) \geq (s_1, \ldots, s_i), \\ & a_{i+1} = \cdots = a_{e+1} \geq s_{i+1} - 1, \ a_i - a_{i+1} \geq 2\}. \end{aligned}$$

In particular, we have

$$A_0 = \{\underline{a} \in \mathcal{N}'(e+1) \ : \ a_1 = \cdots = a_{e+1} \geq s_1 - 1\}$$

and

$$A_e = \{\underline{a} \in \mathcal{N}'(e+1) \ : \ (a_1, \ldots, a_e) \geq (s_1, \ldots, s_e), \ a_e - a_{e+1} \geq 2\}.$$

For $0 \leq i < j \leq e$, the sequences in A_i satisfy $a_j = a_{j+1}$, while those in A_j satisfy $a_j - a_{j+1} \geq 2$; hence, $A_i \cap A_j = \emptyset$; thus, $A(G)$ is partitioned by the A_i.

(ii) For $p = 2$, letting $0 \leq e' \leq e$ be as defined in Section 4.2, we get

$$A(G) = A_0 \cup A_1 \cup \cdots \cup A_{e'} \cup A'_{e'+1} \cup \cdots \cup A'_e,$$

where for $1 \leq i \leq e$, we let

$$A'_i := \{\underline{a} \in A_i \ : \ a_i - a_{i+1} \in 2\mathbb{N}\}.$$

In particular, for $i = e$, we get

$$A'_e := \{\underline{a} \in A_e \ : \ a_e \in 2\mathbb{N}\}.$$

Note that we have $\gamma(A_e) \subseteq \frac{1}{2}\mathbb{Z}$ and $\gamma(A'_e) \subseteq \mathbb{Z}$, thus we recover Kulkarni's theorem (2.2) in the case of Abelian p-groups. The above partition of $A(G)$ now gives a way to compute the reduced minimum genus of G, but we need some more notation.

5.4. Towards the minimum genus. As we have seen above, for p odd, we have

$$\mu_0(G) = \min\{\min \gamma(A_i) \ : \ 0 \leq i \leq e\},$$

while for $p = 2$, we get

$$\mu_0(G) = \min\left(\{\min \gamma(A_i) \ : \ 0 \leq i \leq e'\} \cup \{\min \gamma(A'_i) \ : \ e' < i \leq e\}\right).$$

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(i) We proceed to derive formulae, in terms of the sequence $\underline{s} = (s_1, \ldots, s_{e+1})$ associated with G, to determine $\min \gamma(A_i)$, for $0 \leq i \leq e$: To this end, let

$$\underline{s}^i := \left(s_1, \ldots, s_i, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor, \ldots, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right) \in \mathcal{N}'(e+1)$$

and

$$\underline{s}^{i+} := \left(s_1, \ldots, s_{i-1}, s_i + \epsilon_i, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor, \ldots, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right) \in \mathcal{N}'(e+1),$$

where $\epsilon_i \in \{0, 1, 2\}$ is chosen minimal such that $s_i + \epsilon_i - 2 \cdot \lfloor \frac{s_{i+1}}{2} \rfloor \geq 2$, that is

$$\epsilon_i := \begin{cases} 0, & \text{if } s_i - s_{i+1} \geq 2, \\ 0, & \text{if } s_i - s_{i+1} = 1,\ s_{i+1} \text{ odd}, \\ 1, & \text{if } s_i - s_{i+1} = 1,\ s_{i+1} \text{ even}, \\ 1, & \text{if } s_i = s_{i+1},\ s_{i+1} \text{ odd}, \\ 2, & \text{if } s_i = s_{i+1},\ s_{i+1} \text{ even}. \end{cases}$$

Note that for $i = e$ we have $s_{e+1} = 1$ and $s_e \geq 2$, and thus $\epsilon_e = 0$; moreover, for $i = 0$ we let $\epsilon_0 = 0$.
It now follows from the description of A_i, and Proposition 1, that $\min \gamma(A_i)$ is attained precisely for the hull sequence

$$\underline{\tilde{s}}^{i+} = \left(\tilde{s}_1, \ldots, \tilde{s}_i, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor, \ldots, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right) \in \mathcal{N}'(e+1),$$

of $\underline{s}^{i+}$, where the prefix $(\tilde{s}_1, \ldots, \tilde{s}_i)$ of length i is determined as follows:
For $i \geq 1$, let $0 \leq i'' \leq i' < i$ be both maximal such that $s_{i'} - s_i \geq 1$ and $s_{i''} - s_i \geq 2$; hence, if $i'' < i'$, then we have $s_{i'} - s_{i'+1} = 1$, and $i' = 0$ and $i'' = 0$ refer to the cases $s_1 = s_i$ and $s_1 - s_i \leq 1$, respectively. Then, $(\tilde{s}_1, \ldots, \tilde{s}_i)$ is given as

$$\begin{array}{cl} (s_1, \ldots, s_i), & \text{if } \epsilon_i = 0, \\ (s_1, \ldots, s_{i'}, s_{i'+1} + 1, \ldots, s_i + 1), & \text{if } \epsilon_i = 1, \\ (s_1, \ldots, s_{i''}, s_{i''+1} + 1, \ldots, s_{i'} + 1, s_{i'+1} + 2, \ldots, s_i + 2), & \text{if } \epsilon_i = 2. \end{array}$$

Thus, letting

$$\mu_i := \gamma(\underline{s}^i) = -p^e + p^{e-i} \cdot \left(\left\lfloor \frac{s_{i+1}}{2} \right\rfloor + \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_i) \right),$$

we get

$$\min \gamma(A_i) = \gamma(\underline{\tilde{s}}^{i+}) = \begin{cases} \mu_i, & \text{if } \epsilon_i = 0, \\ \mu_i + \frac{1}{2} \cdot (p^{e-i'} - p^{e-i}), & \text{if } \epsilon_i = 1, \\ \mu_i + \frac{1}{2} \cdot (p^{e-i''} + p^{e-i'} - 2p^{e-i}), & \text{if } \epsilon_i = 2. \end{cases}$$

In particular, we have

$$\min \gamma(A_e) = \mu_e = -p^e + \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e),$$

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being attained precisely for $(s_1, \ldots, s_e, 0)$, and

$$\min \gamma(A_0) = \mu_0 = \left(\left\lfloor \frac{s_1}{2} \right\rfloor - 1\right) \cdot p^e,$$

being attained precisely for $(2 \cdot \lfloor \frac{s_1}{2} \rfloor, \ldots, 2 \cdot \lfloor \frac{s_1}{2} \rfloor)$.

(ii) It remains to consider $\min \gamma(A'_i)$, for $e' < i \leq e$, in the case $p = 2$: For $e' < i < e$, we have $s_i = s_{i+1} = 2$; hence , $\tilde{s}_i = 4$ and $2 \cdot \lfloor \frac{s_{i+1}}{2} \rfloor = 2$, while for $e' < i = e$, we have $s_e = 2$ and $s_{e+1} = 1$; hence, $\tilde{s}_e = 2$ and $2 \cdot \lfloor \frac{s_{e+1}}{2} \rfloor = 0$. Thus, the above description for $e' < i \leq e$ yields

$$\min \gamma(A'_i) = \min \gamma(A_i) = \gamma(\underline{\tilde{s}}^{i+}),$$

implying that the reduced minimum genus of G, just as for p odd, is given as

$$\mu_0(G) = \min\{\min \gamma(A_i) \ : \ 0 \leq i \leq e\}.$$

We are now prepared to prove the following.

THEOREM 5. *Keeping the above notation, we have*

$$\mu_0(G) = \min\{\mu_i \ : \ i \in \mathcal{I}(G)\},$$

where, letting $s_0 := \infty$*,*

$$\mathcal{I}(G) := \{0 \leq i \leq e \ : \ s_i - s_{i+1} \geq 2\} \cup \{0 \leq i \leq e \ : \ s_i - s_{i+1} = 1, \ s_{i+1} \ \textit{odd}\}.$$

In particular, we always have $\{0, e\} \subseteq \mathcal{I}(G)$*, but if* s_1 *is even, then to find* $\mu_0(G)$*, it suffices to consider* $i \in \mathcal{I}(G) \setminus \{0\}$ *only.*

Proof. We have already seen that $\mu_0(G) = \min\{\min \gamma(A_i) \ : \ 0 \leq i \leq e\}$, where $\min \gamma(A_i) = \gamma(\underline{\tilde{s}}^{i+})$. We aim to show that $\mu_0(G)$ can be determined by taking the minimum over a suitably chosen subset of indices $0 \leq i \leq e$. To this end, we consider the cases where $\epsilon_i \neq 0$; hence, we have $1 \leq i \leq e - 1$.

(i) If s_{i+1} is even and $s_i = s_{i+1}$, then we have

$$\begin{aligned} \underline{s}^{i+} &= (s_1, \ldots, s_{i-1}, s_i + 2, s_i, \ldots, s_i), \\ \underline{s}^{(i-1)+} &= (s_1, \ldots, s_{i-1} + \epsilon_{i-1}, s_i, s_i, \ldots, s_i), \end{aligned}$$

where $\epsilon_{i-1} = 0$ whenever $s_{i-1} \geq s_i + 2$, and $s_{i-1} + \epsilon_{i-1} = s_i + 2$ otherwise.

(ii) If s_{i+1} is even and $s_i - s_{i+1} = 1$, then we have

$$\begin{aligned} \underline{s}^{i+} &= (s_1, \ldots, s_{i-1}, s_i + 1, s_i - 1, \ldots, s_i - 1), \\ \underline{s}^{(i-1)+} &= (s_1, \ldots, s_{i-1} + \epsilon_{i-1}, s_i - 1, s_i - 1, \ldots, s_i - 1), \end{aligned}$$

where $\epsilon_{i-1} = 0$ whenever $s_{i-1} \geq s_i + 1$, and $s_{i-1} + \epsilon_{i-1} = s_i + 1$ otherwise.

(iii) If s_{i+1} is odd and $s_i = s_{i+1}$, then we have

$$\begin{aligned} \underline{s}^{i+} &= (s_1, \ldots, s_{i-1}, s_i + 1, s_i - 1, \ldots, s_i - 1), \\ \underline{s}^{(i-1)+} &= (s_1, \ldots, s_{i-1} + \epsilon_{i-1}, s_i - 1, s_i - 1, \ldots, s_i - 1), \end{aligned}$$

where $\epsilon_{i-1} = 0$ whenever $s_{i-1} \geq s_i + 1$, and $s_{i-1} + \epsilon_{i-1} = s_i + 1$ otherwise.

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In either of these cases, going over to hull sequences yields $\tilde{\underline{s}}^{i+} \geq \tilde{\underline{s}}^{(i-1)+}$, implying

$$\min \gamma(A_i) = \gamma(\tilde{\underline{s}}^{i+}) \geq \gamma(\tilde{\underline{s}}^{(i-1)+}) = \min \gamma(A_{i-1}),$$

so that $\min \gamma(A_i)$ need not be considered in finding $\mu_0(G)$.

Thus, we are left with the cases $0 \leq i \leq e$ such that $\epsilon_i = 0$, that is precisely those in $\mathcal{I}(G)$; for the latter we indeed have $\min \gamma(A_i) = \mu_i$. Recalling that $\epsilon_0 = \epsilon_e = 0$, we always have $\{0, e\} \subseteq \mathcal{I}(G)$. But, if s_1 is even, then since $s_1 \geq \cdots \geq s_e \geq 2$, we have

$$\mu_e = -p^e + \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e) \leq -p^e + \frac{s_1}{2} \cdot (p^e - 1) < \left(\frac{s_1}{2} - 1\right) \cdot p^e = \mu_0;$$

hence, in this case, $\min \gamma(A_0)$ need not be considered in finding $\mu_0(G)$. □

In other words, finding $\mu_0(G)$ is reduced to computing the minimum of $|\mathcal{I}(G)| \leq e+1$ numbers, which are given explicitly in terms of known invariants of G. In particular, this method to determine $\mu_0(G)$ will feature prominently in the proof of our main result, i.e., Main Theorem 1. Moreover, to underline the effectiveness of these techniques, in Sections 8 and 9, we give detailed example treatments of the Abelian p-groups of rank at most 2, and of the Abelian p-groups of exponent at most p^2, respectively.

5.5. Translating back. We translate the results back, to express $\mu_i = \min \gamma(A_i)$, for $i \in \mathcal{I}(G)$, in terms of the p-datum giving rise to μ_i, which by Section 5.2 is given as

$$\underline{x}^i = (x_1, \ldots, x_e; h) := \alpha^{-1}\left(s_1, \ldots, s_i, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor, \ldots, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right).$$

(i) If $r_i = s_i - s_{i+1} \geq 2$ and s_{i+1} is even, then we have

$$\underline{x}^i = \left(r_1, \ldots, r_i, 0, \ldots, 0; \frac{s_{i+1}}{2}\right),$$

yielding

$$\mu_i = p^e \cdot \left(\frac{s_{i+1}}{2} - 1 + \frac{1}{2} \cdot \sum_{j=1}^{i} r_j \left(1 - \frac{1}{p^j}\right)\right).$$

(ii) If $r_i = s_i - s_{i+1} \geq 1$ and s_{i+1} is odd, then we have

$$\underline{x}^i = \left(r_1, \ldots, r_{i-1}, r_i + 1, 0, \ldots, 0; \frac{s_{i+1} - 1}{2}\right),$$

yielding

$$\mu_i = p^e \cdot \left(\frac{s_{i+1} - 1}{2} - 1 + \frac{1}{2} \cdot \sum_{j=1}^{i} r_j (1 - \frac{1}{p^j}) + \frac{1}{2} \cdot (1 - \frac{1}{p^i})\right).$$

In particular, the case $i = 0$ is encompassed by the above cases, depending on whether s_1 is even or odd, respectively, by $\underline{x}^0 = (0, \ldots, 0; \lfloor \frac{s_1}{2} \rfloor)$, where this case need not be considered if s_1 is even. Moreover, the case $i = e$, since $s_{e+1} = 1$, is subsumed in the second of the above cases, by $\underline{x}^e = (r_1, \ldots, r_{e-1}, r_e + 1; 0)$.

Finally, we observe that the various $\mu_i = \min \gamma(A_i)$ to be considered belong to pairwise distinct orbit genera, as follows.

PROPOSITION 4. *The map*

$$\mathcal{I}(G) \longrightarrow \mathbb{Z} : i \mapsto \left\lfloor \frac{s_{i+1}}{2} \right\rfloor$$

is strictly decreasing, hence in particular is injective.

Proof. If both $i-1, i \in \mathcal{I}(G)$, then we have $s_i - s_{i+1} \geq 1$ anyway, where for s_i odd and s_{i+1} even from $s_i - s_{i+1} \geq 2$, we get $\lfloor \frac{s_i}{2} \rfloor = \frac{s_i - 1}{2} > \frac{s_{i+1}}{2} = \lfloor \frac{s_{i+1}}{2} \rfloor$. □

5.6. Maclachlan's method. We compare our approach with the method to compute the minimum genus for arbitrary non-cyclic Abelian groups given in [**8**].

To this end, let for a moment $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, where $s \geq 2$ and $1 < n_1 \mid n_2 \mid \cdots \mid n_s$; hence, the exponent of G equals n_s. Let $\nu_h \in \mathbb{N}_0$ be the reduced minimum genus afforded by all signatures of G with fixed orbit genus $h \geq 0$. Then, by [**8**, Theorem 4], the reduced minimum genus of G equals

$$\mu_0(G) = \min \left\{ \nu_h \ : \ 0 \leq h \leq \left\lfloor \frac{s}{2} \right\rfloor \right\},$$

where the numbers ν_h can be computed explicitly as

$$\nu_h = n_s \cdot \left(h - 1 + \frac{1}{2} \cdot \sum_{k=1}^{s-2h} \left(1 - \frac{1}{n_k} \right) + \frac{1}{2} \cdot \left(1 - \frac{1}{n_{s-2h}} \right) \right).$$

In our case of Abelian p-groups this reads as follows: We have

$$(n_1, \ldots, n_s) = \left(p, \ldots, p,\ p^2, \ldots, p^2,\ \ldots,\ p^e, \ldots, p^e\right),$$

where the entry p^i occurs r_i times, for $1 \leq i \leq e$; hence, we have $s = \sum_{i=1}^{e} r_i = s_1 - 1$. Thus, we are able to improve [**8**, Theorem 4], for non-cyclic Abelian p-groups, as follows.

By the injectivity of the map $\mathcal{I}(G) \longrightarrow \mathbb{Z} \ : \ i \mapsto \lfloor \frac{s_{i+1}}{2} \rfloor$, see Proposition 4, for $i \in \mathcal{I}(G)$ we have $\nu_{\lfloor \frac{s_{i+1}}{2} \rfloor} = \mu_i$, and thus by Theorem 5, we may compute $\mu_0(G)$ as a minimum over a set of cardinality $|\mathcal{I}(G)| \leq e + 1$ instead of one of cardinality $\lfloor \frac{s_1 - 1}{2} \rfloor + 1$, as

$$\mu_0(G) = \min\{\nu_{\lfloor \frac{s_{i+1}}{2} \rfloor} \ : \ i \in \mathcal{I}(G)\}.$$

Recall that whenever s_1 is even, the case $i = 0$ need not be considered, so that we always get a subset of the indices used in [**8**], where from the formulae in Section 5.5 to compute μ_i in terms of p-data, we recover the formulae for $\nu_{\lfloor \frac{s_{i+1}}{2} \rfloor}$ given in there. Moreover, our approach is also valid for cyclic p-groups, which are excluded in [**8**]. And since only genus $g \geq 2$ is considered there, the case $s = 2$ and some small Abelian groups have to be treated as exceptions; these reappear in Theorem 7, where we consider Abelian p-groups of non-positive reduced minimum genus.

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6. The main result. We keep the notation introduced in Section 5, in particular, let

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e},$$

where $e \geq 1$, $r_i \geq 0$ for $1 \leq i \leq e-1$, and $r_e \geq 1$.

PROPOSITION 5. *Suppose that*

$$\wp(r_{i+1}, \ldots, r_e) \geq p^{e-i} - 1,$$

for all $0 \leq i \leq e-1$ *such that* s_{i+1} *is odd. Then, we have* $\mu_0(G) = \mu_e$.

If $s_i > s_{i+1}$ *for all* $1 \leq i \leq e-1$ *such that* s_{i+1} *is odd, then the converse also holds.*

Proof. By Section 5.4, we have $\min \gamma(A_e) = \mu_e$ and $\min \gamma(A_0) = \mu_0$, while for $1 \leq i \leq e-1$ we have $\min \gamma(A_i) \geq \mu_i$. Moreover, for $p = 2$ and $e' < i \leq e$ we have $\min \gamma(A_i') = \min \gamma(A_i)$. Thus, it is sufficient to show that under the assumptions made we have $\mu_i \geq \mu_e$, for $0 \leq i \leq e-1$.

Now, $\mu_i \geq \mu_e$ is equivalent to saying

$$2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor \cdot p^{e-i} \geq (p-1) \cdot \wp(s_{i+1}, \ldots, s_e).$$

The right-hand side of this inequality being equal to

$$s_{i+1}p^{e-i} - s_e + \sum_{j=i+1}^{e-1} (s_{j+1} - s_j)p^{e-j} = s_{i+1}p^{e-i} - 1 - \wp(r_{i+1}, \ldots, r_e),$$

we thus have $\mu_i \geq \mu_e$, if and only if

$$\left(s_{i+1} - 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right) \cdot p^{e-i} \leq 1 + \wp(r_{i+1}, \ldots, r_e).$$

The latter inequality clearly holds if s_{i+1} is even, while if s_{i+1} is odd then it holds if and only if $\wp(r_{i+1}, \ldots, r_e) \geq p^{e-i} - 1$. This proves the first assertion.

For the second assertion, let $0 \leq i \leq e-1$ such that s_{i+1} is odd. Then, for $i \neq 0$ the assumption $s_i - s_{i+1} \geq 1$ implies $\epsilon_i = 0$, using the notation of Section 5.4, while we have $\epsilon_0 = 0$ anyway. Thus, we get $\mu_i = \min \gamma(A_i) \geq \mu_0(G) = \mu_e$, which by the above observation implies the second assertion. □

We are now in a position to prove our main result.

MAIN THEOREM 1. *Let G be a non-trivial abelian p-group of form*

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e},$$

such that

$$r_i \geq p-1 \quad \textit{for} \quad 1 \leq i \leq e-1, \quad \textit{and} \quad r_e \geq \max\{p-2, 1\}.$$

(*a*) *Then, the reduced minimum and stable upper genus of G is given as*

$$\mu_0(G) = \sigma_0(G) = \frac{1}{2} \cdot \left(-1 - p^e + \sum_{i=1}^{e} (p^e - p^{e-i}) \cdot r_i\right).$$

In particular, the reduced spectral gap is empty.

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(*b*) *Letting* $0 \leq j \leq e$ *be chosen smallest such that* $(r_{j+1}, \ldots, r_e) = (p-1, \ldots, p-1)$, *where* $j = e$ *refers to the case* $r_e \neq p-1$, *the reduced minimum genus* $\mu_0(G)$ *is afforded precisely by the p-data*

$$\left(r_1, \ldots, r_{i-1}, r_i + 1, 0, \ldots, 0; \frac{1}{2}(e-i)(p-1)\right),$$

where $j \leq i \leq e$ *is arbitrary for p odd, but restricted to the cases where* $e - i$ *is even for* $p = 2$. *In particular,* $\mu_0(G)$ *is always afforded by*

$$(r_1, \ldots, r_{e-1}, r_e + 1; 0).$$

Proof.

(*a*) By Sections 5.4 and 5.5, we have

$$\frac{1}{2} \cdot \left(-1 - p^e + \sum_{i=1}^{e} (p^e - p^{e-i}) \cdot r_i\right) = -p^e + \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e) = \mu_e.$$

Note that $\mu_e \in \frac{1}{2}\mathbb{Z}$, where $\mu_e \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ if and only if $p = 2$ and s_e is odd. Since, $\mu_0(G) \leq \sigma_0(G)$ anyway, it suffices to prove $\sigma_0(G) \leq \mu_e$ and $\mu_e \leq \mu_0(G)$.

(*i*) We first show $\sigma_0(G) \leq \mu_e$: By assumption, we have $s_i - s_{i+1} = r_i \geq p-1$ for $1 \leq i \leq e-1$, that is $||(s_1, \ldots, s_e)|| \geq p-1$. Hence, for any $m \in \mathbb{N}_0$, by Theorem 2, there is a sequence $(a_1, \ldots, a_e) \in \mathcal{N}(e)$ such that $(a_1, \ldots, a_e) \geq (s_1, \ldots, s_e)$ and $\wp(a_1, \ldots, a_e) = \wp(s_1, \ldots, s_e) + m$.
Let first p be odd, and $\sigma \in \mathbb{Z}$ such that $\sigma \geq \mu_e$. Then, there are $m \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$ such that $r < \frac{p-1}{2}$ and

$$\sigma = \mu_e + m \cdot \frac{p-1}{2} + r = -p^e + \frac{p-1}{2} \cdot (\wp(s_1, \ldots, s_e) + m) + r.$$

Let $(a_1, \ldots, a_e)$ such that $\wp(a_1, \ldots, a_e) = \wp(s_1, \ldots, s_e) + m$, and $a_{e+1} := 2r$, then, $a_e - a_{e+1} \geq (r_e + 1) - 2 \cdot \frac{p-3}{2} \geq 2$ implies $(a_1, \ldots, a_{e+1}) \in A_e$. Since $\gamma(a_1, \ldots, a_{e+1}) = -p^e + r + \frac{p-1}{2} \cdot \wp(a_1, \ldots, a_e) = \sigma$, from Section 5.3, we get $\sigma \in \mathrm{sp}_0(G)$.
Let now $p = 2$, and $\sigma \in \frac{1}{2}\mathbb{Z}$, such that $\sigma \geq \mu_e$. Let $m := 2(\sigma - \mu_e) \in \mathbb{N}_0$. Let $(a_1, \ldots, a_e)$ be as above such that $\wp(a_1, \ldots, a_e) = \wp(s_1, \ldots, s_e) + m$, and $a_{e+1} := 0$, then, $a_e - a_{e+1} \geq r_e + 1 \geq 2$ implies $(a_1, \ldots, a_{e+1}) \in A_e$. Since $\gamma(a_1, \ldots, a_{e+1}) = -2^e + \frac{1}{2} \cdot \wp(a_1, \ldots, a_e) = \sigma$, thus, if $e' = e$ from Section 5.3, we get $\sigma \in \mathrm{sp}_0(G)$.
If $e' < e$, then we have $e' = e - 1$ and $s_e = 2$, and hence $\gamma(A(G)) = \gamma(A_0) \cup \gamma(A_1) \cup \cdots \cup \gamma(A_{e-1}) \cup \gamma(A'_e) \subseteq \mathbb{Z}$. Since $\mu_e = \min \gamma(A'_e)$, we may assume that $\sigma \in \mathbb{Z}$, thus $m := 2(\sigma - \mu_e) \in \mathbb{N}_0$ is even. Hence, we get

$$a_e \equiv \wp(a_1, \ldots, a_e) = \wp(s_1, \ldots, s_e) + m \equiv s_e + m \equiv 0 \pmod{2},$$

implying that $(a_1, \ldots, a_{e+1}) \in A'_e$, and from Section 5.3, we get $\sigma \in \mathrm{sp}_0(G)$.

(*ii*) We show $\mu_e \leq \mu_0(G)$: Since $s_i - s_{i+1} = r_i \geq 1$, for all $1 \leq i \leq e-1$, by Proposition 5, we have to show $\wp(r_{i+1}, \ldots, r_e) \geq p^{e-i} - 1$, for all $0 \leq i \leq e-1$ such that s_{i+1} is odd.

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For p odd we have $r_j \geq p-1$ for $1 \leq j \leq e-1$, and $r_e \geq p-2$, where $\sum_{j=i+1}^{e} r_j = s_{i+1} - 1$ being even implies that $(r_{i+1}, \ldots, r_{e-1}, r_e) \neq (p-1, \ldots, p-1, p-2)$. Thus,

$$\wp(r_{i+1}, \ldots, r_e) > -1 + (p-1) \cdot \sum_{j=i+1}^{e} p^{e-j} = p^{e-i} - 2.$$

For $p = 2$ we have $r_j \geq 1$ for $1 \leq j \leq e$, directly yielding

$$\wp(r_{i+1}, \ldots, r_e) = \sum_{j=i+1}^{e} r_j \cdot 2^{e-j} \geq \sum_{j=i+1}^{e} 2^{e-j} = 2^{e-i} - 1.$$

(*iii*) We determine when $\mu_0(G)$ is attained: By (5.4), $\min \gamma(A_e) = \mu_e$ is attained precisely for $(s_1, \ldots, s_e, 0)$, corresponding to the p-datum $(r_1, \ldots, r_{e-1}, r_e + 1; 0)$.
Now, for $0 \leq i \leq e-1$, by the proof of Proposition 5, we have $\mu_i \geq \mu_e$. Moreover, replacing inequalities by equalities in the proof of Proposition 5 shows that $\mu_i = \mu_e$ is equivalent to s_{i+1} being odd and $\wp(r_{i+1}, \ldots, r_e) = p^{e-i} - 1$. Since $(r_{i+1}, \ldots, r_{e-1}, r_e) \geq (p-1, \ldots, p-1, \max\{p-2, 1\})$, the latter equality holds if and only if $(r_{i+1}, \ldots, r_e) = (p-1, \ldots, p-1)$. Since, in this case, $s_{i+1} - 1 = \sum_{j=i+1}^{e} r_j = (e-i)(p-1)$, we have s_{i+1} odd if and only if p is odd or $e-i$ is even. Hence, we conclude, by Section 5.4 again, that in these cases $\min \gamma(A_i) = \mu_i$ is attained precisely for

$$\left(s_1, \ldots, s_i, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor, \ldots, 2 \cdot \left\lfloor \frac{s_{i+1}}{2} \right\rfloor\right) = (s_1, \ldots, s_i, s_{i+1} - 1, \ldots, s_{i+1} - 1),$$

corresponding to the p-datum, using the notation of Section 5.5,

$$\underline{x}^i = (r_1, \ldots, r_{i-1}, r_i + 1, 0, \ldots, 0; \frac{1}{2}(e-i)(p-1)).$$

Note that we have $\mathcal{I}(G) = \{0, \ldots, e\}$ for p odd, while for $p = 2$ we at least get $\{0\} \cup \{e - 2 \cdot \lfloor \frac{e-j}{2} \rfloor, \ldots, e-2, e\} \subseteq \mathcal{I}(G)$; hence, the indices $0 \leq i \leq e$ affording $\mu_0(G)$ are indeed elements of the index set $\mathcal{I}(G)$, in accordance with Theorem 5.

$\square$

EXAMPLE 1.

(*i*) For p odd and $(r_1, \ldots, r_{e-1}, r_e) = (p-1, \ldots, p-1, p-2)$, that is the extremal case, we get

$$\mu_0(G) = \sigma_0(G) = \frac{1}{2} \cdot \Big(\big(e(p-1) - 3\big) \cdot p^e + 1 \Big).$$

Thus, we recover [**13**, Corollary 3.7], where $\sigma_0(G)$ is determined.

(*ii*) For p arbitrary and $(r_1, \ldots, r_{e-1}, r_e) = (p-1, \ldots, p-1, p-1)$, we get

$$\mu_0(G) = \sigma_0(G) = \frac{1}{2} \cdot \big(e(p-1) - 1\big) \cdot p^e,$$

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which for $p = 2$ specialises to $\mu_0(G) = \frac{e-2}{2} \cdot 2^e$.

As an immediate consequence of Main Theorem 1, invoking Kulkarni's theorem 2.2, we are able to describe the complete (reduced) spectrum of the groups in question.

COROLLARY 1.

(*a*) *The reduced spectrum of G is given as*

$$\mathrm{sp}_0(G) = \begin{cases} \mu_0(G) + \mathbb{N}_0, & \text{if } p \text{ odd or } r_e = 1, \\ \mu_0(G) + \frac{1}{2}\mathbb{N}_0, & \text{if } p = 2 \text{ and } r_e \geq 2. \end{cases}$$

(*b*) *Letting* $\delta = \delta(G) := \sum_{i=1}^{e}(ir_i - 1)$ *be the cyclic deficiency of G, then, the minimum genus and the spectrum of G are given as* $\mu(G) = 1 + p^\delta \cdot \mu_0(G)$ *and*

$$\mathrm{sp}(G) = \begin{cases} 1 + p^\delta \cdot \mu_0(G) + p^\delta \cdot \mathbb{N}_0, & \text{if } p \text{ odd or } r_e = 1, \\ 1 + 2^\delta \cdot \mu_0(G) + 2^{\delta-1} \cdot \mathbb{N}_0, & \text{if } p = 2 \text{ and } r_e \geq 2. \end{cases}$$

Moreover, for certain suitable co-finite sets of positive integers we are conversely able to provide Abelian p-groups having the specified set as their reduced spectrum:

THEOREM 6. *Let p be a prime, let* $e \geq 1$, *and let* $m \in \mathbb{N}$ *such that*

$$m \geq \begin{cases} (2e-1)p^e - 2 \cdot \frac{p^e - 1}{p-1} + 1, & \text{if } p \text{ odd}, \\ (e-1) \cdot 2^{e+1} + 2, & \text{if } p = 2. \end{cases}$$

Then, there is a group G of exponent p^e *such that* $\mu_0(G) = -p^e + \frac{p-1}{2} \cdot m$ *and*

$$\mathrm{sp}_0(G) = \begin{cases} \mu_0(G) + \mathbb{N}_0, & \text{if } p \text{ odd or } m \text{ even}, \\ \mu_0(G) + \frac{1}{2}\mathbb{N}_0, & \text{if } p = 2 \text{ and } m \text{ odd}. \end{cases}$$

Proof. We consider the sequence $(a_1, \ldots, a_e) \in \mathcal{N}$ given by $a_e := \max\{p-1, 2\}$, and $a_{e-i} := a_e + i \cdot 2(p-1)$ for $1 \leq i \leq e-1$.

(i) We first show that the lower bound for m given above coincides with $\wp(a_1, \ldots, a_e)$: To this end, let first $s_e(p) := \sum_{i=1}^{e} ip^i$. Then, we have

$$s_e(p) = \frac{p}{p-1} \cdot \left(ep^e - \sum_{i=0}^{e-1} p^i \right),$$

which is seen by induction: The case $e = 1$ is clear, and $s_{e+1}(p) = (e+1)p^{e+1} + s_e(p) = \frac{p}{p-1} \cdot \left((e+1)(p-1)p^e + ep^e - \sum_{i=0}^{e-1} p^i \right) = \frac{p}{p-1} \cdot \left((e+1)p^{e+1} - \sum_{i=0}^{e} p^i \right)$. In particular, for $p = 2$, we get $s_e(2) = (e-1) \cdot 2^{e+1} + 2$.
Now, for p odd, we have

$$\begin{aligned} \wp(a_1, \ldots, a_e) &= (p-1) \cdot \sum_{i=1}^{e} (2(e-i)+1)p^{e-i} \\ &= \frac{2(p-1)}{p} \cdot \sum_{i=1}^{e} ip^i - (p-1) \cdot \sum_{i=0}^{e-1} p^i, \end{aligned}$$

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which using the above expression for $s_e(p)$ can be rewritten as

$$\wp(a_1, \dots, a_e) = 2 \cdot \left(ep^e - \sum_{i=0}^{e-1} p^i \right) - p^e + 1 = (2e-1)p^e + 1 - 2 \cdot \sum_{i=0}^{e-1} p^i.$$

For $p = 2$, we get $\wp(a_1, \dots, a_e) = 2 \cdot \sum_{i=1}^{e} (e-i+1) \cdot 2^{e-i} = \sum_{i=1}^{e} i \cdot 2^i = s_e(2)$.

(ii) The strategy of proof now is reminiscent of the proof of Theorem 2: Given $m \geq \wp(a_1, \dots, a_e)$, then we write $m - \wp(a_1, \dots, a_e)$ in a partial p-adic expansion as $m - \wp(a_1, \dots, a_e) = \sum_{i=1}^{e} b_i p^{e-i}$, where $b_i \geq 0$ such that $b_2, \dots, b_e < p$, but b_1 might be arbitrarily large. Hence, letting $s_i := a_i + b_i$ for $1 \leq i \leq e$, we have $m = \sum_{i=1}^{e} s_i p^{e-i}$. Thus, for $1 \leq i \leq e-1$ we get

$$r_i := s_i - s_{i+1} = 2(p-1) + (b_i - b_{i+1}) \geq p - 1,$$

and $r_e := s_e - 1 \geq a_e - 1 = \max\{p-2, 1\}$. Hence, by Main Theorem 1, for the Abelian group of the form $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \cdots \oplus \mathbb{Z}_{p^e}^{r_e}$ we have

$$\sigma_0(G) = \mu_0(G) = \mu_e = -p^e + \frac{p-1}{2} \cdot \wp(s_1, \dots, s_e) = -p^e + \frac{p-1}{2} \cdot m.$$

Moreover, for $p = 2$, we have $a_e = 2$, and thus if m is even, we get $b_e = 0$ and hence $r_e = 1$, while if m is odd, we get $b_e = 1$ and hence $r_e = 2$. Thus, the statement on $\mathrm{sp}_0(G)$ follows from Corollary 1. □

7. Talu's conjecture. In general, we wonder which invariants of a non-trivial Abelian p-group G are determined by its spectrum. Given the latter, this determines the Kulkarni invariant $N = N(G)$, and hence the cyclic deficiency $\delta = \delta(G) = \log_p(N)$ is known as well whenever p is odd, while $\delta \in \{\log_p(N), 1 + \log_p(N)\}$ for $p = 2$. Thus, the spectrum also determines the reduced minimum and stable upper genus whenever p is odd, while the latter are known up to a factor of 2 for $p = 2$.

In this spirit, Talu's conjecture says that if p is odd, then even the isomorphism type of G is determined by its spectrum. Here, we include the case $p = 2$ as well, by conjecturing this to hold true up to finitely many finite sets of exceptions; we cannot possibly expect more, for example, in view of the sets of groups $\{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_8\}$, $\{\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2^3, \mathbb{Z}_2 \oplus \mathbb{Z}_8\}$, and $\{\mathbb{Z}_2^2 \oplus \mathbb{Z}_4, \mathbb{Z}_2^4\}$ presented in Section 8.1 and Table 3.

As for evidence, restricting to certain classes of Abelian p-groups, in Sections 8 and 9, we show that Talu's conjecture (including the case $p = 2$) holds within the class of cyclic p-groups with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$; within the class of Abelian p-groups of rank 2 with the exception of the groups $\{\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8\}$; within the class of elementary Abelian p-groups with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_2^2\}$; and within the class of p-groups of exponent p^2 without exception.

We proceed to prove a further finiteness result.

Proposition 6. *Let $\mathcal{G}$ be a set of groups (up to isomorphism) fulfilling the assumptions of Main Theorem 1 and having the same reduced minimum genus. Then, $\mathcal{G}$ is finite.*

Proof. Since the only admissible cyclic groups are $\mathbb{Z}_2$ and $\mathbb{Z}_3$, we may assume that the groups under consideration are non-cyclic, that is have an associated sequence

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$(s_1, \ldots, s_e) \neq (2, \ldots, 2)$. We show that, given any $m \geq 0$, there are only finitely many $e \geq 1$ and sequences $s_1 \geq \cdots \geq s_e \geq 2$, where $s_1 \geq 3$, such that

$$\mu_e = -p^e + \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e) \leq m.$$

This is seen as follows: The above inequality is equivalent to

$$\wp(s_1 - 2, \ldots, s_e - 2) = \wp(s_1, \ldots, s_e) - 2 \cdot \frac{p^e - 1}{p-1} \leq \frac{2(m+1)}{p-1}.$$

This implies $(s_1 - 2) \cdot p^{e-1} \leq \frac{2(m+1)}{p-1}$; hence, since $s_1 \geq 3$, we infer that e is bounded. Fixing e, we get $(s_i - 2) \cdot p^{e-i} \leq \frac{2(m+1)}{p-1}$, bounding s_i as well, for $1 \leq i \leq e$. □

In view of this, there necessarily are groups fulfilling the assumptions of main Theorem 1 whose reduced minimum genus exceeds any given bound. Hence, the point of Theorem 6 is to add some precision to this observation. But here positive results come to an end.

In Sections 7.1–7.4, we are going to construct counterexamples to Talu's conjecture (both for p odd and $p = 2$), consisting of pairs of groups having the same order and exponent, and pairs where these invariants are different, respectively. Moreover, by the results in Section 7.1, there cannot be an absolute bound on the cardinality of a set of Abelian p-groups having the same spectrum, even if we restrict to groups having the same order and exponent.

7.1. Counterexamples with fixed exponent. We construct non-isomorphic Abelian p-groups G and $\tilde{G}$ having the same order, exponent and spectrum, thus, in particular, having the same Kulkarni invariant, cyclic deficiency, minimum genus and reduced minimum genus.

In view of the results in Propositions 7 and 8, we let $e := 3$, and look at groups

$$G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2} \oplus \mathbb{Z}_{p^3}^{r_3} \quad \text{and} \quad \tilde{G} \cong \mathbb{Z}_p^{\tilde{r}_1} \oplus \mathbb{Z}_{p^2}^{\tilde{r}_2} \oplus \mathbb{Z}_{p^3}^{\tilde{r}_3}$$

of exponent p^3 fulfilling the assumptions of Main Theorem 1, that is coming from sequences $\underline{r} = (r_1, r_2, r_3)$ and $\underline{\tilde{r}} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)$ such that $r_1, r_2, \tilde{r}_1, \tilde{r}_2 \geq p-1$ and $r_3, \tilde{r}_3 \geq \max\{p-2, 1\}$. Then, by Corollary 1, the groups G and $\tilde{G}$ are as desired if and only if they are non-isomorphic such that $|G| = |\tilde{G}|$ and $\mu_0(G) = \mu_0(\tilde{G})$, and in case $p = 2$, we have $r_3 = 1$ if and only if $\tilde{r}_3 = 1$.

Now, $|G| = |\tilde{G}|$ translates into

$$r_1 + 2r_2 + 3r_3 = \log_p(|G|) = \log_p(|\tilde{G}|) = \tilde{r}_1 + 2\tilde{r}_2 + 3\tilde{r}_3,$$

and $\mu_0(G) = \mu_0(\tilde{G})$ translates into

$$\sum_{i=1}^{3} (p^3 - p^{3-i}) \cdot r_i = \sum_{i=1}^{3} (p^3 - p^{3-i}) \cdot \tilde{r}_i.$$

Hence, we conclude that we have $|G| = |\tilde{G}|$ and $\mu_0(G) = \mu_0(\tilde{G})$ if and only if $\underline{\tilde{r}} - \underline{r} \in \mathbb{Z}^3$ is an element of the row kernel of the matrix

$$P := \begin{bmatrix} 1 & p^3 - p^2 \\ 2 & p^3 - p \\ 3 & p^3 - 1 \end{bmatrix} \in \mathbb{Z}^{3\times 2} \subseteq \mathbb{Q}^{3\times 2}.$$

Now, P has $\mathbb{Q}$-rank 2, and its row kernel is given as $\ker(P) = \langle \underline{\rho} \rangle_{\mathbb{Q}}$, where

$$\underline{\rho} := (p+2, -2p-1, p) \in \mathbb{Z}^3.$$

Since $\gcd(p+2, -2p-1, p) = 1$, we conclude that $\ker(P) \cap \mathbb{Z}^3 = \langle \underline{\rho} \rangle_{\mathbb{Z}}$.

In conclusion, we have $|G| = |\tilde{G}|$ and $\mu_0(G) = \mu_0(\tilde{G})$ if and only if $\underline{\tilde{r}} = \underline{r} + k \cdot \underline{\rho}$ for some $k \in \mathbb{Z}$, where G and $\tilde{G}$ are non-isomorphic if and only if $k \neq 0$. Thus, this provides a complete picture of the counterexamples to Talu's conjecture in the realm of Abelian p-groups of exponent p^3 fulfilling the assumptions of Main Theorem 1.

In particular, for any $l \in \mathbb{N}$, there is a set of isomorphism types of cardinality at least $l+1$ consisting of groups having the same order and reduced minimum genus: Given $r_1 \geq p-1$ and $r_3 \geq p-2$, such that $r_3 \geq 2$ for $p=2$, and letting $r_2 := (p-1) + l \cdot (2p+1)$, all the sequences $\underline{r} + k \cdot \underline{\rho}$, where $0 \leq k \leq l$, give rise to groups as desired. The smallest counterexamples, in terms of group order, are given by choosing $\underline{r}$ as small as possible for the case $l = 1$.

(i) For p odd, this yields

$$\underline{r} = (p-1, 3p, p-2) \quad \text{and} \quad \underline{\tilde{r}} := \underline{r} + \underline{\rho} = (2p+1, p-1, 2p-2),$$

giving rise to groups such that

$$|G| = |\tilde{G}| = p^{10p-7} \quad \text{and} \quad \mu_0(G) = \mu_0(\tilde{G}) = \frac{1}{2} \cdot (5p^4 - 5p^3 - 2p^2 - p + 1).$$

Hence, in particular, for $p = 3$, we get $\underline{r} = (2, 9, 1)$ and $\underline{\tilde{r}} = (7, 2, 4)$, giving rise to groups such that $|G| = |\tilde{G}| = 3^{23}$ and $\mu_0(G) = \mu_0(\tilde{G}) = 125$.

(ii) In order to cover the case $p = 2$ as well, for p arbitrary, we may let

$$\underline{r} = (p-1, 3p, p) \quad \text{and} \quad \underline{\tilde{r}} = (2p+1, p-1, 2p),$$

giving rise to groups such that

$$|G| = |\tilde{G}| = p^{10p-1} \quad \text{and} \quad \mu_0(G) = \mu_0(\tilde{G}) = \frac{1}{2} \cdot (5p^4 - 3p^3 - 2p^2 - p - 1).$$

Hence, in particular, for $p = 2$, we get $\underline{r} = (1, 6, 2)$ and $\underline{\tilde{r}} = (5, 1, 4)$, giving rise to groups such that $|G| = |\tilde{G}| = 2^{19}$ and $\mu_0(G) = \mu_0(\tilde{G}) = \frac{45}{2}$.

7.2. Towards counterexamples with varying exponent. We construct non-isomorphic Abelian p-groups G and $\tilde{G}$ just having the same spectrum, thus, in particular, having the same Kulkarni invariant and minimum genus; hence, for p odd also having the same cyclic deficiency and reduced minimum genus.

To do so, we look at groups afforded by sequences $\underline{r} = (r_1, \ldots, r_e)$ and $\underline{\tilde{r}} = (\tilde{r}_1, \ldots, \tilde{r}_{\tilde{e}})$, where $1 \leq \tilde{e} \leq e$, fulfilling the assumptions of Main Theorem 1, that is $r_i \geq p-1$ for $1 \leq i \leq e-1$, and $\tilde{r}_i \geq p-1$ for $1 \leq i \leq \tilde{e}-1$, as well as $r_e, \tilde{r}_{\tilde{e}} \geq \max\{p-2, 1\}$. We indicate the heuristics we are using.

Let $\delta \geq -2e + \frac{e(e+1)}{2} \cdot (p-1)$ whenever p is odd, and $\delta \geq \frac{e(e-1)}{2}$ for $p = 2$, in each case the lower bound being the cyclic deficiency associated with the smallest admissible sequence $(p-1, \dots, p-1, \max\{p-2, 1\})$; note that smaller values of δ are not achieved at all. We now aim at varying $\underline{r}$ within the set of admissible sequences, such that $\log_p(|G|) = \delta + e = \sum_{i=1}^{e} ir_i$ is kept fixed, but

$$2\mu_e + 1 = -p^e + \sum_{i=1}^{e}(p^e - p^{e-i}) \cdot r_i = -p^e + \sum_{i=1}^{e} \frac{p^e - p^{e-i}}{i} \cdot ir_i$$

is maximised and minimised, respectively.

To this end, we observe that the arithmetic mean of the first i entries of the sequence $(p^{e-1}, \dots, p, 1)$ is given as $\frac{1}{i} \cdot \sum_{j=e-i}^{e-1} p^j = \frac{1}{i} \cdot \frac{p^e - p^{e-i}}{p-1}$, for $1 \leq i \leq e$; hence, the sequence $(\frac{p^e-p^{e-1}}{1}, \frac{p^e-p^{e-2}}{2}, \dots, \frac{p^e-1}{e})$ is strictly decreasing. Thus, $2\mu_e + 1$ becomes largest (respectively smallest) by choosing the last (respectively first) $e-1$ entries of $\underline{r}$ as small as possible, and adjusting the first (respectively last) entry such that $\underline{r}$ has cyclic deficiency δ associated with it.

We now distinguish the cases p odd and $p = 2$.

7.3. Counterexamples with varying exponent for p odd. We keep the setting of Section 7.2, and let p be odd.

Then, maximizing yields $2\mu_e + 1 \leq 2\mu_e(a, p-1, \dots, p-1, p-2) + 1$, where

$$a := \delta + 2e - \frac{(e+2)(e-1)}{2} \cdot (p-1).$$

Note that by the choice of δ, we conclude that $a \geq p-1$; hence, the right-hand side of the above inequality is achieved. By a straightforward computation, we get

$$\begin{aligned} 2\mu_e + 1 \leq & \left(\delta + \tfrac{(e-1)(e+6)}{2} - \tfrac{e(e-1)(p-1)}{2}\right) \cdot p^e \\ & - \left(\delta + \tfrac{e(e+5)}{2}\right) \cdot p^{e-1} + 2 \end{aligned}.$$

Similarly, minimizing yields $2\mu_e + 1 \geq 2\mu_e(p-1, p-1, \dots, p-1, b) + 1$, where

$$b := \frac{\delta}{e} - \frac{e-1}{2} \cdot (p-1) + 1.$$

Note that here b in general is not integral, so that the right-hand side of the above inequality might not be achieved. By a straightforward computation, we get

$$2\mu_e + 1 \geq \left(\frac{\delta}{e} + \frac{(e-1)(p-1)}{2} - 1\right) \cdot p^e + \frac{(e+1)(p-1)}{2} - \frac{\delta}{e}.$$

Hence, we have to ensure that the above upper bound for $2\mu_{\tilde{e}} + 1$, applied to some $1 \leq \tilde{e} < e$, is at least as large as the lower bound for $2\mu_e + 1$.

Viewing the upper and lower bounds as linear functions in δ, in order to have an unbounded range of candidates δ to check, the slope of the upper bound function should exceed the slope of the lower bound function. This yields

$$(p-1)p^{\tilde{e}-1} \geq \frac{p^e-1}{e};$$

in other words

$$e \geq \sum_{i=1}^{e} p^{(e-i)-(\tilde{e}-1)} = \sum_{i=0}^{e-\tilde{e}} p^i + \sum_{i=1}^{\tilde{e}-1} p^{-i} = \frac{p^{e-\tilde{e}+1}-1}{p-1} + \sum_{i=1}^{\tilde{e}-1} p^{-i},$$

implying

$$e \geq \frac{p^{e-\tilde{e}+1}-1}{p-1} + 1.$$

Thus, we are led to consider the case $\tilde{e} = e-1$. Then, the smallest possible choices, so that the upper bound function actually is at least as large as the lower bound function, are $e := p+2$ and $\tilde{e} := p+1$. This leads to the following specific examples, which actually have been found by running an explicit search for odd $p \leq 11$, using the computer algebra system GAP [3], and observing the pattern arising:

Let

$$\underline{r} := (p-1, \ldots, p-1, p, p^3+p^2-2),$$

thus having p consecutive entries $p-1$, and for $p \geq 5$, let

$$\underline{\tilde{r}} := (p^4+3p^3+2p^2-p-1, p-1, \ldots, p-1, p, p, p-1, p-2),$$

thus having $p-4$ consecutive entries $p-1$, while for $p=3$, let

$$\underline{\tilde{r}} := (p^4+3p^3+2p^2-p, p, p-1, p-2)|_{p=3} = (177, 3, 2, 1);$$

a few explicit cases are given in Table 1.

Then, by a straightforward computation, we indeed have

$$\delta = \tilde{\delta} = p^4 + \frac{7}{2}p^3 + 3p^2 - \frac{5}{2}p - 6,$$

and

$$\mu_e(\underline{r}) = \mu_{\tilde{e}}(\underline{\tilde{r}}) = \frac{1}{2} \cdot \left((p^3+2p^2-4) \cdot p^{p+2} - p^3 - p^2 + 1\right).$$

Thus, $\underline{r}$ and $\underline{\tilde{r}}$ give rise to groups G and $\tilde{G}$, respectively, by Corollary 1 having the same spectrum, but having distinct exponents p^{p+2} and p^{p+1}, respectively.

7.4. Counterexamples with varying exponent for $p=2$. We keep the setting of Section 7.2, and let $p=2$. Since our approach involves sequences $\underline{r}$, such that $r_e \geq 2$, for $\underline{\tilde{r}}$ we distinguish the cases $\tilde{r}_{\tilde{e}} \geq 2$ and $\tilde{r}_{\tilde{e}} = 1$:

(i) Let first $\tilde{r}_{\tilde{e}} \geq 2$. Then, by Corollary 1, the groups G and $\tilde{G}$ associated with these sequences have the same spectrum if and only if they have the same cyclic

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Table 1. Counterexamples with varying exponent for p odd

p	$\underline{r}$	$\tilde{\underline{r}}$
3	(2, 2, 2, 3, 34)	(177, 3, 2, 1)
5	(4, 4, 4, 4, 4, 5, 148)	(1044, 4, 5, 5, 4, 3)
7	(6, 6, 6, 6, 6, 6, 6, 7, 390)	(3520, 6, 6, 6, 7, 7, 6, 5)
11	(10, ..., 10, 11, 1450)	(18864, 10, ..., 10, 11, 11, 10, 9)
13	(12, ..., 12, 13, 2364)	(35476, 12, ..., 12, 13, 13, 12, 11)
17	(16, ..., 16, 17, 5200)	(98820, 16, ..., 16, 17, 17, 16, 15)

p	e	δ	μ_e
3	5	189	4964
5	7	1119	6679613
7	9	3725	8817262934
11	13	19629	27083067676913144
13	15	36719	64775747609331851801
17	19	101535	655895227302212659718161655

deficiency and reduced minimum genus. Thus, a similar analysis, as in Section 7.3, yields $2\mu_e + 1 \leq 2\mu_e(a, 1, \dots, 1) + 1$, where $a := \delta - \frac{(e-2)(e+1)}{2}$; hence, we get

$$2\mu_e + 1 \leq \left(\delta - \frac{(e-2)(e-3)}{2} - 1\right) \cdot 2^{e-1} + 1.$$

Similarly, we get $2\mu_e + 1 \geq 2\mu_e(1, \dots, 1, b) + 1$, where $b := \frac{\delta}{e} - \frac{e-3}{2}$, yielding

$$2\mu_e + 1 \geq \left(\frac{\delta}{e} + \frac{e-3}{2}\right) \cdot 2^e + \frac{e+1}{2} - \frac{\delta}{e}.$$

Again comparing slopes with respect to δ of the upper and lower bound functions yields $2^{\tilde{e}-1} \geq \frac{2^e-1}{e}$, which is the same formula, as in section 7.3, specialised to $p = 2$. Hence, we obtain the condition $e \geq 2^{e-\tilde{e}+1}$. Hence we are led to consider the case $\tilde{e} = e - 1$, where the smallest possible choices turn out to be $e := 4$ and $\tilde{e} = 3$:

An explicit search using GAP yields, as smallest cases with respect to δ,

$$\underline{r} := (1, 1, 1, 18) \quad \text{and} \quad \tilde{\underline{r}} := (69, 1, 2).$$

Then, we get

$$\delta = \tilde{\delta} = 74 \quad \text{and} \quad \mu_e(\underline{r}) = \mu_{\tilde{e}}(\tilde{\underline{r}}) = \frac{287}{2}.$$

Thus, $\underline{r}$ and $\tilde{\underline{r}}$ give rise to groups G and $\tilde{G}$, respectively, by Corollary 1 having the same spectrum, and both fulfilling the '$e' = e$' property, but having distinct exponents 16 and 8, respectively.

(ii) Let now $\tilde{r}_{\tilde{e}} = 1$. Then, by Corollary 1, the groups G and $\tilde{G}$ associated with the sequences $\underline{r}$ and $\tilde{\underline{r}}$ have the same spectrum if and only if for the associated cyclic deficiency and reduced minimum genus, we have

$$\tilde{\delta} = \delta - 1 \quad \text{and} \quad \mu_{\tilde{e}}(\tilde{\underline{r}}) = 2\mu_e(\underline{r}).$$

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Considering again the slopes with respect to δ of the upper and lower bound functions, from $2\mu_{\tilde{e}}(\underline{\tilde{r}}) + 1 = 4\mu_e(\underline{r}) + 1 = 2 \cdot (2\mu_e(\underline{r}) + 1) - 1$, we get $2^{\tilde{e}-1} \geq 2 \cdot \frac{2^e-1}{e}$, implying $e \geq 2^{e-\tilde{e}+2}$, thus leading us to consider the case $\tilde{e} = e - 1$, where the smallest possible choices turn out to be $e := 8$ and $\tilde{e} = 7$:
An explicit search using GAP yields, as smallest cases with respect to δ,

$$\underline{r} := (1, 1, 1, 1, 1, 1, 1, 1025) \quad \text{and} \quad \underline{\tilde{r}} := (8199, 1, 1, 1, 1, 1, 1).$$

Then, we get

$$\delta = 8220 = \tilde{\delta} + 1 \quad \text{and} \quad \mu_e(\underline{r}) = 131328 = \frac{1}{2} \cdot \mu_{\tilde{e}}(\underline{\tilde{r}}).$$

Thus, $\underline{r}$ and $\underline{\tilde{r}}$ give rise to groups G and $\tilde{G}$, respectively, by Corollary 1 having the same spectrum, precisely one of them fulfilling the '$e' = e$' property, and having distinct exponents 256 and 128, respectively.

7.5. Counterexamples with fixed exponent again. We keep the setting of Section 7.4(ii), in order to remark that the above approach can also be used to find counterexamples fulfilling $\tilde{e} = e$:

Actually, by Propositions 7 and 8, there cannot be counterexamples for $1 \leq \tilde{e} = e \leq 2$, except the groups $\{\mathbb{Z}_2, \mathbb{Z}_2^2\}$. Indeed, the latter is a specific counterexample, for $\delta = 1$, but our approach aims at finding $1 \leq \tilde{e} \leq e$ allowing for an infinite range of candidates δ. Moreover, it turns out that for $\tilde{e} = e = 3$ and any $\delta \geq 0$ the upper bound for $2\mu_e + 1$ is smaller than the lower bound for $2 \cdot (2\mu_e + 1) - 1$, excluding this case. Hence, we are led to consider the case $e := 4$:

An explicit search using GAP yields, as smallest cases with respect to δ,

$$\underline{r} := (1, 1, 1, 21) \quad \text{and} \quad \underline{\tilde{r}} := (80, 1, 1, 1).$$

Then, we get

$$\delta = 86 = \tilde{\delta} + 1 \quad \text{and} \quad \mu_e(\underline{r}) = 166 = \frac{1}{2} \cdot \mu_{\tilde{e}}(\underline{\tilde{r}}).$$

Thus, $\underline{r}$ and $\underline{\tilde{r}}$ give rise to groups G and $\tilde{G}$, respectively, by Corollary 1 having the same spectrum, precisely one of them fulfilling the '$e' = e$' property, and having the same exponent 16.

8. Examples: Small rank. In the remaining two sections, in order to show that the method developed in Section 5 actually is efficient to find the minimum reduced genus, and in suitable cases even all of the reduced genus spectrum, we explicitly work out some 'small' examples. Moreover, we show that Talu's conjecture (including the case $p = 2$) holds within the various classes of Abelian p-groups considered.

In this section, now, we deal with the Abelian p-groups of minimum genus at most 1, with those of rank at most 2, for which we are particularly interested in finding the smallest positive reduced genus, and with a few explicit Abelian 2-groups and 3-groups, whose genus spectrum we determine completely.

Table 2. Non-positive reduced minimum genus

	G	μ_0	p-datum
$s_1 = s_e = 2$	$\mathbb{Z}_{p^e}$	0	$(0, \ldots, 0; 1)$
$s_1 = s_e = 3$	$\mathbb{Z}_{p^e}^2$	0	$(0, \ldots, 0; 1)$
$s_1 = 3 > s_e = 2$	$\mathbb{Z}_{p^{e'}} \oplus \mathbb{Z}_{p^e}$	0	$(0, \ldots, 0; 1)$

		G	μ_e	p-datum
	$s_1 = s_e = 2$	$\mathbb{Z}_{p^e}$	-1	$(0, \ldots, 0, 2; 0)$
$p = 3,\ e = 1,$	$s_1 = 3$	$\mathbb{Z}_3^2$	0	$(3; 0)$
$p = 2,\ e = 1,$	$s_1 = 3$	$\mathbb{Z}_2^2$	$-\frac{1}{2}$	$(3; 0)$
$p = 2,\ e = 1,$	$s_1 = 4$	$\mathbb{Z}_2^3$	0	$(4; 0)$
$p = 2,\ e = 2,$	$s_1 = 3 > s_2 = 2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	0	$(1, 2; 0)$

THEOREM 7. *The non-trivial Abelian p-groups G such that $\mu(G) \in \{0, 1\}$ are given as follows:*

(*a*) *We have $\mu(G) = 0$, that is $\mu_0(G) \in \{-1, -\frac{1}{2}\}$, if and only if*

$$G \cong \mathbb{Z}_{p^e} \quad \text{or} \quad G \cong \mathbb{Z}_2^2.$$

(*b*) *We have $\mu(G) = 1$, that is $\mu_0(G) = 0$, if and only if*

$$G \cong \mathbb{Z}_{p^{e'}} \oplus \mathbb{Z}_{p^e} \text{ for } e' < e, \quad \text{or} \quad G \cong \mathbb{Z}_{p^e}^2 \text{ for } p^e \neq 2, \quad \text{or} \quad G \cong \mathbb{Z}_2^3.$$

Proof. We have $\mu_i \leq 0$, for $i \in \mathcal{I}(G)$, if and only if

$$\frac{p-1}{2} \cdot \wp(s_1, \ldots, s_i) \leq p^i - \lfloor \frac{s_{i+1}}{2} \rfloor.$$

From $s_1 \geq \cdots \geq s_i \geq 2 \cdot \lfloor \frac{s_{i+1}}{2} \rfloor + 2$, we get

$$(\lfloor \frac{s_{i+1}}{2} \rfloor + 1) \cdot (p^i - 1) \leq \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_i),$$

hence assuming $\mu_i \leq 0$ yields

$$(\lfloor \frac{s_{i+1}}{2} \rfloor + 1) \cdot (p^i - 1) \leq p^i - \lfloor \frac{s_{i+1}}{2} \rfloor,$$

that is $\lfloor \frac{s_{i+1}}{2} \rfloor \cdot p^i \leq 1$, a contradiction for $1 \leq i \leq e - 1$.

For $i = 0$, we get $\mu_0 \leq 0$ if and only if $\lfloor \frac{s_1}{2} \rfloor \leq 1$, or equivalently $2 \leq s_1 \leq 3$, yielding the cases as indicated in the first part of Table 2, where $1 \leq e' < e$.

For $i = e$, we get $\mu_e \leq 0$ if and only if $\frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e) \leq p^e$; hence, since $s_1 \geq \cdots \geq s_e \geq 2$ implies $p^e - 1 = \frac{p-1}{2} \cdot \wp(2, \ldots, 2) \leq \frac{p-1}{2} \cdot \wp(s_1, \ldots, s_e)$, we get the cases indicated in the second part of Table 2. □

Note that the explicit cases for $p \leq 3$ are precisely the non-cyclic Abelian groups of order at most 9, which are treated as exceptional cases in [**8**, Theorem 4].

This compares to the well-known description of finite group actions on compact Riemann surfaces of genus $g \leq 1$, see [**11**, Appendix] or [**2**, Section 6.7], as follows:

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The cases of $\mu_e < 0$ are precisely the Abelian p-groups amongst the groups with signature of positive Euler characteristic, and belong to branched self-coverings of the Riemann sphere. The cases of $\mu_0 = 0$ and $\mu_e = 0$ are precisely the Abelian p-groups being smooth epimorphic images of the groups with finite signature of zero Euler characteristic, belonging to unramified coverings of surfaces of genus 1, and to branched coverings of the Riemann sphere by surfaces of genus 1, respectively.

The groups occurring in Theorem 7 encompass all non-trivial Abelian p-groups of rank at most 2. These we next consider in more detail, and determine their smallest genus $\mu^+(G) \geq 2$, or equivalently their smallest reduced genus $\mu_0^+(G) > 0$.

THEOREM 8. *Let $G \cong \mathbb{Z}_{p^e}$ be a non-trivial cyclic p-group, for some $e \geq 1$. If $p^e \neq 2, 3, 4$, then the smallest genus $\mu^+(G) \geq 2$ is given as*

$$\mu^+(G) = \frac{1}{2} \cdot p^{e-1} \cdot (p-1),$$

while for $p^e \in \{2, 3, 4\}$ we have $\mu^+(G) = 2$.

Proof. Note that $\mu^+(G) = \mu_0^+(G) + 1$, and $(s_1, \dots, s_e) = (2, \dots, 2)$. We have

$$A_0 = \{(2a, \dots, 2a) \ : \ a \geq 1\},$$

and hence $\gamma(2a, \dots, 2a) = (a-1) \cdot p^e$ yields min $\gamma(A_0) = \mu_0 = 0$ and

$$\min \ (\gamma(A_0) \setminus \{0\}) = p^e.$$

For $1 \leq i \leq e-1$, using the notation of Section 5.4, we have $i' = i'' = 0$ and $\epsilon_i = 2$; thus, we have $\mu_i = 0$ and

$$\min \ \gamma(A_i) = p^e - p^{e-i} \geq p^e - p^{e-1} = \min \ \gamma(A_1).$$

Moreover, for $p = 2$, we have $e' = 0$ and min $\gamma(A_i') = \min \ \gamma(A_i)$. Now let $i = e$:

(i) Let first p be odd. Then, we have

$$A_e = \{(a_1, \dots, a_e, 2a) \ : \ a_1 \geq \cdots \geq a_e \geq 2(a+1)\};$$

hence, comparing $\gamma(a_1, \dots, a_e, 2a) = -p^e + a + \frac{p-1}{2} \cdot \wp(a_1, \dots, a_e)$ with $\gamma(2, \dots, 2, 0) = \min \ \gamma(A_e) = \mu_e = -1$ yields

$$\min \ (\gamma(A_e) \setminus \{-1\}) = \frac{1}{2} \cdot p^{e-1} \cdot (p-1) - 1 \geq 0,$$

being attained precisely for $(3, 2, \dots, 2, 0)$. We have $p^{e-1} \cdot (p-1) = 2$ if and only if $p = 3$ and $e = 1$. Thus, if $p^e \neq 3$, then we have $\mu_0^+(G) = \frac{1}{2} \cdot p^{e-1} \cdot (p-1) - 1$. The case $p^e = 3$ is presented in Section 8.2) and Table 4.

(ii) Let, now, $p = 2$. We have

$$A_e' = \{(a_1, \dots, a_e, 2a) \ : \ a_1 \geq \cdots \geq a_e \geq 2(a+1), \ a_e \text{ even}\}.$$

We assume that $e \geq 3$. Comparing $\gamma(a_1, \dots, a_e, 2a) = -2^e + a + \frac{1}{2} \cdot \wp(a_1, \dots, a_e)$ with $\gamma(2, \dots, 2, 0) = \mu_e = -1$, we get

$$\min \ (\gamma(A_e') \setminus \{-1\}) = 2^{e-2} - 1 > 0,$$

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being attained precisely for $(3, 2, \ldots, 2, 0)$. Hence, we conclude $\mu_0^+(G) = 2^{e-2} - 1$. The cases $p^e = 2, 4$ are presented in Section 8.1 and Table 3. □

Hence, we recover the results in [**4**] and [**6**, Proposition 3.3]. Moreover, we conclude that a cyclic p-group is uniquely determined by its smallest genus $\mu^+(G) \geq 2$, with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$, which by Section 8.1 and Table 3, indeed have the same spectrum. In particular, Talu's conjecture (including the case $p = 2$) holds within the class of cyclic p-groups.

THEOREM 9. *Let $G \cong \mathbb{Z}_{p^{e'}} \oplus \mathbb{Z}_{p^e}$ be an Abelian p-group of rank 2, for $1 \leq e' \leq e$.*

(*a*) *If $e' < e$ and $(p^{e'}, p^e) \neq (2, 4)$, then the smallest genus $\mu^+(G) \geq 2$ is given as*

$$\mu^+(G) = \left(\frac{1}{2} \cdot p^e - 1\right) \cdot (p^{e'} - 1),$$

while $\mu^+(\mathbb{Z}_2 \oplus \mathbb{Z}_4) = 3$.

(*b*) *If $e' = e$, that is $G \cong \mathbb{Z}_{p^e}^2$, and $p^e \neq 2, 3$, then $\mu^+(G)$ is given as*

$$\mu^+(G) = \frac{1}{2} \cdot p^e \cdot (p^e - 3) + 1,$$

while $\mu^+(\mathbb{Z}_2^2) = 2$ and $\mu^+(\mathbb{Z}_3^2) = 4$.

Proof. Note that $\mu^+(G) = p^{e'} \cdot \mu_0^+(G) + 1$, and

$$(s_1, \ldots, s_{e'}, s_{e'+1}, \ldots, s_e) = (3, \ldots, 3, 2, \ldots, 2).$$

We have

$$A_0 = \{(2a, \ldots, 2a) \ : \ a \geq 1\},$$

and hence $\gamma(2a, \ldots, 2a) = (a - 1) \cdot p^e$ yields $\min\ \gamma(A_0) = \mu_0 = 0$ and

$$\min\ (\gamma(A_0) \setminus \{0\}) = p^e.$$

Let $1 \leq i \leq e - 1$. Using the notation of Section 5.4, for $1 \leq i \leq e'$, we have

$$\mu_i = -p^e + p^{e-i} \cdot \left(1 + \frac{3}{2} \cdot (p^i - 1)\right) = \frac{1}{2} \cdot p^{e-i} \cdot (p^i - 1);$$

hence, from $i' = 0$ and $\epsilon_i = 1$, we get

$$\min\ \gamma(A_i) = \mu_i + \frac{1}{2} \cdot p^{e-i} \cdot (p^i - 1) = p^{e-i} \cdot (p^i - 1).$$

For $e' < i \leq e - 1$, we have

$$\mu_i = -p^e + p^{e-i} \cdot \left(p^i + \frac{1}{2} \cdot p^{i-e'} \cdot (p^{e'} - 1)\right) = \frac{1}{2} \cdot p^{e-e'} \cdot (p^{e'} - 1);$$

hence, from $i' = e'$ and $i'' = 0$, as well as $\epsilon_i = 2$, we get

$$\min\ \gamma(A_i) = \mu_i + \frac{1}{2} \cdot p^{e-e'} \cdot (p^{e'} + 1) - p^{e-i} = p^{e-i} \cdot (p^i - 1).$$

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Thus, for all $1 \leq i \leq e-1$, we have

$$\min\ \gamma(A_i) = p^e - p^{e-i} \geq p^e - p^{e-1} = \min\ \gamma(A_1).$$

Moreover, for $p=2$ and $e' < i \leq e-1$, we have $\min\ \gamma(A_i') = \min\ \gamma(A_i)$.

Hence, let $i = e$. We have

$$\min\ \gamma(A_e) = \mu_e = -1 + \frac{1}{2} \cdot p^{e-e'} \cdot (p^{e'} - 1),$$

where $\mu_e \leq 0$ if and only if $p^{e-e'} \cdot (p^{e'} - 1) \leq 2$, which holds if and only if $e' = 1$ and $p^e \in \{2, 3, 4\}$. Hence, for $p^e > 4$, or $p^e = 4$ and $e' = e$, we have $\mu_e > 0$.

Assume that $p^e - p^{e-1} < \mu_e = -1 + \frac{1}{2} \cdot (p^e - p^{e-e'})$, then we have $p^e \cdot (1 - \frac{2}{p} + \frac{1}{p^{e'}}) < -2$, implying that $1 - \frac{2}{p} + \frac{1}{p^{e'}} < 0$, or equivalently $\frac{2}{p} - \frac{1}{p^{e'}} > 1$, a contradiction.

Thus, for $1 \leq e' < e$ and $(p^{e'}, p^e) \neq (2, 4)$, we conclude that

$$\mu_0^+(\mathbb{Z}_{p^{e'}} \oplus \mathbb{Z}_{p^e}) = -1 + \frac{1}{2} \cdot p^{e-e'} \cdot (p^{e'} - 1),$$

and for $p^e \geq 4$, we have

$$\mu_0^+(\mathbb{Z}_{p^e}^2) = \frac{1}{2} \cdot (p^e - 3).$$

The cases $(p^{e'}, p^e) = (2, 4), (2, 2)$ are presented in Section 8.1 and Table 3, and the case $(p^{e'}, p^e) = (3, 3)$ is presented in Section 8.2 and Table 4. □

Note that the cases with $e \leq 2$ will reappear in Section 9. Moreover, this improves the general bound given in [**6**, Proposition 3.4]; and for the cases of cyclic deficiency $\delta = 1$, where p is odd, we recover the relevant part of [**9**, Theorem 5.4] and [**9**, Corollary 5.5].

We conclude that an Abelian p-group of rank 2 is uniquely determined by its smallest genus $\mu^+(G) \geq 2$, with the exception of the groups $\{\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z}_4^2\}$; where by (8.1), Table 3, the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ indeed have the same spectrum, which differs from that of $\mathbb{Z}_4^2$. In particular, Talu's conjecture (including the case $p = 2$) holds within the class of Abelian p-groups of rank 2.

8.1. Small 2-groups. We compute the reduced genus spectrum of the non-trivial Abelian 2-groups of order at most 16. The results are collected in Table 3. We consider the cases not covered by Corollary 1 in turn:

(*i*) Let $G \cong \mathbb{Z}_4$; hence, $(e', e) = (0, 2)$, that is $(s_1, s_2) = (2, 2)$. We have

$$A_2' = \{(a_1, a_2, 2a)\ :\ a_1 \geq a_2 \geq 2(a+1),\ a_2 \text{ even}\}$$

and $\gamma(a_1, a_2, 2a) = -4 + a + a_1 + \frac{a_2}{2}$. From $\gamma(a_1, 2, 0) = a_1 - 3$, for $a_1 \geq 2$, we conclude that $\gamma(A_2') = \{-1\} \cup \mathbb{N}_0$, thus we get

$$\mathrm{sp}_0(\mathbb{Z}_4) = \{-1\} \cup \mathbb{N}_0 \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_4) = \mathbb{N}_0;$$

in particular, we recover a special case of [**7**, Corollary 6.3].

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(*ii*) Let $G \cong \mathbb{Z}_8$; hence, $(e', e) = (0, 3)$, that is $(s_1, s_2, s_3) = (2, 2, 2)$. We have

$$A'_3 = \{(a_1, a_2, a_3, 2a) \ : \ a_1 \geq a_2 \geq a_3 \geq 2(a+1), \ a_3 \text{ even}\}$$

and $\gamma(a_1, a_2, a_3, 2a) = -8 + a + 2a_1 + a_2 + \frac{a_2}{2}$. From $\gamma(a_1, 2, 2, 0) = 2a_1 - 5$ for $a_1 \geq 2$, and $\gamma(a_1, 3, 2, 0) = 2a_1 - 4$ for $a_1 \geq 3$, we conclude that $\gamma(A'_3) = \{-1\} \cup \mathbb{N}$. Since, by Theorem 8, we have $\gamma(A_0) = 8\mathbb{N}_0$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_8) = \{-1\} \cup \mathbb{N}_0 \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_8) = \mathbb{N}_0;$$

in particular, we recover a special case of [**7**, Corollary 6.3].

(*iii*) Let $G \cong \mathbb{Z}_{16}$; hence, $(e', e) = (0, 4)$, that is $(s_1, s_2, s_3, s_4) = (2, 2, 2, 2)$. We have

$$A'_4 = \{(a_1, a_2, a_3, a_4, 2a) \ : \ a_1 \geq a_2 \geq a_3 \geq a_4 \geq 2(a+1), \ a_4 \text{ even}\}$$

and $\gamma(a_1, a_2, a_3, a_4, 2a) = -16 + a + 4a_1 + 2a_2 + a_3 + \frac{a_4}{2}$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -15 + 4 \cdot \frac{m+9}{4} + 2 \cdot 2 + 2, & \text{if } m \equiv 3 \pmod 4, \\ -15 + 4 \cdot \frac{m+7}{4} + 2 \cdot 3 + 2, & \text{if } m \equiv 1 \pmod 4, \\ -15 + 4 \cdot \frac{m+6}{4} + 2 \cdot 3 + 3, & \text{if } m \equiv 2 \pmod 4, \\ -15 + 4 \cdot \frac{m+4}{4} + 2 \cdot 4 + 3, & \text{if } m \equiv 0 \pmod 4) \end{cases}$$

shows that any $m \in (\{-1\} \cup \mathbb{N}_0) \setminus \{0, 1, 2, 4, 8\}$ can be written as $m = -16 + 4a_1 + 2a_2 + a_3 + \frac{2}{2}$ for some $a_1 \geq a_2 \geq a_3 \geq 2$, while none of $\{0, 1, 2, 4, 8\}$ is of the form $-16 + a + 4a_1 + 2a_2 + a_3 + \frac{a_4}{2}$ for any $(a_1, a_2, a_3, a_4, 2a) \in A'_4$. Thus, we have

$$\gamma(A'_4) = (\{-1\} \cup \mathbb{N}_0) \setminus \{0, 1, 2, 4, 8\}.$$

Since, by Theorem 8, we have $\gamma(A_0) = 16\mathbb{N}_0$, and $\min\ \gamma(A'_i) = \min\ \gamma(A_i) = 16 - 2^{4-i}$ for $1 \leq i \leq 3$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_{16}) = (\{-1\} \cup \mathbb{N}_0) \setminus \{1, 2, 4\} \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_{16}) = \mathbb{N}_0 \setminus \{2, 3, 5\};$$

in particular, we recover a special case of [**7**, Corollary 6.3].

(*iv*) Let $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$; hence, $(e', e) = (1, 3)$, that is $(s_1, s_2, s_3) = (3, 2, 2)$. We have

$$A'_3 = \{(a_1, a_2, a_3, 2a) \ : \ a_1 \geq \max\{3, a_2\}, \ a_2 \geq a_3 \geq 2(a+1), \ a_3 \text{ even}\}$$

and $\gamma(a_1, a_2, a_3, 2a) = -8 + a + 2a_1 + a_2 + \frac{a_3}{2}$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -7 + 2 \cdot \frac{m+5}{2} + 2, & \text{if } m \text{ odd}, \\ -7 + 2 \cdot \frac{m+4}{2} + 3, & \text{if } m \text{ even} \end{cases}$$

shows that $m = -8 + 2a_1 + a_2 + \frac{2}{2}$ for some $a_1 \geq a_2 \geq 2$ such that $a_1 \geq 3$. Thus, we have $\gamma(A'_3) = \mathbb{N}$. Since, by Theorem 9, we have $\gamma(A_0) = 8\mathbb{N}_0$, and $\min\ \gamma(A_i) = 8 - 2^{3-i}$ for $1 \leq i \leq 2$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_2 \oplus \mathbb{Z}_8) = \mathbb{N}_0 \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_2 \oplus \mathbb{Z}_8) = 1 + 2\mathbb{N}_0.$$

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Table 3. Small 2-groups

G	$\mathrm{sp}_0(G)$	$\mathrm{sp}(G)$
$\mathbb{Z}_2$	$\{-1\} \cup \mathbb{N}_0$	$\mathbb{N}_0$
$\mathbb{Z}_4$	$\{-1\} \cup \mathbb{N}_0$	$\mathbb{N}_0$
$\mathbb{Z}_2^2$	$\{-\frac{1}{2}\} \cup \frac{1}{2}\mathbb{N}_0$	$\mathbb{N}_0$
$\mathbb{Z}_8$	$\{-1\} \cup \mathbb{N}_0$	$\mathbb{N}_0$
$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{N}_0$	$1 + 2\mathbb{N}_0$
$\mathbb{Z}_2^3$	$\frac{1}{2}\mathbb{N}_0$	$1 + 2\mathbb{N}_0$
$\mathbb{Z}_{16}$	$(\{-1\} \cup \mathbb{N}_0) \setminus \{1, 2, 4\}$	$\mathbb{N}_0 \setminus \{2, 3, 5\}$
$\mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\mathbb{N}_0$	$1 + 2\mathbb{N}_0$
$\mathbb{Z}_4^2$	$(\frac{1}{2}\mathbb{N}_0) \setminus \{1\}$	$(1 + 2\mathbb{N}_0) \setminus \{5\}$
$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$\mathbb{N}$	$5 + 4\mathbb{N}_0$
$\mathbb{Z}_2^4$	$\frac{1}{2}\mathbb{N}$	$5 + 4\mathbb{N}_0$

(v) Let $G \cong \mathbb{Z}_4^2$; hence, $(e', e) = (2, 2)$, that is $(s_1, s_2) = (3, 3)$. We have

$$A_2 = \{(a_1, a_2, 2a) \ : \ a_1 \geq a_2 \geq \max\{3, 2(a+1)\}\}$$

and $\gamma(a_1, a_2, 2a) = \frac{1}{2} \cdot (-8 + 2a + 2a_1 + a_2)$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -8 + 2 \cdot \frac{m+5}{2} + 3, & \text{if } m \text{ odd}, \\ -8 + 2 \cdot \frac{m+4}{2} + 4, & \text{if } m \text{ even} \end{cases}$$

shows that any $m \in \mathbb{N} \setminus \{2\}$ can be written as $m = -8 + 2a_1 + a_2$ for some $a_1 \geq a_2 \geq 3$, while 2 is not of the form $-8 + 2a + 2a_1 + a_2$ for any $(a_1, a_2, 2a) \in A_2$. Thus, we have

$$\gamma(A_2) = \left(\frac{1}{2}\mathbb{N}\right) \setminus \{1\}.$$

Since, by Theorem 9, we have $\gamma(A_0) = 4\mathbb{N}_0$ and $\min\ \gamma(A_1) = 2$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_4^2) = \left(\frac{1}{2}\mathbb{N}_0\right) \setminus \{1\} \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_4^2) = (1 + 2\mathbb{N}_0) \setminus \{5\}.$$

8.2. Small 3-groups. We compute the reduced genus spectrum of the non-trivial Abelian 3-groups of order at most 27. The results are presented in Table 4. We consider the cases not covered by Corollary 1 in turn:

(i) Let $G \cong \mathbb{Z}_9$; hence, $(e', e) = (0, 2)$, that is $(s_1, s_2) = (2, 2)$. We have

$$A_2 = \{(a_1, a_2, 2a) \ : \ a_1 \geq a_2 \geq 2(a+1)\}$$

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and $\gamma(a_1, a_2, 2a) = -9 + a + 3a_1 + a_2$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -9 + 3 \cdot \frac{m+7}{3} + 2, & \text{if } m \equiv 2 \pmod 3, \\ -9 + 3 \cdot \frac{m+6}{3} + 3, & \text{if } m \equiv 0 \pmod 3, \\ -9 + 3 \cdot \frac{m+5}{3} + 4, & \text{if } m \equiv 1 \pmod 3 \end{cases}$$

shows that any $m \in (\{-1\} \cup \mathbb{N}_0) \setminus \{0, 1, 4\}$ can be written as $m = -9 + 3a_1 + a_2$ for some $a_1 \geq a_2 \geq 2$, while none of $\{0, 1, 4\}$ is of the form $-9 + a + 3a_1 + a_2$ for any $(a_1, a_2, 2a) \in A_2$. Thus, we have

$$\gamma(A_2) = (\{-1\} \cup \mathbb{N}_0) \setminus \{0, 1, 4\}.$$

Since, by Theorem 8, we have $\gamma(A_0) = 9\mathbb{N}_0$ and $\min\ \gamma(A_1) = 6$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_9) = (\{-1\} \cup \mathbb{N}_0) \setminus \{1, 4\} \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_9) = \mathbb{N}_0 \setminus \{2, 5\};$$

in particular, we recover a special case of [**7**, Corollary 5.3].

(*ii*) Let $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_9$; hence, $(e', e) = (1, 2)$, that is $(s_1, s_2) = (3, 2)$. We have

$$A_2 = \{(a_1, a_2, 2a) \ : \ a_1 \geq \max\{3, a_2\},\ a_2 \geq 2(a + 1)\}$$

and $\gamma(a_1, a_2, 2a) = -9 + a + 3a_1 + a_2$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -9 + 3 \cdot \frac{m+7}{3} + 2, & \text{if } m \equiv 2 \pmod 3, \\ -9 + 3 \cdot \frac{m+6}{3} + 3, & \text{if } m \equiv 0 \pmod 3, \\ -9 + 3 \cdot \frac{m+5}{3} + 4, & \text{if } m \equiv 1 \pmod 3 \end{cases}$$

shows that any $m \in \mathbb{N}_0 \setminus \{0, 1, 4\}$ can be written as $m = -9 + 3a_1 + a_2$ for some $a_1 \geq \max\{3, a_2\}$ and $a_2 \geq 2$, while none of $\{0, 1, 4\}$ is of the form $-9 + a + 3a_1 + a_2$ for any $(a_1, a_2, 2a) \in A_2$. Thus, we have

$$\gamma(A_2) = \mathbb{N}_0 \setminus \{0, 1, 4\}.$$

Since, by Theorem 9, we have $\gamma(A_0) = 9\mathbb{N}_0$ and $\min\ \gamma(A_1) = 6$, we conclude that

$$\mathrm{sp}_0(\mathbb{Z}_3 \oplus \mathbb{Z}_9) = \mathbb{N}_0 \setminus \{1, 4\} \quad \text{and} \quad \mathrm{sp}(\mathbb{Z}_3 \oplus \mathbb{Z}_9) = (1 + 3\mathbb{N}_0) \setminus \{4, 13\},$$

thus recovering [**9**, Corollary 5.5].

(*iii*) Let $G \cong \mathbb{Z}_{27}$; hence, $(e', e) = (0, 3)$, that is $(s_1, s_2, s_3) = (2, 2, 2)$. We have

$$A_3 = \{(a_1, a_2, a_3, 2a) \ : \ a_1 \geq a_2 \geq a_3 \geq 2(a + 1)\}$$

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and $\gamma(a_1, a_2, a_3, 2a) = -27 + a + 9a_1 + 3a_2 + a_3$. Writing $m \in \mathbb{Z}$ as

$$m = \begin{cases} -27 + 9 \cdot \frac{m+19}{9} + 3 \cdot 2 + 2, & \text{if } m \equiv 8 \pmod 9, \\ -27 + 9 \cdot \frac{m+16}{9} + 3 \cdot 3 + 2, & \text{if } m \equiv 2 \pmod 9, \\ -27 + 9 \cdot \frac{m+15}{9} + 3 \cdot 3 + 3, & \text{if } m \equiv 3 \pmod 9, \\ -27 + 9 \cdot \frac{m+13}{9} + 3 \cdot 4 + 2, & \text{if } m \equiv 5 \pmod 9, \\ -27 + 9 \cdot \frac{m+12}{9} + 3 \cdot 4 + 3, & \text{if } m \equiv 6 \pmod 9, \\ -27 + 9 \cdot \frac{m+11}{9} + 3 \cdot 4 + 4, & \text{if } m \equiv 7 \pmod 9, \\ -27 + 9 \cdot \frac{m+9}{9} + 3 \cdot 5 + 3, & \text{if } m \equiv 0 \pmod 9, \\ -27 + 9 \cdot \frac{m+8}{9} + 3 \cdot 5 + 4, & \text{if } m \equiv 1 \pmod 9, \\ -27 + 9 \cdot \frac{m+5}{9} + 3 \cdot 6 + 4, & \text{if } m \equiv 4 \pmod 9) \end{cases}$$

shows that any $m \in (\{-1\} \cup \mathbb{N}_0) \setminus \mathcal{S}'$, where

$$\mathcal{S}' := \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 18, 19, 22, 27, 28, 31, 40\}$$

can be written as $m = -27 + 9a_1 + 3a_2 + a_3$ for some $a_1 \geq a_2 \geq a_3 \geq 2$. Hence, it remains to check which of the elements of the finite set $\mathcal{S}'$ are contained in $\bigcup_{i=0}^{3} \gamma(A_i)$, where A_3 and $\gamma(a_1, a_2, a_3, 2a)$ are as given above, and

$$\begin{aligned} A_2 &= \{(a_1, a_2, 2a, 2a) \ : \ a_1 \geq a_2 \geq 2(a+1) \geq 4\}, \\ A_1 &= \{(a_1, 2a, 2a, 2a) \ : \ a_1 \geq 2(a+1) \geq 4\}, \\ A_0 &= \{(2a, 2a, 2a, 2a) \ : \ a \geq 1\}, \end{aligned}$$

and

$$\begin{aligned} \gamma(a_1, a_2, 2a, 2a) &= -27 + 3a + 9a_1 + 3a_2, \\ \gamma(a_1, 2a, 2a, 2a) &= -27 + 9a + 9a_1, \\ \gamma(2a, 2a, 2a, 2a) &= -27 + 27a. \end{aligned}$$

Since all integers a_1, a_2, a_3, a occurring are non-negative and bounded above by $\lfloor \frac{40+27}{9} \rfloor = 7$, this amounts to a finite number of checks, which are straightforwardly done using GAP. It turns out that $\mathcal{S}' \cap \gamma(A_3) = \emptyset = \mathcal{S}' \cap \gamma(A_2)$, while $\mathcal{S}' \cap \gamma(A_1) = \{18, 27\}$ and $\mathcal{S}' \cap \gamma(A_0) = \{0, 27\}$. Thus, we conclude that $\mathrm{sp}_0(\mathbb{Z}_{27}) = (\{-1\} \cup \mathbb{N}_0) \setminus \mathcal{S}$ and $\mathrm{sp}(\mathbb{Z}_{27}) = \{g_0 + 1 \in \mathbb{N}; g_0 \in \mathrm{sp}_0(\mathbb{Z}_{27})\}$, where

$$\mathcal{S} := \mathcal{S}' \setminus \{0, 18, 27\} = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 19, 22, 28, 31, 40\};$$

in particular, we recover a special case of [**7**, Corollary 5.3].

9. Examples: Small exponents. In this section, we consider Abelian p-groups of exponent at most p^2. In particular, we ask ourselves whether their reduced minimum genus has a 'generic' description in terms of the defining invariants of the group in question.

Table 4. Small 3-groups

G	$\mathrm{sp}_0(G)$	$\mathrm{sp}(G)$
$\mathbb{Z}_3$	$\{-1\} \cup \mathbb{N}_0$	$\mathbb{N}_0$
$\mathbb{Z}_9$	$(\{-1\} \cup \mathbb{N}_0) \setminus \{1, 4\}$	$\mathbb{N}_0 \setminus \{2, 5\}$
$\mathbb{Z}_3^2$	$\mathbb{N}_0$	$1 + 3\mathbb{N}_0$
$\mathbb{Z}_{27}$	$(\{-1\} \cup \mathbb{N}_0) \setminus \mathcal{S}$	$\mathbb{N}_0 \setminus \{m + 1 \ : \ m \in \mathcal{S}\}$
$\mathbb{Z}_3 \oplus \mathbb{Z}_9$	$\mathbb{N}_0 \setminus \{1, 4\}$	$(1 + 3\mathbb{N}_0) \setminus \{4, 13\}$
$\mathbb{Z}_3^3$	$\mathbb{N}$	$10 + 9\mathbb{N}_0$

$\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 19, 22, 28, 31, 40\}$

THEOREM 10. *Let $G \cong \mathbb{Z}_p^r$ be a non-trivial elementary Abelian p-group, and let $s := r + 1$. Then, the reduced minimum genus $\mu_0(G)$ is given as*

$$\mu_0(G) = \begin{cases} \frac{p}{2} \cdot (s-3), & \text{if } s \text{ odd and } s \leq p, \\ \frac{p}{2}(s-2) - \frac{s}{2}, & \text{if } s \text{ even or } s \geq p. \end{cases}$$

Proof. We have $e = 1$ and $\mathcal{I}(G) = \{0, 1\}$, where Theorem 5 says that $0 \in \mathcal{I}(G)$ can be ignored whenever $s \geq 2$ is even. Still, by Section 5.4, we have

$$\min \gamma(A_0) = \mu_0 = \begin{cases} \frac{ps}{2} - p, & \text{if } s \text{ even}, \\ \frac{ps}{2} - \frac{3p}{2}, & \text{if } s \text{ odd}, \end{cases}$$

and

$$\min \gamma(A_1) = \mu_1 = \frac{ps}{2} - \frac{s}{2} - p.$$

Thus, we have $\mu_0 < \mu_1$ if and only if s is odd and $s < p$, with equality if and only if $s = p$ is odd. □

In particular, for p odd, we thus recover, and at the same time correct [**9**, Section 7, Remark], where $\mu_0(G)$ is erroneously stated for $s < p$.

We call the cases where s is odd such that $s < p$ the 'exceptional' ones, and the remaining the 'generic' ones; then, there are only finitely many 'exceptional' cases, which do not occur at all for $p = 2$. In particular, as part of the 'generic' region we have $\mu_0(G) = \mu_1$ for $s \geq \max\{p - 1, 2\}$, in accordance with Main Theorem 1.

PROPOSITION 7. *We keep the notation of Theorem 10. Then, G is uniquely determined by its minimum genus $\mu(G)$, with the exception of the groups $\{\mathbb{Z}_2, \mathbb{Z}_2^2\}$.*

Proof. We distinguish the cases p odd and $p = 2$:

(i) For p odd, viewing μ_0 and μ_1 as linear functions in s, with positive slope $\frac{p}{2}$ and $\frac{p-1}{2}$, respectively, and since $\mu_0(s+1) - \mu_1(s) = \frac{s}{2} > 0$, for $2 \leq s < p$ even, we conclude that $\mu_0(G)$ is strictly increasing with s, and thus $\mu(G) = 1 + p^{s-2} \cdot \mu_0(G)$ is as well. A few values are given in the first part of Table 5, where the 'exceptional' cases are given in bold face.

Table 5. Elementary Abelian p-groups

s	2	**3**	4	**5**	...	$p-3$	$\mathbf{p-2}$	$p-1$	p	$p+1$
$\mu_0(G)$	-1	**0**	$p-2$	**p**	...	$\frac{p(p-6)+3}{2}$	$\mathbf{\frac{p(p-5)}{2}}$	$\frac{p(p-4)+1}{2}$	$\frac{p(p-3)}{2}$	$\frac{p(p-2)-1}{2}$

s	2	3	4	5	6	7	8
$\mu_0(G)$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\mu(G)$	0	0	1	5	17	49	129

(ii) For $p=2$, we have $\mu_0(G)=\mu_1=\frac{s}{2}-2$ for all $s \geq 2$; thus, $\mu_0(G)$ is strictly increasing with s, and hence $\mu(G)=1+2^{s-2}\cdot\mu_0(G)$ is as well for $s \geq 3$. A few values are given in the second part of Table 5. □

By Section 8.1 and Table 3, the exceptions mentioned indeed have the same spectrum. In particular, Talu's conjecture (including the case $p=2$) holds within the class of elementary Abelian p-groups.

THEOREM 11. *Let $G \cong \mathbb{Z}_p^{r_1} \oplus \mathbb{Z}_{p^2}^{r_2}$, where $r_1 \geq 0$ and $r_2 \geq 1$, be an Abelian p-group of exponent p^2, and let $s := r_1 + r_2 + 1$ and $t := r_2 + 1$. Then, the reduced minimum genus $\mu_0(G)$ is given as*

$$\mu_0(G) = \begin{cases} \frac{p^2}{2}\cdot(s-3), & \text{if } \big(s \text{ odd and } p(s-t)+t \leq p^2\big), \\ \frac{p^2}{2}\cdot(s-2)-\frac{p}{2}\cdot(s-t+1), & \text{if } \big(s \text{ even or } s-t \geq p-1\big) \\ & \text{and } \big(t \text{ odd and } t \leq p\big), \\ \frac{p^2}{2}\cdot(s-2)-\frac{p}{2}\cdot(s-t)-\frac{t}{2}, & \text{if } \big(s \text{ even or } p(s-t)+t \geq p^2\big) \\ & \text{and } \big(t \text{ even or } t=s \text{ or } t \geq p\big). \end{cases}$$

Proof. We have $e=2$ and $\{0,2\} \subseteq \mathcal{I}(G) \subseteq \{0,1,2\}$. Moreover, we have $s=r_1+r_2+1=s_1$ and $t=r_2+1=s_2$; hence, $1 \in \mathcal{I}(G)$ if and only if $s-t \geq 2$, or $s-t=1$ and t is odd. Additionally, Theorem 5 says that $0 \in \mathcal{I}(G)$ can be ignored whenever s is even.

Still, in order to obtain a complete overview, by Section 5.4, we explicitly have

$$\min \gamma(A_0) = \mu_0 = \begin{cases} \frac{p^2 s}{2} - p^2, & \text{if } s \text{ even}, \\ \frac{p^2 s}{2} - \frac{3p^2}{2}, & \text{if } s \text{ odd}, \end{cases}$$

and

$$\min \gamma(A_1) = \begin{cases} \frac{p^2 s}{2} - \frac{p(s-t)}{2} - p^2, & \text{if } t \text{ even},\ s-t \geq 2, \\ \frac{p^2 s}{2} - \frac{p(s-t)}{2} - \frac{p}{2} - p^2, & \text{if } t \text{ odd},\ s-t \geq 2, \\ \frac{p^2 s}{2} - p - \frac{p^2}{2}, & \text{if } t \text{ even},\ s-t = 1, \\ \frac{p^2 s}{2} - p - p^2, & \text{if } t \text{ odd},\ s-t = 1, \\ \frac{p^2 s}{2} - p, & \text{if } t \text{ even},\ s = t, \\ \frac{p^2 s}{2} - p - \frac{p^2}{2}, & \text{if } t \text{ odd},\ s = t, \end{cases}$$

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and

$$\min \gamma(A_2) = \mu_2 = \frac{p^2 s}{2} - \frac{p(s-t)}{2} - \frac{t}{2} - p^2.$$

Now, we compare the various minima:

0 vs. 2: We have $\mu_0 < \mu_2$ if and only if s is odd and $p(s-t)+t < p^2$, with equality if and only if s is odd and $p(s-t)+t = p^2$.

1 vs. 2: We have $(\min \gamma(A_1)) < \mu_2$ if and only if t is odd and $t < \min\{p, s\}$, with equality if and only if $t = p$ is odd and $t < s$.

0 vs. 1: We have $\mu_0 < (\min \gamma(A_1))$ if and only if we are in one of the following cases:

$$\begin{cases} s = t \text{ even}, \\ s \text{ odd}, \ t \text{ even}, \ s-t < p, \\ s \text{ odd}, \ t \text{ odd}, \ s-t < p-1, \end{cases}$$

with equality if and only if s is odd, and $s-t = p$ odd or $s-t = p-1$ even.

In particular, we have equality $\mu_0 = (\min \gamma(A_1)) = \mu_2$ throughout if and only if $t = p$ odd and $s = 2p-1$.

Thus, there are three cases, in which $\mu_0(G)$ coincides with either of μ_0, μ_1 and μ_2, where the intersection of these cases is described by equating the associated μ_i:

(i) Let s be odd such that $p(s-t)+t \leq p^2$, in particular, implying $s-t < p$. Then, we have $\mu_0 \leq \mu_2$ and $\mu_0 \leq (\min \gamma(A_1))$, hence we get $\mu_0(G) = \mu_0$.

(ii) Let t be odd such that $t \leq p$, and let s be even or $s-t \geq p-1$. Then, we have $(\min \gamma(A_1)) \leq \mu_2$ and $(\min \gamma(A_1)) \leq \mu_0$, hence we get

$$\mu_0(G) = \mu_1 = \frac{p^2}{2} \cdot (s-2) - \frac{p}{2} \cdot (s-t+1).$$

(iii) Let s be even or $p(s-t)+t \geq p^2$, and let t be even or $t = s$ or $t \geq p$. Then, we have $\mu_2 \leq \mu_0$ and $\mu_2 \leq (\min \gamma(A_1))$, hence we get $\mu_0(G) = \mu_2$. □

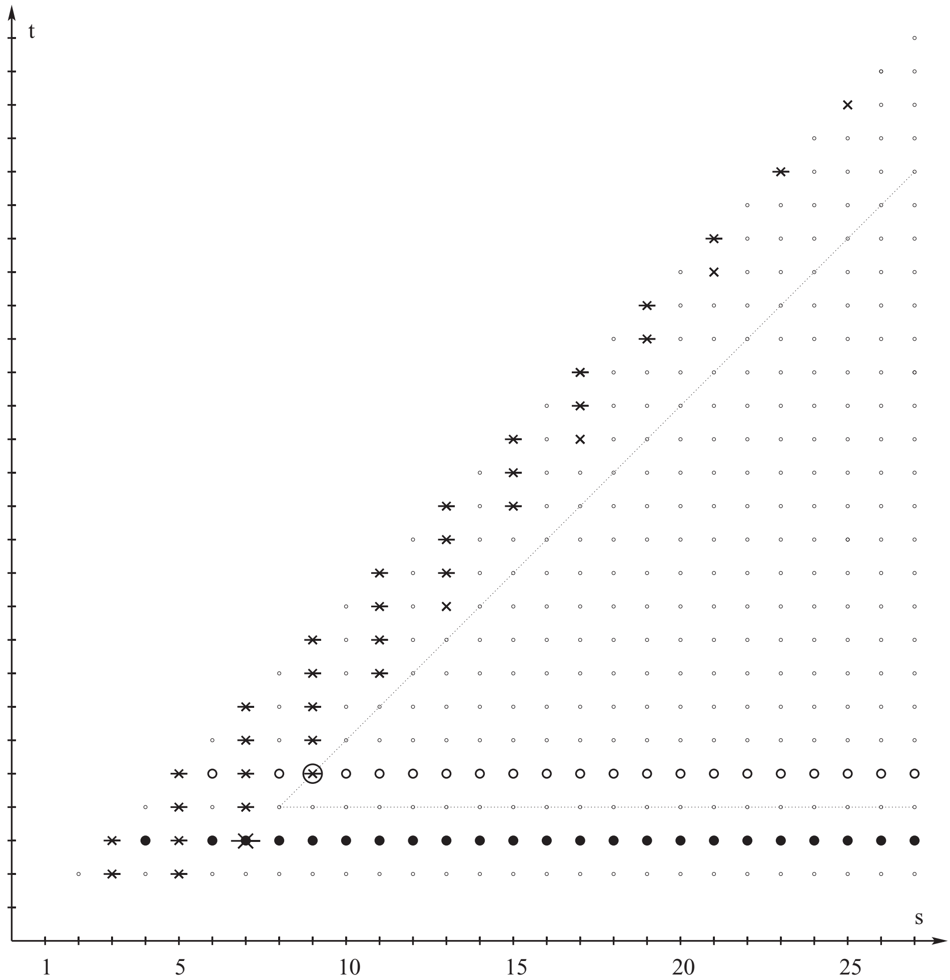
Then, case (i) consists of finitely many pairs (s, t), while in case (ii) s is unbounded, but t is still bounded. Hence, we again call these the 'exceptional' cases, as opposed to the 'generic' case (iii), where both s and t are unbounded. In particular, as part of the 'generic' region, we have $\mu_0(G) = \mu_2$ for $t \geq \max\{p-1, 2\}$ and $s-t \geq p-1$, in accordance with Main Theorem 1.

In particular, for $p = 2$, case (i) consists of the pairs $(s, t) = (3, 3)$ and $(s, t) = (3, 2)$, that is $G \cong \mathbb{Z}_4^2$ and $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$, respectively, case (ii) does not occur at all, and all pairs except $(s, t) = (3, 3)$ belong to case (iii).

To further illustrate the idea of distinguishing between 'generic' and 'exceptional' cases, the various cases for $p = 5$ and $2 \leq t \leq s \leq 27$ are presented in Table 6: The Cases (i), (ii) and (iii) are depicted by '$*$', '$\bullet$' and '$\cdot$', respectively, the intersections 'i)∩iii)', 'ii)∩iii)' and 'i)∩ii)' are indicated by '$\times$', '$\circ$' and '$\circledast$', respectively, and 'i)∩ii)∩iii)' is denoted by '$\otimes$'. In particular, 'i)∩ii)' consists of $(s, t) \in \{(7, 3), (9, 5)\}$, and 'i)∩ii)∩iii)' consists of $(s, t) = (9, 5)$.

The closed interior of the cone emanating from $(s, t) = (8, 4)$ indicates the realm of applicability of Main Theorem 1; actually, this turns out to be the largest cone being contained in the 'generic' region, saying that in a certain sense this result is

Table 6. 'Generic' and 'exceptional' cases for $p = 5$



best possible, at least for the cases considered here. Moreover, within this cone, the 'generic' case (iii) refers to the case $j = 2$ in the notation of Main Theorem 1, while the 'exceptional' intersection 'ii)∩iii)' refers to $j \leq 1$, that is the pairs $(s, 5)$ such that $s \geq 9$, and finally the intersection 'i)∩ii)∩iii)' refers to $j = 0$, that is $(s, t) = (9, 5)$.

PROPOSITION 8. *We keep the notation of Theorem 11. Then, G is uniquely determined by its Kulkarni invariant $N = N(G)$ and its minimum genus $\mu(G)$, with the exception of the groups $\{\mathbb{Z}_4^2, \mathbb{Z}_2 \oplus \mathbb{Z}_4\}$.*

Proof. We distinguish the cases p odd and $p = 2$:

(i) Let first p be odd. The cyclic deficiency $\delta = \delta(G)$ and the reduced minimum genus $\mu_0(G)$ of G are known from $\delta = \log_p(N)$ and $\mu_0(G) = \frac{\mu(G)-1}{p^\delta}$. We have $\delta = r_1 + 2r_2 - 2 = s + t - 4$; thus, we may view μ_0 in case of Theorem 11(i), μ_1 in case of Theorem 11(ii), and μ_2 in case of Theorem 11(iii) as linear functions in

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s, depending on parameter δ:

$$\begin{aligned}
\mu_0 &= \frac{p^2}{2}\cdot s - \frac{3p^2}{2},\\
\mu_1 &= \left(\frac{p^2}{2} - p\right)\cdot s + \frac{p(\delta+3)}{2} - p^2,\\
\mu_2 &= \frac{(p-1)^2}{2}\cdot s + \frac{(p-1)(\delta+4)}{2} - p^2.
\end{aligned}$$

As these functions have positive slopes, they are strictly increasing; hence, we look for coincidences across the cases (Theorem 11(i), (ii), (iii)):
(i) vs. (ii): Let, first, $\mu_1(s,t) = \mu_0(\tilde{s},\tilde{t})$, where (s,t) belongs to case Theorem 11(ii), and $(\tilde{s},\tilde{t})$ belongs to case Theorem 11(i). Then, we conclude that $\tilde{s} = s - \frac{s-t+1}{p} + 1$; hence, we have $s - t = kp - 1$ for some $k \geq 1$. From this, we get $s = \frac{1}{2}\cdot(\delta + 3 + kp)$ and $t = \frac{1}{2}\cdot(\delta + 5 - kp)$, implying $\tilde{s} = s - k + 1 = \frac{1}{2}\cdot(\delta + 5 - 2k + kp)$ and $\tilde{t} = \delta + 4 - \tilde{s} = \frac{1}{2}\cdot(\delta + 3 + 2k - kp)$. Thus, we get $\tilde{s} - \tilde{t} = 1 + k(p-2)$. Hence, $\tilde{s} - \tilde{t} \leq p - 1$ yields $k = 1$, and thus $\tilde{s} = s$ and $\tilde{t} = t$. Note that in this case both s and t are odd such that $t \leq p$ and $s - t = p - 1$, indeed yielding $\mu_1(s,t) = \mu_0(s,t)$.

(i) vs. (iii): Let, next, $\mu_2(s,t) = \mu_0(\tilde{s},\tilde{t})$, where (s,t) belongs to case Theorem 11(iii), and $(\tilde{s},\tilde{t})$ belongs to case Theorem 11(i). Then, we conclude that $\tilde{s} = s + \frac{(p-1)t-ps}{p^2} + 1$; hence, we have $t = kp$ for some $k \geq 1$. Thus, we infer that p divides $k(p-1) - s$; hence, we get $s = k(p-1) + lp$ for some $l \geq 1$. This yields $\tilde{s} = (k+l)(p-1) + 1$ and $\tilde{t} = s - \tilde{s} + t = kp + l - 1$. Hence, we have $p(\tilde{s} - \tilde{t}) + \tilde{t} = l(p-1)^2 + 2p - 1 \leq p^2$, implying $l = 1$; thus, $\tilde{s} = s = (k+1)p - k$ and hence $\tilde{t} = t$. Note that in this case s is odd, where $s - t = p - k$ and $t = kp \geq p$, hence $p(s-t) + t = p^2$, indeed yielding $\mu_2(s,t) = \mu_0(s,t)$.
(ii) vs. (iii): Let, finally, $\mu_2(s,t) = \mu_1(\tilde{s},\tilde{t})$, where (s,t) belongs to case Theorem 11(iii), and $(\tilde{s},\tilde{t})$ belongs to case Theorem 11(ii). Then, we conclude that $(p-1)\tilde{s} + \tilde{t} - 1 = (p-1)s + \frac{p-1}{p}\cdot t$; hence, we have $t = kp$ for some $k \geq 1$, and thus $\tilde{t} - 1 = (p-1)(s + k - \tilde{s}) \geq p - 1$. This yields $\tilde{s} = s + k - 1$ and $\tilde{t} = p$. Hence, we get $s + kp = s + t = \delta + 4 = \tilde{s} + \tilde{t} = s + k - 1 + p$, implying $(k-1)p = k - 1$; thus, $k = 1$, and hence $\tilde{s} = s$ and $\tilde{t} = t$. Note that in this case $t = p$ is odd, and s is even or $s \geq 2p - 1$, in particular, yielding $\mu_2(s,t) = \mu_1(s,t)$.

(ii) Let now $p = 2$. We first consider case Theorem 11(iii), where, using $s + t = \delta - 4$ again, we have

$$\mu_2 = s + \frac{t}{2} - 4 = \frac{s}{2} + \frac{\delta}{2} - 2.$$

We distinguish the cases $t = 2$ and $t > 2$: If $t = 2$, then we have $\log_2(N) = \delta = s - 2 = \mu_2 + 1$; thus,

$$\mu(G) = \mu_2 \cdot 2^\delta + 1 = (\log_2(N) - 1)\cdot N + 1,$$

while if $t > 2$, then we have $\log_2(N) = \delta - 1$; thus,

$$\mu(G) = \mu_2 \cdot 2^\delta + 1 = (\log_2(N) + s - 3)\cdot N + 1.$$

Hence, we are able to decide in which of these cases we are, and to determine δ and subsequently s, in the former case from N and in the latter case from N and $\mu(G)$.

Finally, we consider the pair $(3, 3)$, that is $G \cong \mathbb{Z}_4^2$, which is the only pair not belonging to case Theorem 11(iii), but just to case Theorem 11(i): We have $\mu_0(\mathbb{Z}_4^2) = \mu_0(3, 3) = 0$; hence, its minimum genus equals $\mu(\mathbb{Z}_4^2) = 1$. For pairs (s, t) belonging to case Theorem 11(iii), the statement $\mu(G) = 1$ translates into $\mu_2(s, t) = 0$, that is $s + \frac{t}{2} = 4$, being equivalent to $(s, t) = (3, 2)$, that is $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$; note that $(3, 2)$ is the other pair belonging to case Theorem 11(i). Moreover, for $G \cong \mathbb{Z}_4^2$, we have $\log_2(N) = \delta - 1 = 1$, and for $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$, we also have $\log_2(N) = \delta = 1$. Thus, $\{\mathbb{Z}_4^2, \mathbb{Z}_2 \oplus \mathbb{Z}_4\}$ are the only groups under consideration, which cannot be distinguished by N and $\mu(G)$. $\square$

By Section 8.1 and Table 3, the exceptions mentioned can be distinguished by their spectrum. In particular, Talu's conjecture (including the case $p = 2$) holds within the class of Abelian p-groups of exponent p^2; thus, for p odd, we recover [**13**, Theorem 3.8].

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