Nagoya Math. J., **231** (2018), 23–54
DOI 10.1017/nmj.2017.10

LYUBEZNIK NUMBERS OF LOCAL RINGS AND LINEAR STRANDS OF GRADED IDEALS

JOSEP ÀLVAREZ MONTANER AND KOHJI YANAGAWA

Abstract. In this work, we introduce a new set of invariants associated to the linear strands of a minimal free resolution of a $\mathbb{Z}$-graded ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$. We also prove that these invariants satisfy some properties analogous to those of Lyubeznik numbers of local rings. In particular, they satisfy a consecutiveness property that we prove first for the Lyubeznik table. For the case of squarefree monomial ideals, we get more insight into the relation between Lyubeznik numbers and the linear strands of their associated Alexander dual ideals. Finally, we prove that Lyubeznik numbers of Stanley–Reisner rings are not only an algebraic invariant but also a topological invariant, meaning that they depend on the homeomorphic class of the geometric realization of the associated simplicial complex and the characteristic of the base field.

§1. Introduction

Let A be a Noetherian local ring that admits a surjection from an n-dimensional regular local ring $(R, \mathfrak{m})$ containing its residue field $\Bbbk$, and let $I \subseteq R$ be the kernel of the surjection. In [13], Lyubeznik introduced a new set of invariants $\lambda_{p,i}(A)$ as the pth Bass number of the local cohomology module $H_I^{n-i}(R)$. That is,

$$\lambda_{p,i}(A) := \mu^p(\mathfrak{m}, H_I^{n-i}(R)) = \dim_{\Bbbk} \operatorname{Ext}_R^p(\Bbbk, H_I^{n-i}(R)),$$

and they depend only on A, i and p, but not on the choice of R or the surjection $R \longrightarrow A$. In the seminal works of Huneke and Sharp [10] and Lyubeznik [13], it is proven that these Bass numbers are all finite. Denoting $d = \dim A$, Lyubeznik numbers satisfy the following properties[1].

(i) $\lambda_{p,i}(A) \neq 0$ implies $0 \leqslant p \leqslant i \leqslant d$.
(ii) $\lambda_{d,d}(A) \neq 0$.

Received December 5, 2014. Revised March 30, 2016. Accepted June 16, 2016.
2010 Mathematics subject classification. 13D45, 13D02, 13F55.
The first author was partially supported by Generalitat de Catalunya 2014SGR-634 project and Spanish Ministerio de Economía y Competitividad MTM2015-69135-P. The second author was partially supported by JSPS KAKENHI 25400057

[1] Property (iii) was shown to us by García-López (see [2] for details).

(iii) *Euler characteristic:*

$$\sum_{0\leqslant p,i\leqslant d}(-1)^{p-i}\lambda_{p,i}(A)=1.$$

Therefore, we can collect them in the so-called *Lyubeznik table*:

$$\Lambda(A)=\begin{pmatrix}\lambda_{0,0} & \cdots & \lambda_{0,d}\\ & \ddots & \vdots\\ & & \lambda_{d,d}\end{pmatrix},$$

and we say that the Lyubeznik table is *trivial* if $\lambda_{d,d}=1$ and the rest of these invariants vanish.

Despite their algebraic nature, Lyubeznik numbers also provide some geometrical and topological information, as was already pointed out in [13]. For instance, in the case of isolated singularities, Lyubeznik numbers can be described in terms of certain singular cohomology groups in characteristic zero (see [6]) or étale cohomology groups in positive characteristic (see [4, 5]). The highest Lyubeznik number $\lambda_{d,d}(A)$ can be described using the so-called Hochster and Huneke graph, as has been proved in [15, 31]. However, very little is known about the possible configurations of Lyubeznik tables except for low-dimension cases [12, 24] or the just mentioned case of isolated singularities.

In Section 2, we give some new constraints to the possible configurations of Lyubeznik tables. Namely, the main result, Theorem 2.1, establishes some consecutiveness of the nonvanishing superdiagonals of the Lyubeznik tables using spectral sequence arguments.

In Section 3, we introduce a new set of invariants associated to the linear strands of a minimal free resolution of a $\mathbb{Z}$-graded ideal $I\subseteq R=\Bbbk[x_1,\ldots,x_n]$. It turns out that these new invariants satisfy some analogous properties to those of Lyubeznik numbers, including the aforementioned consecutiveness property. Moreover, we provide a Thom–Sebastiani type formula for these invariants, which is a refinement of the formula for Betti numbers given by Jacques and Katzman in [11]. This section should be of independent interest, and we hope it can be further developed in future work.

In the rest of the paper, we treat the case where I is a monomial ideal in a polynomial ring $R=\Bbbk[x_1,\ldots,x_n]$, and $\mathfrak{m}=(x_1,\ldots,x_n)$ is the graded maximal ideal. Bass numbers are invariant with respect to completion, so we consider $\lambda_{p,i}(R/I) = \lambda_{p,i}(\widehat{R}/I\widehat{R})$, where $\widehat{R} = \Bbbk[\![x_1,\dots,x_n]\!]$. In this sense, our study on the Lyubeznik tables of monomial ideals is a (very) special case of that for local rings. However, advanced techniques in combinatorial commutative algebra are very effective in this setting, and we can go much further than the general case, so we hope that monomial ideals are good test cases for the study of Lyubeznik tables.

Since local cohomology modules satisfy $H^i_I(R) \cong H^i_{\sqrt{I}}(R)$, we often assume that a monomial ideal I is *squarefree*, that is, $I = \sqrt{I}$. In this case, I coincides with the *Stanley–Reisner ideal* I_Δ of a simplicial complex $\Delta \subseteq 2^{\{1,\dots,n\}}$. More precisely,

$$I = I_\Delta := \left(\prod_{i \in F} x_i \mid F \subseteq \{1,\dots,n\}, F \notin \Delta \right).$$

The Stanley–Reisner ring R/I_Δ is one of the most fundamental tools in combinatorial commutative algebra, and it is known that R/I_Δ reflects topological properties of the geometric realization $|\Delta|$ of Δ in several ways.

In Section 4, we get a deeper insight to the relation, given by the first author and Vahidi in [1], between Lyubeznik numbers of monomial ideals and the linear strands of their associated Alexander dual ideals. In particular, we give a different approach to the fact proved in [2] that if R/I_Δ is sequentially Cohen–Macaulay, then its Lyubeznik table is trivial. We also provide a Thom–Sebastiani type formula for Lyubeznik numbers.

One of the main results of this paper is left for Section 5. Namely, Theorem 5.3 states that Lyubeznik numbers of Stanley–Reisner rings are not only algebraic invariants but also topological invariants, meaning that the Lyubeznik numbers of R/I_Δ depend on the homeomorphic class of the geometric realization $|\Delta|$ of Δ and the characteristic of the base field.

The proof of this result is quite technical and is irrelevant to the other parts of the paper, so we decided to put it in the final section. We also remark that this result holds in a wider setting. More precisely, if R is a normal simplicial semigroup ring that is Gorenstein, and I is a monomial ideal, then the corresponding result holds. We work in this general setting, since the proof is the same as in the polynomial ring case.

§2. Consecutiveness of nontrivial superdiagonals of the Lyubeznik table

It seems to be a very difficult task to give a full description of the possible configurations of Lyubeznik tables of any local ring, and only a few results can be found in the literature. The aim of this section is to give some constraints to the possible configurations of Lyubeznik tables, aside from the Euler characteristic formula.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing its residue field $\Bbbk$, and let $I \subseteq R$ be any ideal with $\dim R/I = d$. For each $j \in \mathbb{N}$ with $0 \leqslant j \leqslant d$, set

$$\rho_j(R/I) = \sum_{i=0}^{d-j} \lambda_{i,i+j}(R/I).$$

For example, $\rho_0(R/I)$ (resp. $\rho_1(R/I)$) is the sum of the entries in the diagonal (resp. superdiagonal) of the Lyubeznik table $\Lambda(R/I)$. Clearly, $\sum_{j\in\mathbb{N}}(-1)^j \rho_j(R/I) = 1$. We say that $\rho_j(R/I)$ is *nontrivial* if

$$\rho_j(R/I) \geqslant \begin{cases} 2, & \text{if } j = 0, \\ 1, & \text{if } j \geqslant 1. \end{cases}$$

Clearly, $\Lambda(R/I)$ is nontrivial if and only if $\rho_j(R/I)$ is nontrivial for some j.

It is easy to see that $\lambda_{0,d}(R/I) = 0$ if $d \geqslant 1$ and $\lambda_{0,d}(R/I) = 1$ if $d = 0$. That is, $\rho_d(R/I)$ is always trivial.

A key fact that we use in this section is that local cohomology modules have a natural structure over the ring of k-linear differential operators $D_{R|k}$ (see [13, 14]). In fact, they are $D_{R|k}$-modules of finite length (see [3, Theorem 2.7.13] and [13, Example 2.2] for the case of characteristic zero and [14, Theorem 5.7] in positive characteristic). In particular, Lyubeznik numbers are nothing but the length as a $D_{R|k}$-module of the local cohomology modules $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. That is,

$$\lambda_{p,i}(R/I) = \mathrm{length}_{D_{R|k}}(H^p_{\mathfrak{m}}(H^{n-i}_I(R))).$$

The $D_{R|k}$-module length, which will be denoted simply as $e(-)$, is an additive function. That is, given a short exact sequence of holonomic $D_{R|k}$-modules $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$, we have

$$e(M_2) = e(M_1) + e(M_3).$$

The main result of this section is the following.

THEOREM 2.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing its residue field $\Bbbk$, and let $I \subseteq R$ be any ideal with* $\dim R/I = d$. *Then, we have the following.*

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

- *If $\rho_j(R/I)$ is nontrivial for some j with $0<j<d$, then either $\rho_{j-1}(R/I)$ or $\rho_{j+1}(R/I)$ is nontrivial.*
- *If $\rho_0(R/I)$ is nontrivial, then so is $\rho_1(R/I)$.*

Proof. Consider Grothendieck's spectral sequence

$$E_2^{p,n-i} = H_{\mathfrak{m}}^p(H_I^{n-i}(R)) \Longrightarrow H_{\mathfrak{m}}^{p+n-i}(R).$$

This is a spectral sequence of $D_{R|k}$-modules where $\lambda_{p,i}(R/I) = e(E_2^{p,n-i})$. Notice also that the local cohomology modules $H_{\mathfrak{m}}^r(R)$ vanish for all $r \neq n$, and in this case $e(H_{\mathfrak{m}}^n(R)) = 1$.

We prove the assertion by contradiction. Therefore, assume that $\rho_j(R/I)$ is nontrivial for some $0<j<d$, but both $\rho_{j-1}(R/I)$ and $\rho_{j+1}(R/I)$ are trivial. (The case $j=0$ can be proved by a similar argument.) We have some p, i with $i=p+j$ such that $\lambda_{p,i}(R/I) \neq 0$ (equivalently, $E_2^{p,n-i} \neq 0$). Consider the maps on E_2-terms

$$E_2^{p-2,n-i+1} \xrightarrow{d_2} E_2^{p,n-i} \xrightarrow{d_2'} E_2^{p+2,n-i-1}.$$

We show that $d_2 = d_2' = 0$.

Consider first the case $j>1$. We have $E_2^{p-2,n-i+1} = E_2^{p+2,n-i-1} = 0$ just because $e(E_2^{p-2,n-i+1}) = \lambda_{p-2,i-1}(R/I)$ and $e(E_2^{p+2,n-i-1}) = \lambda_{p+2,i+1}(R/I)$ concern $\rho_{j+1}(R/I)$ and $\rho_{j-1}(R/I)$, respectively. Therefore, $d_2 = d_2' = 0$ is satisfied trivially. When $j=1$, that is, the case when $(p+2, n-i-1) = (d, n-d)$, we have

$$E_2^{d-4,n-d+2} \xrightarrow{d_2} E_2^{d-2,n-d+1} \xrightarrow{d_2'} E_2^{d,n-d}.$$

The triviality of $\rho_2(R/I)$ and $\rho_0(R/I)$ means that $E_2^{d-4,n-d+2} = 0$ and $\lambda_{d,d} = e(E_2^{d,n-d}) = 1$, so $d_2 = 0$. Now, we assume that the map $d_2' : E_2^{d-2,n-d+1} \to E_2^{d,n-d}$ is nonzero. Then, $\operatorname{Im} d_2' = E_2^{d,n-d}$ due to the fact that $E_2^{d,n-d}$ is a simple $D_{R|k}$-module. It follows that $E_3^{d,n-d} = E_2^{d,n-d}/\operatorname{Im} d_2' = 0$, so

$$0 = E_3^{d,n-d} = E_4^{d,n-d} = \cdots = E_\infty^{d,n-d}.$$

On the other hand, since $\rho_0(R/I)$ is trivial, we have

$$0 = E_2^{i,n-i} = E_3^{i,n-i} = \cdots = E_\infty^{i,n-i},$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

for all $i < d$. Therefore, we get a contradiction since, by the general theory of spectral sequences, there exists a filtration

$$0 \subseteq \mathcal{F}^n_n \subseteq \cdots \subseteq \mathcal{F}^n_1 \subseteq H^n_{\mathfrak{m}}(R), \tag{2.1}$$

where $E^{i,n-i}_{\infty} = \mathcal{F}^n_i / \mathcal{F}^n_{i+1}$.

Anyway, we have shown that $d_2 = d'_2 = 0$ in all cases, and this implies that $E^{p,n-i}_3 = E^{p,n-i}_2 \neq 0$. Now, we consider the maps on E_3-terms,

$$E^{p-3,n-i+2}_3 \xrightarrow{d_3} E^{p,n-i}_3 \xrightarrow{d'_3} E^{p+3,n-i-2}_3.$$

Since $E^{p-3,n-i+2}_3$ and $E^{p+3,n-i-2}_3$ concern $\rho_{j+1}(R/I)$ and $\rho_{j-1}(R/I)$, respectively, we have $d_3 = d'_3 = 0$ by the same argument as above. Hence, we have $E^{p,n-i}_4 = E^{p,n-i}_3 \neq 0$. Repeating this argument, we have $0 \neq E^{p,n-i}_2 = E^{p,n-i}_3 = \cdots = E^{p,n-i}_{\infty}$, so we get a contradiction with the fact that $H^{p+n-i}_{\mathfrak{m}}(R) = 0$. (Recall that $j = i - p \neq 0$.) ∎

The behavior of the consecutive superdiagonals is reflected in the following example.

EXAMPLE 2.2. Let $I \subseteq R = \Bbbk[x_1, \ldots, x_8]$ be the Alexander dual ideal of the edge ideal of an 8-cycle; that is, $I^{\vee} = (x_1x_2, x_2x_3, \ldots, x_7x_8, x_8x_1)$. Using the results of [1], we get the Lyubeznik table

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 1 \end{pmatrix}.$$

Notice that $\rho_0(R/I)$ being trivial does not imply that $\rho_1(R/I) = 0$.

REMARK 2.3. Using similar spectral sequence arguments to those considered in Theorem 2.1, Kawasaki [12] and Walther [24] described the possible Lyubeznik tables for rings up to dimension two. Namely, their result is as follows.

- If $d = 2$, then $\lambda_{2,2}(R/I) - 1 = \lambda_{0,1}(R/I)$, and the other Lyubeznik numbers are 0.

If we take a careful look at the spectral sequence we can also obtain the following.

- If $d \geqslant 3$, then $\lambda_{2,d}(R/I) = \lambda_{0,d-1}(R/I)$, and

$$\begin{aligned}\lambda_{1,d-1}(R/I) \leqslant \lambda_{3,d}(R/I) &\leqslant \lambda_{1,d-1}(R/I) + \lambda_{0,d-2}(R/I)\\ &\leqslant \lambda_{3,d}(R/I) + \lambda_{2,d-1}(R/I).\end{aligned}$$

For $d = 3$, we can refine the last inequality; that is,

$$\lambda_{1,2}(R/I) + \lambda_{0,1}(R/I) = \lambda_{3,3}(R/I) + \lambda_{2,2}(R/I) - 1.$$

Indeed, using the filtration (2.1), we have

$$\sum_{i=0}^{d} e(E_\infty^{i,n-r-i}) = e(H_{\mathfrak{m}}^{n-r}(R)) = \begin{cases} 1, & \text{if } r = 0,\\ 0, & \text{otherwise.}\end{cases}$$

Then, the result follows considering the differentials $d_2 : E_2^{0,n-d+1} \longrightarrow E_2^{2,n-d}$, $d_2 : E_2^{1,n-d+1} \longrightarrow E_2^{3,n-d}$, $d_2 : E_2^{0,n-d+2} \longrightarrow E_2^{2,n-d+1}$ and $d_3 : E_3^{0,n-d+2} \longrightarrow E_3^{3,n-d}$. Finally, we point out that $E_3^{0,n-d+1} = E_\infty^{0,n-d+1}$, $E_3^{1,n-d+1} = E_\infty^{1,n-d+1}$, $E_3^{2,n-d} = E_\infty^{2,n-d}$, $E_4^{0,n-d+2} = E_\infty^{0,n-d+2}$ and $E_4^{3,n-d} = E_\infty^{3,n-d}$.

§3. Linear strands of minimal free resolutions of $\mathbb{Z}$-graded ideals

Throughout this section, we consider $\mathbb{Z}$-graded ideals I in the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$. In particular, I is not necessarily a monomial ideal. For simplicity, we assume that $I \neq 0$. The minimal $\mathbb{Z}$-graded free resolution of I is an exact sequence of free $\mathbb{Z}$-graded modules:

$$(3.1)\quad L_\bullet(I): \quad 0 \longrightarrow L_n \xrightarrow{d_n} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow I \longrightarrow 0,$$

where the ith term is of the form

$$L_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(I)},$$

and the matrices of the morphisms $d_i : L_i \longrightarrow L_{i-1}$ do not contain invertible elements. The *Betti numbers* of I are the invariants $\beta_{i,j}(I)$. Notice that

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

$L_i \cong R^{\beta_i(I)}$ as underlying R-modules, where, for each i, we set $\beta_i(I) := \sum_{j\in\mathbb{Z}} \beta_{i,j}(I)$. Hence, (3.1) implies that

$$\sum_{0\leqslant i\leqslant n} (-1)^i \beta_i(I) = \operatorname{rank}_R(I) = 1.$$

Given $r \in \mathbb{N}$, we also consider the r-*linear strand* of $L_\bullet(I)$:

$$\mathbb{L}_\bullet^{<r>}(I): \quad 0 \longrightarrow L_n^{<r>} \xrightarrow{d_n^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0,$$

where

$$L_i^{<r>} = R(-i-r)^{\beta_{i,i+r}(I)},$$

and the differential $d_i^{<r>}: L_i^{<r>} \longrightarrow L_{i-1}^{<r>}$ is the corresponding component of d_i.

REMARK 3.1. Sometimes, we also consider the minimal $\mathbb{Z}$-graded free resolution $L_\bullet(R/I)$ of the quotient ring R/I:

(3.2)

$$L_\bullet(R/I): \quad 0 \longrightarrow L_n \xrightarrow{d_n} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 = R \longrightarrow R/I \longrightarrow 0.$$

Its truncation at the first term $L_{\geqslant 1}(R/I)$ gives a minimal free resolution $L_\bullet(I)$ of I. For $r \geqslant 2$, $\mathbb{L}_\bullet^{<r>}(I)$ is isomorphic to the $(r-1)$-linear strand $\mathbb{L}_\bullet^{<r-1>}(R/I)$ up to translation. However, this is not true for $r = 1$, since $\mathbb{L}_\bullet^{<0>}(R/I)$ starts from the 0th term R, which is irrelevant to $\mathbb{L}_\bullet^{<1>}(I)$.

To the minimal $\mathbb{Z}$-graded free resolution of I we may associate a set of invariants that measure the acyclicity of the linear strands as follows. Let $\mathbb{K}$ denote the field of fractions $Q(R)$ of R, and set

$$\nu_{i,j}(I) := \dim_{\mathbb{K}}[H_i(\mathbb{L}_\bullet^{<j-i>}(I) \otimes_R \mathbb{K})].$$

Since the complex $\mathbb{L}_\bullet^{<r>}(I) \otimes_R \mathbb{K}$ is of the form

$$0 \longrightarrow \mathbb{K}^{\beta_{n,n+r}(I)} \xrightarrow{\partial_n^{<r>}} \cdots \longrightarrow \mathbb{K}^{\beta_{1,1+r}(I)} \xrightarrow{\partial_1^{<r>}} \mathbb{K}^{\beta_{0,r}(I)} \longrightarrow 0,$$

we have $\nu_{i,j}(I) \leqslant \beta_{i,j}(I)$ for all i, j (if $i > j$ then $\nu_{i,j}(I) = \beta_{i,j}(I) = 0$), and

$$\sum_{i=0}^{n} (-1)^i \nu_{i,i+r}(I) = \sum_{i=0}^{n} (-1)^i \beta_{i,i+r}(I)$$

for each r. If we mimic the construction of the *Betti table*, we may also consider the ν-*table* of I:

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

$\nu_{i,i+r}(I)$	0	1	2	$\cdots$
0	$\nu_{0,0}(I)$	$\nu_{1,1}(I)$	$\nu_{2,2}(I)$	$\cdots$
1	$\nu_{0,1}(I)$	$\nu_{1,2}(I)$	$\nu_{2,3}(I)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Next, we consider some basic properties of ν-numbers. It turns out that they satisfy analogous properties to those of Lyubeznik numbers. For instance, these invariants satisfy the following *Euler characteristic* formula.

LEMMA 3.2. *For a $\mathbb{Z}$-graded ideal I, we have*

$$\sum_{i,j\in\mathbb{N}} (-1)^i \nu_{i,j}(I) = 1.$$

Proof. The assertion follows from the computation below:

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} (-1)^i \nu_{i,j}(I) &= \sum_{i,r\in\mathbb{N}} (-1)^i \nu_{i,i+r}(I) \\ &= \sum_{r\in\mathbb{N}} \sum_{0\leqslant i\leqslant n} (-1)^i \nu_{i,i+r}(I) \\ &= \sum_{r\in\mathbb{N}} \sum_{0\leqslant i\leqslant n} (-1)^i \beta_{i,i+r}(I) \\ &= \sum_{0\leqslant i\leqslant n} \sum_{r\in\mathbb{N}} (-1)^i \beta_{i,i+r}(I) \\ &= \sum_{0\leqslant i\leqslant n} (-1)^i \beta_i(I) \\ &= 1.\end{aligned}$$

□

We can also single out a particular nonvanishing ν-number. For each $i \in \mathbb{N}$, let $I_{<i>}$ denote the ideal generated by the homogeneous component $I_i = \{f \in I \mid \deg(f) = i\} \cup \{0\}$. Then, we have the following.

LEMMA 3.3. *If I is a $\mathbb{Z}$-graded ideal with $l := \min\{i \mid I_i \neq 0\}$, then we have $\nu_{0,l}(I) \neq 0$.*

Proof. It is easy to see that there is a surjection $H_0(\mathbb{L}_\bullet^{<l>}(I)) \twoheadrightarrow I_{<l>}$. Since $\dim_R I_{<l>} = n$, we have $H_0(\mathbb{L}_\bullet^{<l>}(I) \otimes_R \mathbb{K}) \cong H_0(\mathbb{L}_\bullet^{<l>}(I)) \otimes_R \mathbb{K} \neq 0$. □

This fact allows us to consider the following notion.

DEFINITION 3.4. Let I be a $\mathbb{Z}$-graded ideal, and set $l := \min\{i \mid I_i \neq 0\}$. We say that I has a *trivial ν-table* if $\nu_{0,l}(I) = 1$ and the rest of these invariants vanish.

3.1 Componentwise linear ideals

It might be an interesting problem to find necessary and/or sufficient conditions for a $\mathbb{Z}$-graded ideal to have a trivial ν-table. In this direction, we have the following relation to the notion of *componentwise linear* ideals.

DEFINITION 3.5. (Herzog and Hibi [8]) We say that a $\mathbb{Z}$-graded ideal I is *componentwise linear* if $I_{<r>}$ has a linear resolution for all $r \in \mathbb{N}$; that is, $\beta_{i,j}(I_{<r>}) = 0$ unless $j = i + r$.

Römer [20] and the second author [25, Theorem 4.1] independently showed that I is componentwise linear if and only if $H_i(\mathbb{L}_\bullet^{<r>}(I)) = 0$ for all r and all $i \geqslant 1$.

PROPOSITION 3.6. *A componentwise linear ideal I has a trivial ν-table.*

Proof. Since I is componentwise linear, we have $H_i(\mathbb{L}_\bullet^{<r>}(I)) = 0$ for all r and all $i \geqslant 1$, and hence $\nu_{i,j}(I) = 0$ for all j and all $i \geqslant 1$. Now, the assertion follows from Lemmas 3.2 and 3.3. $\square$

The converse of the above proposition is not true. For example, in Corollary 3.13 below, we show that if $I_1 \neq 0$, then it has a trivial ν-table. However, there is no relation between being componentwise linear and $I_1 \neq 0$.

3.2 Consecutiveness of nontrivial columns of the ν-tables

For a $\mathbb{Z}$-graded ideal $I \subseteq R$ and $i \in \mathbb{N}$, set

$$\nu_i(I) = \sum_{j \in \mathbb{N}} \nu_{i,j}(I).$$

If we denote $\mathbb{L}_\bullet(I) := \bigoplus_{r \in \mathbb{N}} \mathbb{L}_\bullet^{<r>}(I)$, then

$$\nu_i(I) = \dim_{\mathbb{K}} H_i(\mathbb{L}_\bullet(I) \otimes_R \mathbb{K}).$$

By Lemma 3.2, we have $\sum_{i=0}^{n} (-1)^i \nu_i(I) = 1$. We say that $\nu_i(I)$ is *nontrivial* if

$$\nu_i(I) \geqslant \begin{cases} 2, & \text{if } i = 0, \\ 1, & \text{if } i \geqslant 1. \end{cases}$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

Clearly, the ν-table of I is nontrivial if and only if $\nu_i(I)$ is nontrivial for some i. If $n \geqslant 1$, we have $\mathrm{proj.dim}_R I \leqslant n-1$, and hence $\nu_n(I) = 0$. In particular, $\nu_n(I)$ is always trivial.

The main result of this subsection is the following.

THEOREM 3.7. *Let I be a $\mathbb{Z}$-graded ideal of R. Then, we have the following.*

- *If $\nu_j(I)$ is nontrivial for $1 \leqslant j \leqslant n-1$, then either $\nu_{j-1}(I)$ or $\nu_{j+1}(I)$ is nontrivial.*
- *If $\nu_0(I)$ is nontrivial, then so is $\nu_1(I)$.*

In order to prove the theorem, we reconstruct $\mathbb{L}_\bullet(I)$ using a spectral sequence. Let $L_\bullet(I)$ be the minimal free resolution of I as before. Consider the $\mathfrak{m}$-adic filtration $L_\bullet(I) = F_0 L_\bullet \supset F_1 L_\bullet \supset \cdots$ of $L_\bullet(I)$, where $F_i L_\bullet$ is a subcomplex whose component of homological degree j is $\mathfrak{m}^i L_j$. For any given R-module M, we regard $\mathrm{gr}\,(M) := \bigoplus_{i \in \mathbb{N}} \mathfrak{m}^i M/\mathfrak{m}^{i+1} M$ as an R-module via the isomorphism $\mathrm{gr}\, R = \bigoplus_{i \in \mathbb{N}} \mathfrak{m}^i/\mathfrak{m}^{i+1} \cong R = \Bbbk[x_1, \ldots, x_n]$. Since each L_j is a free R-module,

$$\bigoplus_{p+q=-j} E_0^{p,q} = \left(\bigoplus_{p \geqslant 0} \mathfrak{m}^p L_j/\mathfrak{m}^{p+1} L_j \right) = \mathrm{gr}\, L_j$$

is isomorphic to L_j (if we identify $\mathrm{gr}\, R$ with R), while we have to forget the original $\mathbb{Z}$-grading of L_j. Since $L_\bullet(I)$ is a minimal free resolution, $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ is the zero map for all p, q, and hence $E_0^{p,q} = E_1^{p,q}$. It follows that

$$\mathbb{E}_j^{(1)} := \bigoplus_{p+q=-j} E_1^{p,q} = \bigoplus_{p+q=-j} E_0^{p,q}$$

is isomorphic to L_j under the identification $R \cong \mathrm{gr}\, R$. Collecting the maps

$$d_1^{p,q} : E_1^{p,q} (= \mathfrak{m}^p L_j/\mathfrak{m}^{p+1} L_j) \longrightarrow E_1^{p+1,q} (= \mathfrak{m}^{p+1} L_{j-1}/\mathfrak{m}^{p+2} L_{j-1})$$

for p, q with $p + q = -j$, we have the R-morphism $d_j^{(1)} : \mathbb{E}_j^{(1)} \to \mathbb{E}_{j-1}^{(1)}$, and these morphisms make $\mathbb{E}_\bullet^{(1)}$ a chain complex of R-modules. Under the isomorphism $\mathbb{E}_j^{(1)} \cong L_j$, $\mathbb{E}_\bullet^{(1)}$ is isomorphic to $\mathbb{L}_\bullet(I) = \bigoplus_{r \in \mathbb{N}} \mathbb{L}_\bullet^{<r>}(I)$. Hence, we have

$$\mathbb{E}_j^{(2)} := \bigoplus_{p+q=-j} E_2^{p,q} \cong H_j(\mathbb{L}_\bullet(I))$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

and $\nu_j(I) = \dim_{\mathbb{K}}(\mathbb{E}_j^{(2)} \otimes_R \mathbb{K})$. Collecting the maps $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$, we have the R-morphism

$$d_j^{(2)} : \mathbb{E}_j^{(2)} \ (\cong H_j(\mathbb{L}_\bullet(I))) \longrightarrow \mathbb{E}_{j-1}^{(2)} \ (\cong H_{j-1}(\mathbb{L}_\bullet(I))).$$

Moreover, we have the chain complex

$$\cdots \longrightarrow \mathbb{E}_{j+1}^{(2)} \xrightarrow{d_{j+1}^{(2)}} \mathbb{E}_j^{(2)} \xrightarrow{d_j^{(2)}} \mathbb{E}_{j-1}^{(2)} \longrightarrow \cdots$$

of R-modules whose jth homology is isomorphic to $\mathbb{E}_j^{(3)} := \bigoplus_{p+q=-j} E_3^{p,q}$. For all $r \geqslant 4$, $\mathbb{E}_j^{(r)} := \bigoplus_{p+q=-j} E_r^{p,q}$ satisfies the same property.

By the construction of spectral sequences, if

$$r > \max\{k \mid \beta_{j,k}(I) \neq 0\} - \min\{k \mid \beta_{j-1,k}(I) \neq 0\},$$

then the map $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is zero for all p, q with $p+q = -j$, and hence $d_j^{(r)} : \mathbb{E}_j^{(r)} \to \mathbb{E}_{j-1}^{(r)}$ is zero. This implies that $\mathbb{E}_j^{(r)}$ is isomorphic to

$$\mathbb{E}_j^{(\infty)} := \bigoplus_{p+q=-j} E_\infty^{p,q}$$

for $r \gg 0$.

Proof of Theorem 3.7. We prove the assertion by contradiction using the spectral sequence introduced above. First, assume that $\nu_j(I)$ is nontrivial for some $2 \leqslant j \leqslant n-1$, but both $\nu_{j-1}(I)$ and $\nu_{j+1}(I)$ are trivial. (The cases $j = 0, 1$ can be proved using similar arguments, and we make a few remarks later.) Then, we have $\mathbb{E}_j^{(2)} \otimes_R \mathbb{K} \neq 0$ and $\mathbb{E}_{j+1}^{(2)} \otimes_R \mathbb{K} = \mathbb{E}_{j-1}^{(2)} \otimes_R \mathbb{K} = 0$. It follows that $\mathbb{E}_j^{(3)} \otimes_R \mathbb{K} \neq 0$, since it is the homology of

$$\mathbb{E}_{j+1}^{(2)} \otimes_R \mathbb{K} \longrightarrow \mathbb{E}_j^{(2)} \otimes_R \mathbb{K} \longrightarrow \mathbb{E}_{j-1}^{(2)} \otimes_R \mathbb{K}.$$

Similarly, we have $\mathbb{E}_{j-1}^{(3)} \otimes_R \mathbb{K} = \mathbb{E}_{j+1}^{(3)} \otimes_R \mathbb{K} = 0$. Repeating this argument, we have $\mathbb{E}_j^{(r)} \otimes_R \mathbb{K} \neq 0$ for all $r \geqslant 4$. Hence, $E_\infty^{p,q} \neq 0$ for some p, q with $p+q = -j$. However, this contradicts the facts that

$$E_r^{p,q} \Longrightarrow H_{-p-q}(L_\bullet(I))$$

and $H_j(L_\bullet(I)) = 0$. (Recall that $j > 0$ now.)

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

Next, we assume that $\nu_1(I)$ is nontrivial, but $\nu_0(I)$ and $\nu_2(I)$ are trivial. That is,

$$\mathbb{E}_1^{(2)} \otimes_R \mathbb{K} \neq 0, \qquad \mathbb{E}_0^{(2)} \otimes_R \mathbb{K} \cong \mathbb{K} \qquad \text{and} \qquad \mathbb{E}_2^{(2)} \otimes_R \mathbb{K} = 0.$$

As we have seen above, we must have $\mathbb{E}_1^{(r)} = 0$ for $r \gg 0$. Since $\mathbb{E}_2^{(r)} \otimes_R \mathbb{K} = 0$ for all r now, if $d_1^{(r)} \otimes_R \mathbb{K} : \mathbb{E}_1^{(r)} \otimes_R \mathbb{K} \longrightarrow \mathbb{E}_0^{(r)} \otimes_R \mathbb{K}$ are the zero maps for all r, then $\mathbb{E}_1^{(r)} \otimes_R \mathbb{K} \cong \mathbb{E}_1^{(2)} \otimes_R \mathbb{K} \neq 0$ for all r, and this is a contradiction. Therefore, there is some $r \geqslant 2$ such that $d_1^{(r)} \otimes_R \mathbb{K}$ is not zero. If s is the minimum among these r, $d_1^{(s)} \otimes_R \mathbb{K} : \mathbb{E}_1^{(s)} \otimes_R \mathbb{K} \longrightarrow (\mathbb{E}_0^{(s)} \otimes_R \mathbb{K}) \cong \mathbb{K}$ is surjective. Hence, $\mathbb{E}_0^{(r)} \otimes_R \mathbb{K} = 0$ for all $r > s$, and $\mathbb{E}_0^{(\infty)} \otimes_R \mathbb{K} = 0$. However, since $\mathbb{E}_0^{(\infty)} \cong \mathrm{gr}\, H_0(L_\bullet(I)) \cong \mathrm{gr}\,(I)$ and $\dim_R I = n$, we have $\dim_R(\mathrm{gr}\,(I)) = n$ and hence $\mathbb{E}_0^{(\infty)} \otimes_R \mathbb{K} \neq 0$. This is a contradiction. The case when $\nu_0(I)$ is nontrivial can be proved in a similar way. ∎

3.3 Thom–Sebastiani type formulas

Let I, J be $\mathbb{Z}$-graded ideals in two disjoint sets of variables, say $I \subseteq R = \Bbbk[x_1, \ldots, x_m]$ and $J \subseteq S = \Bbbk[y_1, \ldots, y_n]$. The aim of this subsection is to describe the ν-numbers of $IT + JT$, where $T = R \otimes_\Bbbk S = \Bbbk[x_1, \ldots, x_m, y_1, \ldots, y_n]$, in terms of those of I and J, respectively. When we just consider Betti numbers, we have the following results due to Jacques and Katzman [11].

PROPOSITION 3.8. (Cf. [11, Lemma 2.1]) *Let $L_\bullet(R/I)$ and $L_\bullet(S/J)$ be minimal graded free resolutions of R/I and S/J, respectively. Then,*

$$(L_\bullet(R/I) \otimes_R T) \otimes_T (L_\bullet(S/J) \otimes_S T)$$

is a minimal graded free resolution of $T/IT + JT$.

Hence, Betti numbers satisfy the following relation.

COROLLARY 3.9. (Cf. [11, Corollary 2.2]) *The Betti numbers of $T/IT + JT$ have the following form:*

$$\beta_{i,j}(T/IT + JT) = \sum_{\substack{k+k'=i \\ l+l'=j}} \beta_{k,l}(T/IT)\beta_{k',l'}(T/JT).$$

Hence, we have

$$\beta_{i,j}(IT + JT) = \beta_{i,j}(IT) + \beta_{i,j}(JT) + \sum_{\substack{k+k'=i-1 \\ l+l'=j}} \beta_{k,l}(IT)\beta_{k',l'}(JT).$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

Our aim is to extend the result in [11] to the case of ν-numbers. To such purpose, it is more convenient to consider separately the case of ideals with degree one elements. Thus, let $I \subseteq R$ be any $\mathbb{Z}$-graded ideal and assume for simplicity that J is principally generated by an element of degree 1; for example, $J = (y) \subseteq S$.

LEMMA 3.10. *Let* $I \subseteq R = \Bbbk[x_1, \dots, x_m]$ *and* $J = (y) \subseteq S = \Bbbk[y]$ *be* $\mathbb{Z}$*-graded ideals, and set* $T = R \otimes_\Bbbk S = \Bbbk[x_1, \dots, x_m, y]$. *For* $r \geqslant 2$, *the* r*-linear strand* $\mathbb{L}_\bullet^{<r>}(IT + JT)$ *is the mapping cone of the chain map*

$$\times y : (\mathbb{L}_\bullet^{<r>}(IT))(-1) \to \mathbb{L}_\bullet^{<r>}(IT).$$

Proof. It is easy to see that a minimal T-free resolution $L_\bullet(T/IT + JT)$ of $T/IT + JT$ is given by the mapping cone of the chain map $\times y : L_\bullet(T/IT)(-1) \to L_\bullet(T/IT)$, where $L_\bullet(T/IT)$ is a minimal T-free resolution of T/IT. Since the operation of taking the r-linear strand commutes with the operation of taking the mapping cone, we are done. □

The general case is more involved. Assume now that $I \subseteq R = \Bbbk[x_1, \dots, x_m]$ and $J \subseteq S = \Bbbk[y_1, \dots, y_n]$ are $\mathbb{Z}$-graded ideals such that $I_1 = 0$ and $J_1 = 0$. Let $L_\bullet(I)$ be a minimal graded R-free resolution of I, and let $L_\bullet(J)$ be a minimal graded S-free resolution of J, and consider their extensions $L_\bullet(IT)$ and $L_\bullet(JT)$ to $T = R \otimes_\Bbbk S = \Bbbk[x_1, \dots, x_m, y_1, \dots, y_n]$.

LEMMA 3.11. *Under the previous assumptions, the* r*-linear strand* $\mathbb{L}_\bullet^{<r>}(IT + JT)$ *is*

$$\mathbb{L}_\bullet^{<r>}(IT + JT) = \mathbb{L}_\bullet^{<r>}(IT) \oplus \mathbb{L}_\bullet^{<r>}(JT) \oplus \left(\bigoplus_{a+b=r+1} (\mathbb{L}_\bullet^{<a>}(IT) \otimes_T \mathbb{L}_\bullet^{<b>}(JT))[-1] \right).$$

Here, for a chain complex $C_\bullet$, $C_\bullet[-1]$ *denotes the translated complex whose component of homological degree* j *is* C_{j-1}.

Proof. Consider the minimal $\mathbb{Z}$-graded free resolutions of R/I and S/J, respectively,

$$L_\bullet(R/I): \quad 0 \longrightarrow L_m \xrightarrow{d_m} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow R/I \longrightarrow 0,$$

$$L'_\bullet(S/J): \quad 0 \longrightarrow L'_n \xrightarrow{d'_n} \cdots \longrightarrow L'_1 \xrightarrow{d'_1} L'_0 \longrightarrow S/J \longrightarrow 0,$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

where $L_0 = R$ and $L'_0 = S$. According to Proposition 3.8, the minimal $\mathbb{Z}$-graded free resolution $L_\bullet(T/IT + JT)$ has the form[2]

$$\cdots \longrightarrow \begin{array}{c} L_2 \otimes L'_0 \\ \oplus \\ L_1 \otimes L'_1 \\ \oplus \\ L_0 \otimes L'_2 \end{array} \xrightarrow{\partial_2} \begin{array}{c} L_1 \otimes L'_0 \\ \oplus \\ L_0 \otimes L'_1 \end{array} \xrightarrow{\partial_1} L_0 \otimes L'_0 \longrightarrow T/IT + JT \longrightarrow 0,$$

where, for any given $x_i \otimes y_{p-i} \in L_i \otimes L'_{p-i}$, we have

$$\begin{aligned} \partial_p(x_i \otimes y_{p-i}) = d_i(x_i) \otimes y_{p-i} + (-1)^i x_i \otimes d'_{p-i}(y_{p-i}) &\in (L_{i-1} \otimes L'_{p-i}) \\ &\oplus (L_i \otimes L'_{p-i-1}). \end{aligned}$$

To describe the r-linear strand $\mathbb{L}_\bullet^{<r>}(IT + JT)$ of the ideal $IT + JT$, we must consider the truncation at the first term of the above resolution and take a close look at the free modules and the components of the corresponding differentials. Recall that $L_\bullet^{<r-1>}(R/I)$ corresponds to $L_\bullet^{<r>}(I)$ for all $r \geqslant 2$. It is easy to see that both

$$\begin{aligned} \mathbb{L}_\bullet^{<r>}(IT) : 0 \longrightarrow L_m^{<r-1>} \otimes L'_0 \longrightarrow \cdots \longrightarrow L_2^{<r-1>} \otimes L'_0 \\ \longrightarrow L_1^{<r-1>} \otimes L'_0 \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \mathbb{L}_\bullet^{<r>}(JT) : 0 \longrightarrow L_0 \otimes {L'}_n^{<r-1>} \longrightarrow \cdots \longrightarrow L_0 \otimes {L'}_2^{<r-1>} \\ \longrightarrow L_0 \otimes {L'}_1^{<r-1>} \longrightarrow 0 \end{aligned}$$

are subcomplexes of $\mathbb{L}_\bullet^{<r>}(IT + JT)$. Moreover, $\mathbb{L}_\bullet^{<r>}(IT)$ and $\mathbb{L}_\bullet^{<r>}(JT)$ are direct summands of $\mathbb{L}_\bullet^{<r>}(IT + JT)$. In fact, since $I_1 = J_1 = 0$, the linear parts of the maps $L_i \otimes L'_1 \to L_i \otimes L'_0$ and $L_1 \otimes L'_j \to L_0 \otimes L'_j$ vanish.

In order to obtain the remaining components of $\mathbb{L}_\bullet^{<r>}(IT + JT)$, we must consider the r-linear strand of

$$\cdots \longrightarrow \begin{array}{c} L_3 \otimes L'_1 \\ \oplus \\ L_2 \otimes L'_2 \\ \oplus \\ L_1 \otimes L'_3 \end{array} \xrightarrow{\partial_4} \begin{array}{c} L_2 \otimes L'_1 \\ \oplus \\ L_1 \otimes L'_2 \end{array} \xrightarrow{\partial_3} L_1 \otimes L'_1 \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} 0.$$

[2] By an abuse of notation, we denote $(L_i \otimes_R T) \otimes_T (L'_j \otimes_S T)$ simply as $L_i \otimes L'_j$.

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

This complex starts at the second term (i.e., the term of homological degree 1), and the first term of the r-linear strand is $\bigoplus_{a+b=r+1}\mathbb{L}_0^{<a>}(IT)\otimes_T \mathbb{L}_0^{<b>}(JT)$. If we take a close look at the free summands of these components and their differentials, we obtain the following description:

$$\bigoplus_{a+b=r+1}(\mathbb{L}_\bullet^{<a>}(IT)\otimes_T \mathbb{L}_\bullet^{<b>}(JT))[-1].$$

Therefore, we are done. □

The main result of this subsection is the following.

PROPOSITION 3.12. *The ν-numbers of $IT+JT$ have the following form.*

(i) *If $I_1\neq 0$ or $J_1\neq 0$, then $IT+JT$ has a trivial ν-table.*
(ii) *If $I_1=0$ and $J_1=0$, then we have*

$$\nu_{i,j}(IT+JT)=\nu_{i,j}(IT)+\nu_{i,j}(JT)+\sum_{\substack{k+k'=i-1\\ l+l'=j}}\nu_{k,l}(IT)\nu_{k',l'}(JT).$$

Proof. (i) If $J_1\neq 0$, we may assume that $y_n\in J$ without loss of generality. Now, we have $J=(f_1,\dots,f_r,y_n)$, where $f_1\dots,f_r$ are homogeneous polynomials in $\Bbbk[y_1,\dots.y_{n-1}]$. Set $R':=\Bbbk[x_1,\dots,x_m,y_1,\dots,y_{n-1}]$, $S'=\Bbbk[y_n]$, and let $I'=IR'+(f_1,\dots,f_r)$ be an ideal in R' (note that $f_1\dots,f_r$ are elements in R'), and let $J'=(y_n)$ be an ideal in S'. Then, we have $T=R\otimes_\Bbbk S=R'\otimes_\Bbbk S'$, and $IT+JT=I'T+J'T$. This means that we may assume that $J=(y)\subseteq S=\Bbbk[y]$ from the beginning. For $r\geqslant 2$, the r-linear strand $\mathbb{L}_\bullet^{<r>}(IT+JT)$ is given by the mapping cone of the chain map $\times y:(\mathbb{L}_\bullet^{<r>}(IT))(-1)\to\mathbb{L}_\bullet^{<r>}(IT)$ by Lemma 3.10. Hence, $\mathbb{L}_\bullet^{<r>}(IT+JT)\otimes_T\mathbb{K}$ is given by the mapping cone of the chain map

$$\times y:\mathbb{L}_\bullet^{<r>}(IT)\otimes_T\mathbb{K}\longrightarrow\mathbb{L}_\bullet^{<r>}(IT)\otimes_T\mathbb{K},$$

where $\mathbb{K}$ is the field of fractions of T. Clearly, this is the identity map, and its mapping cone is exact. This means that $H_i(\mathbb{L}_\bullet^{<r>}(IT+JT)\otimes_T\mathbb{K})=0$ for all $r\geqslant 2$ and all i.

On the other hand, $(IT+JT)_{<1>}$ is a complete intersection ideal generated by degree-1 elements, and hence we have $\dim_\mathbb{K} H_i(\mathbb{L}_\bullet^{<1>}(I)\otimes_T\mathbb{K})=\delta_{0,i}$. Summing up, we see that $IT+JT$ has a trivial ν-table.

(ii) Follows immediately from Lemma 3.11. □

The following is just a rephrasing of part (i) of the previous result.

Corollary 3.13. *Let $I \subseteq R$ be a $\mathbb{Z}$-graded ideal with $I_1 \neq 0$, then I has a trivial ν-table.*

The following is another corollary of Proposition 3.12.

Corollary 3.14. *With the same notation as in Proposition 3.12, if $I_1 = J_1 = 0$, then $IT + JT$ always has a nontrivial ν-table.*

Proof. Set $l := \min\{i \mid I_i \neq 0\}$ and $l' := \min\{i \mid J_i \neq 0\}$. Then, we have $\nu_{1,l+l'}(IT + JT) \geqslant \nu_{0,l}(IT)\nu_{0,l'}(JT) > 0$ by Proposition 3.12(ii). ∎

§4. Lyubeznik numbers versus ν-numbers for monomial ideals

In [26], the second author showed that, via Alexander duality, the study of local cohomology modules with supports in monomial ideals can be translated into the study of the minimal free resolutions of squarefree monomial ideals. This fact was later refined by Vahidi and the first author in [1] in order to study Lyubeznik numbers of squarefree monomial ideals in terms of the linear strands of their Alexander dual ideals. The aim of this section is to go further in this direction.

In the following, we only consider monomial ideals in the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$, and $\mathfrak{m} = (x_1, \ldots, x_n)$ denotes the graded maximal ideal. Recall that Lyubeznik numbers are well defined in this nonlocal setting since they are invariant with respect to completion, so we consider $\lambda_{p,i}(R/I) = \lambda_{p,i}(\widehat{R}/I\widehat{R})$, where $\widehat{R} = \Bbbk[\![x_1, \ldots, x_n]\!]$. For a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, set $\mathrm{supp}(\mathbf{a}) := \{i \mid a_i \neq 0\} \subseteq \{1, \ldots, n\}$. For each $1 \leqslant i \leqslant n$, let $\mathbf{e}_i \in \mathbb{Z}^n$ be the ith standard vector. The following notion was introduced by the second author, and serves as a powerful tool for combinatorial commutative algebra.

Definition 4.1. We say that a finitely generated $\mathbb{N}^n$-graded R-module $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree* if the multiplication map $M_{\mathbf{a}} \ni y \longmapsto x_i y \in M_{\mathbf{a}+\mathbf{e}_i}$ is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \mathrm{supp}(\mathbf{a})$.

The theory of squarefree modules is found in [25, 26, 28, 29]. Here, we list some basic properties.

- For a monomial ideal I, it is a squarefree R-module if and only if $I = \sqrt{I}$ (equivalently, the Stanley–Reisner ideal I_Δ for some Δ). The free module R itself and the $\mathbb{Z}^n$-graded canonical module $\omega_R = R(-\mathbf{1})$ are squarefree. Here, $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$. The Stanley–Reisner ring R/I_Δ is also squarefree.

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

- Let M be a squarefree R-module, and let $L_\bullet$ be its $\mathbb{Z}^n$-graded minimal free resolution. Then, the free module L_i and the syzygy module $\mathrm{Syz}_i(M)$ are squarefree for each i. Moreover, $\mathrm{Ext}^i_R(M, \omega_R)$ is squarefree for all i.
- Let $^*\mathrm{mod}\, R$ be the category of $\mathbb{Z}^n$-graded finitely generated R-modules, and let $\mathrm{Sq}\, R$ be its full subcategory consisting of squarefree modules. Then, $\mathrm{Sq}\, R$ is an abelian subcategory of $^*\mathrm{mod}\, R$. We have an exact contravariant functor $\mathbf{A}$ from $\mathrm{Sq}\, R$ to itself. The construction of $\mathbf{A}$ is found in (for example) [29]. Here, we just remark that $\mathbf{A}(R/I_\Delta) \cong I_{\Delta^\vee}$, where $\Delta^\vee := \{F \subseteq \{1, \ldots, n\} \mid (\{1, \ldots, n\} \setminus F) \notin \Delta\}$ is the Alexander dual simplicial complex of Δ.

In this framework, we have the following description of Lyubeznik numbers.

THEOREM 4.2. [26, Corollary 3.10] *Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring, and let I_Δ be a squarefree monomial ideal. Then, we have*

$$\lambda_{p,i}(R/I_\Delta) = \dim_\Bbbk [\mathrm{Ext}^{n-p}_R(\mathrm{Ext}^{n-i}_R(R/I_\Delta, \omega_R), \omega_R)]_0 < \infty.$$

For a squarefree R-module M, the second author defined the cochain complex $\mathbf{D}(M)$ of squarefree R-modules satisfying $H^i(\mathbf{D}(M)) \cong \mathrm{Ext}^{n+i}_R(M, \omega_R)$ for all i (see [29, Section 3]). By [25, Theorem 4.1] or [28, Theorem 3.8], we have the isomorphism

$$\mathbf{A} \circ \mathbf{D}\,(\mathrm{Ext}^{n-i}_R(R/I_\Delta, \omega_R)) \cong (\mathbb{L}_\bullet^{<n-i>}(I_{\Delta^\vee}))[-i] \tag{4.1}$$

of cochain complexes of $\mathbb{Z}^n$-graded R-modules[3]. Here, for a cochain complex $C^\bullet$, $C^\bullet[-i]$ means the $-i$th translation of $C^\bullet$. More precisely, it is the cochain complex whose component of cohomological degree j is C^{j-i}, and we regard a chain complex $C_\bullet$ as the cochain complex whose component of cohomological degree j is C_{-j}.

The following is a variant of a result given by the first author and Vahidi.

THEOREM 4.3. (Cf. [1, Corollary 4.2]) *Let $I_\Delta \subseteq R = \Bbbk[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then, we have*

$$\lambda_{p,i}(R/I_\Delta) = \nu_{i-p,n-p}(I_{\Delta^\vee}).$$

[3]Our situation is closer to that of [25, Theorem 4.1] ([29] works in a wider context). However, [25] does not recognize $\mathbf{D}$ and $\mathbf{A}$ as individual operations, but treats the composition $\mathbf{A} \circ \mathbf{D}$. In fact, $\mathbf{A} \circ \mathbf{D}$ corresponds to the operation $\mathbb{F}_\bullet(-)$ of [25] up to translation.

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

Proof. By (4.1) and the construction of $\mathbf{A}$, we have an isomorphism

$$([\mathbf{D}\ (\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R)]_0)^* \cong (\mathbb{L}_\bullet^{<n-i>}(I_{\Delta^\vee}))_{\mathbf{1}}[-i]$$

of cochain complexes of $\Bbbk$-vector spaces. Here, $(-)^*$ means the $\Bbbk$-dual. We also remark that, for a squarefree module M, we have

$$\dim_\Bbbk M_{\mathbf{1}} = \mathrm{rank}_R M = \dim_{\mathbb{K}} M \otimes_R \mathbb{K}.$$

Thus, we have the following computation:

$$\begin{aligned}
\lambda_{p,i}(R/I_\Delta) &= \dim_\Bbbk [\mathrm{Ext}_R^{n-p}(\mathrm{Ext}_R^{n-i}(R/I, \omega_R), \omega_R)]_0 \\
&= \dim_\Bbbk [H^{-p}(\mathbf{D}\ (\mathrm{Ext}_R^{n-i}(R/I, \omega_R)))]_0 \\
&= \dim_\Bbbk [H_{i-p}(\mathbb{L}_\bullet^{<n-i>}(I_{\Delta^\vee}))]_{\mathbf{1}} \\
&= \dim_{\mathbb{K}} H_{i-p}(\mathbb{L}_\bullet^{<n-i>}(I_{\Delta^\vee})) \otimes_R \mathbb{K} \\
&= \nu_{i-p,n-p}(I_{\Delta^\vee}).
\end{aligned}$$

∎

As mentioned in the introduction, for a local ring A containing a field, we have

$$\sum_{0 \leqslant p,i \leqslant n} (-1)^{p-i} \lambda_{p,i}(A) = 1.$$

In the monomial ideal case, this equation is an immediate consequence of Lemma 3.2 and Theorem 4.3.

As a special case of Theorem 2.1, the Lyubeznik tables of monomial ideals in $R = \Bbbk[x_1, \dots, x_n]$ satisfy the consecutiveness property of nontrivial superdiagonals. However, this also follows from the consecutiveness property of nontrivial columns of the ν-tables (Theorem 3.7) via Theorem 4.3. In this sense, both “consecutiveness theorems” are related.

4.1 Sequentially Cohen–Macaulay rings

Let M be a finitely generated graded module over the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$. We say that M is *sequentially Cohen–Macaulay* if $\mathrm{Ext}_R^{n-i}(M, R)$ is either a Cohen–Macaulay module of dimension i or the 0 module for all i. The original definition is given by the existence of a certain filtration (see [22, III, Definition 2.9]). However, it is equivalent to the above one by [22, III, Theorem 2.11]. The sequentially Cohen–Macaulay property of a finitely generated module over a regular local ring is defined/characterized in the same way.

In [2], the first author showed that the sequentially Cohen–Macaulay property implies the triviality of Lyubeznik tables in positive characteristic as well as in the case of squarefree monomial ideals. Using Proposition 3.6, we can give a new proof/interpretation of this result for the case of monomial ideals.

PROPOSITION 4.4. (Cf. [2, Theorem 3.2]) *Let I be a monomial ideal of the polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ such that R/I is sequentially Cohen–Macaulay. Then, the Lyubeznik table of R/I is trivial.*

Proof. By [9, Theorem 2.6], $R/\sqrt{I}$ is sequentially Cohen–Macaulay again. Hence, we may assume that I is the Stanley–Reisner ideal I_Δ of a simplicial complex Δ. Herzog and Hibi [8] showed that R/I_Δ is sequentially Cohen–Macaulay if and only if $I_{\Delta^\vee}$ is componentwise linear. Now, the assertion immediately follows from Proposition 3.6 and Theorem 4.3. □

The converse of Proposition 4.4 is not true. That is, even if R/I has a trivial Lyubeznik table, it need not be sequentially Cohen–Macaulay. For example, if I is the monomial ideal

$$(x_1, x_2) \cap (x_3, x_4) \cap (x_1, x_5) \cap (x_2, x_5) \cap (x_3, x_5) \cap (x_4, x_5)$$

in $R = \Bbbk[x_1, \ldots, x_5]$, then R/I has a trivial Lyubeznik table, but this ring is *not* sequentially Cohen–Macaulay. Since all associated primes of I have the same height, it is the same to say that R/I is not Cohen–Macaulay. However, R/I does not even satisfy Serre's condition (S_2).

In Proposition 4.5 below, we see that if a monomial ideal I has height 1 (i.e., admits a height one associated prime), then the Lyubeznik table of R/I is trivial. Of course, R/I need not be sequentially Cohen–Macaulay in this situation.

4.2 Thom–Sebastiani type formulas

Let $I \subseteq R = \Bbbk[x_1, \ldots, x_m]$ and $J \subseteq S = \Bbbk[y_1, \ldots, y_n]$ be squarefree monomial ideals in two disjoint sets of variables. Let Δ_1 and Δ_2 be the simplicial complexes associated to I and J by the Stanley–Reisner correspondence; that is, $I = I_{\Delta_1}$ and $J = I_{\Delta_2}$. Then, the sum $IT + JT = I_{\Delta_1 * \Delta_2}$ corresponds to the simplicial *join* of the two complexes. Let $\Delta_1^\vee$ (resp. $\Delta_2^\vee$) be the Alexander dual of Δ_1 (resp. Δ_2) as a simplicial complex on $\{1, 2, \ldots, m\}$ (resp. $\{1, 2, \ldots, n\}$). Set $I^\vee := I_{\Delta_1^\vee} \subseteq R$ and $J^\vee := I_{\Delta_2^\vee} \subseteq S$. Then, it is easy to see that

$$\mathbf{A}(T/IT) \cong I^\vee T, \qquad \mathbf{A}(T/JT) \cong J^\vee T \qquad \text{and}$$
$$\mathbf{A}(T/IT \cap JT) \cong I^\vee T + J^\vee T,$$

where $\mathbf{A}$ denotes the Alexander duality functor of $\operatorname{Sq} T$.

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

PROPOSITION 4.5. *The Lyubeznik numbers of $T/IT \cap JT$ have the following form.*

(i) *If either the height of I or the height of J is 1, then $T/IT \cap JT$ has a trivial Lyubeznik table.*
(ii) *If both the height of I and the height of J are $\geqslant 2$, then we have*

$$\begin{aligned}
\lambda_{p,i}(T/IT \cap JT) &= \lambda_{p,i}(T/IT) + \lambda_{p,i}(T/JT) \\
&\quad + \sum_{\substack{q+r=p+\dim T \\ j+k=i+\dim T-1}} \lambda_{q,j}(T/IT)\lambda_{r,k}(T/JT) \\
&= \lambda_{p-n,i-n}(R/I) + \lambda_{p-m,i-m}(S/J) \\
&\quad + \sum_{\substack{q+r=p \\ j+k=i-1}} \lambda_{q,j}(R/I)\lambda_{r,k}(S/J).
\end{aligned}$$

Proof. The assertion easily follows from Proposition 3.12 and Theorem 4.3, but for completeness we will make a few remarks.

(i) Recall that, for a simplicial complex Δ, the height of I_Δ is 1 if and only if $[I_{\Delta^\vee}]_1 \neq 0$.

(ii) The last equality follows from the fact that

$$\lambda_{p,i}(T/IT) = \lambda_{p-n,i-n}(R/I) \qquad \text{and} \qquad \lambda_{p,i}(T/JT) = \lambda_{p-m,i-m}(S/J),$$

which can be seen from Theorem 4.3 and the construction of linear strands. ∎

EXAMPLE 4.6. It is well known that local cohomology modules as well as free resolutions depend on the characteristic of the base field, so Lyubeznik numbers depend on the characteristic as well. The most recurrent example is the Stanley–Reisner ideal associated to a minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$; that is, the ideal in $R = \Bbbk[x_1, \ldots, x_6]$:

$$\begin{aligned}
I = \big(&x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, \\
&x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6\big).
\end{aligned}$$

Its Lyubeznik table has been computed in [1, Example 4.8]. Namely, in characteristic zero and two respectively, we have

$$\Lambda_{\mathbb{Q}}(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix}, \qquad \Lambda_{\mathbb{Z}/2\mathbb{Z}}(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}.$$

One can slightly modify this example and use Proposition 4.5 to obtain some interesting behavior of Lyubeznik numbers.

• The ideal $J = I \cap (x_7)$ in $R = \Bbbk[x_1, \dots, x_7]$ has a trivial Lyubeznik table in any characteristic, so we obtain an example where the local cohomology modules depend on the characteristic but Lyubeznik numbers do not.

• The ideal $J = I \cap (x_7, x_8) \cap (x_9, x_{10})$ in $R = \Bbbk[x_1, \dots, x_{10}]$ satisfies

$$1 = \lambda_{6,7}^{\mathbb{Q}}(R/J) \neq \lambda_{6,7}^{\mathbb{Z}/2\mathbb{Z}}(R/J) = 2,$$

and both Lyubeznik numbers are different from zero.

§5. Lyubeznik table is a topological invariant

While the other sections treat the case where R is a regular local ring or a polynomial ring, in this section we work in a slightly different situation. Here, the ring R means a normal semigroup ring. When R is simplicial and Gorenstein, the second author proved in [27] that the local cohomology modules $H_I^r(R)$ have finite Bass numbers for radical monomial ideals $I \subset R$. In fact, without these conditions, Bass numbers are out of control and can be infinite (see [7] for details).

Before going to the main result of this section (Theorem 5.3), we introduce the setup that we work with. For more details, we refer to [27].

Let $C \subset \mathbb{Z}^n$ be an affine semigroup (i.e., C is a finitely generated additive submonoid of $\mathbb{Z}^n$), and let $R := \Bbbk[\mathbf{x^c} \mid \mathbf{c} \in C] \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the semigroup ring of C over $\Bbbk$. Here, $\mathbf{x^c}$ denotes the monomial $\prod_{i=1}^n x_i^{c_i}$ for $\mathbf{c} = (c_1, \dots, c_n) \in C$. Regarding C as a subset of $\mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$, let $P := \mathbb{R}_{\geqslant 0} C \subset \mathbb{R}^n$ be the polyhedral cone spanned by C. We always assume that $\mathbb{Z}C = \mathbb{Z}^n$, $\mathbb{Z}^n \cap P = C$ and $C \cap (-C) = \{0\}$. Thus, R is a normal Cohen–Macaulay integral domain of dimension n with the graded maximal ideal $\mathfrak{m} := (\mathbf{x^c} \mid 0 \neq \mathbf{c} \in C)$. We say that R is *simplicial* if the cone P is spanned by n vectors in $\mathbb{R}^n$. The polynomial ring $\Bbbk[x_1, \dots, x_n]$ is a typical example of a simplicial semigroup ring $\Bbbk[C]$ for $C = \mathbb{N}^n$. Clearly, $R = \bigoplus_{\mathbf{c} \in C} \Bbbk\, \mathbf{x^c}$ is a $\mathbb{Z}^n$-graded ring. We say that a $\mathbb{Z}^n$-graded ideal of R is a *monomial ideal*, and we denote by $^*\mathrm{mod}\, R$ the category of finitely generated $\mathbb{Z}^n$-graded R-modules and degree preserving R-homomorphisms.

Let L be the set of nonempty faces of the polyhedral cone P. Note that $\{0\}$ and P itself belong to L. Regarding L as a partially ordered set by inclusion, R is simplicial if and only if L is isomorphic to the power set $2^{\{1,\dots,n\}}$. For $F \in L$, $\mathfrak{p}_F := (\mathbf{x^c} \mid \mathbf{c} \in C \setminus F)$ is a prime ideal of R. Conversely, any monomial prime ideal is of the form $\mathfrak{p}_F$ for some $F \in L$. Note that $R/\mathfrak{p}_F \cong \Bbbk[\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C \cap F]$ for $F \in L$. For a point $\mathbf{c} \in C$, we always have a unique face $F \in L$ whose relative interior contains $\mathbf{c}$. Here, we denote $s(\mathbf{c}) = F$.

The following is a generalization of the notion of squarefree modules (see Definition 4.1) to this setting.

Definition 5.1. [27] We say that a module $M \in {}^*\mathrm{mod}\, R$ is *squarefree* if it is C-graded (i.e., $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin C$), and the multiplication map $M_{\mathbf{a}} \ni y \longmapsto \mathbf{x}^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in C$ with $s(\mathbf{a}+\mathbf{b}) = s(\mathbf{a})$.

For a monomial ideal I, R/I is a squarefree R-module if and only if I is a radical ideal (i.e., $\sqrt{I} = I$). We say that $\Delta \subseteq L$ is an *order ideal* if $\Delta \ni F \supset F' \in L$ implies $F' \in \Delta$. If Δ is an order ideal, then $I_\Delta := (\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C,\ s(\mathbf{c}) \notin \Delta) \subseteq R$ is a radical monomial ideal. Conversely, any radical monomial ideal is of the form I_Δ for some Δ. Clearly,

$$[R/I_\Delta]_{\mathbf{c}} \cong \begin{cases} \Bbbk, & \text{if } \mathbf{c} \in C \text{ and } s(\mathbf{c}) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

If R is simplicial, an order ideal Δ is essentially a simplicial complex on the vertices $1, 2, \ldots, n$. If R is the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, then R/I_Δ is nothing but the Stanley–Reisner ring of the simplicial complex Δ.

For each $F \in L$, take some $\mathbf{c}(F) \in C \cap \text{rel-int}(F)$ (i.e., $s(\mathbf{c}(F)) = F$). For a squarefree R-module M and $F, G \in L$ with $G \supset F$, [27, Theorem 3.3] gives a $\Bbbk$-linear map

$$\varphi^M_{G,F} : M_{\mathbf{c}(F)} \to M_{\mathbf{c}(G)}.$$

These maps satisfy $\varphi^M_{F,F} = \mathrm{Id}$ and $\varphi^M_{H,G} \circ \varphi^M_{G,F} = \varphi^M_{H,F}$ for all $H \supset G \supset F$. We have $M_{\mathbf{c}} \cong M_{\mathbf{c}'}$ for $\mathbf{c}, \mathbf{c}' \in C$ with $s(\mathbf{c}) = s(\mathbf{c}')$. Under these isomorphisms, the maps $\varphi^M_{G,F}$ do not depend on the particular choice of $\mathbf{c}(F)$.

Let $\mathrm{Sq}\, R$ be the full subcategory of ${}^*\mathrm{mod}\, R$ consisting of squarefree modules. As shown in [27], $\mathrm{Sq}\, R$ is an abelian category with enough injectives. For an indecomposable squarefree module M, it is injective in $\mathrm{Sq}\, R$ if and only if $M \cong R/\mathfrak{p}_F$ for some $F \in L$.

Let ω_R be the $\mathbb{Z}^n$-graded canonical module of R. It is well known that ω_R is isomorphic to the radical monomial ideal $(\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C, s(\mathbf{c}) = P)$. As shown in [27, Proposition 3.7], we have $\mathrm{Ext}^i_R(M, \omega_R) \in \mathrm{Sq}\, R$ for $M \in \mathrm{Sq}\, R$.

5.1 Lyubeznik numbers

Let $R = \Bbbk[C]$ be a normal simplicial semigroup ring that is Gorenstein, and let I be a monomial ideal of R. As in the polynomial ring case, we set the Lyubeznik numbers as

$$\lambda_{p,i}(R/I) := \mu^p(\mathfrak{m}, H_I^{n-i}(R)).$$

Work of the second author in [27] states that this set of invariants are well defined in this framework. Namely, Theorem 4.2 holds verbatim in this situation.

THEOREM 5.2. [27, Corollary 5.12] *Let $R = \Bbbk[C]$ be a normal simplicial semigroup ring that is Gorenstein, and let I_Δ be a radical monomial ideal. Then, we have*

$$\lambda_{p,i}(R/I_\Delta) = \dim_\Bbbk [\mathrm{Ext}_R^{n-p}(\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R), \omega_R)]_0 < \infty.$$

Notice that in this setting we have that whenever we have a multigraded isomorphism $\Bbbk[C]/I_\Delta \cong \Bbbk[C']/I_{\Delta'}$ between quotients of Gorenstein normal simplicial semigroup rings by radical monomial ideals, then the corresponding Lyubeznik numbers coincide. This multigraded framework slightly differs from the original situation for regular local rings stated in [13]. However, as stated in [27, Remark 5.14], if $\Delta \cong \Delta'$ as simplicial complexes, then R/I_Δ and $R'/I_{\Delta'}$ have the same Lyubeznik numbers. In this sense, to study the Lyubeznik numbers of a quotient R/I_Δ of a Gorenstein normal simplicial semigroup ring R by a radical monomial ideal I_Δ, we may assume that R is a polynomial ring and R/I_Δ is a Stanley–Reisner ring. In Theorem 5.3, we prove a stronger result.

It is also worth pointing out that several features of Lyubeznik numbers are still true in this setting. In what follows, we assume that I is a monomial ideal of R.

(1) As in the polynomial ring case, we have the Euler characteristic equation

$$\sum_{0 \leqslant p,i \leqslant d} (-1)^{p-i} \lambda_{p,i}(R/I) = 1.$$

Moreover, the statements corresponding to Theorem 2.1 (the consecutiveness of nontrivial lines) still hold. In fact, we may assume that I is a radical ideal, and hence $I = I_\Delta$ for some simplicial complex Δ, and then reduce to the case when R is a polynomial ring as in [27, Remark 5.14 (b)].

If we assume that $I = \sqrt{I}$, Proposition 4.4 also holds in the present situation. However, we cannot drop this assumption, since we have no idea whether the condition of being sequentially Cohen–Macaulay is preserved after taking radicals. What is known is that if R/I is Cohen–Macaulay, then so is $R/\sqrt{I}$ (see [30, Theorem 6.1]). Hence, if R/I is Cohen–Macaulay, then the Lyubeznik table of R/I is trivial.

(2) For a radical monomial ideal I_Δ with $\dim R/I_\Delta = d$, the highest Lyubeznik number

$$\lambda_{d,d}(R/I_\Delta) = \dim_{\Bbbk}[\mathrm{Ext}_R^{n-d}(\mathrm{Ext}_R^{n-d}(R/I_\Delta, \omega_R), \omega_R)]_0$$

has a simple topological (or combinatorial) meaning. In fact, to study this number, we may assume that R is a polynomial ring, and we can use a combinatorial description of

$$\mathrm{Ext}_R^{n-d}(\mathrm{Ext}_R^{n-d}(R/I_\Delta, \omega_R), \omega_R)$$

given in [22, p. 96]. Roughly speaking, $\lambda_{d,d}(R/I_\Delta)$ is the number of "connected in codimension-one components" of $|\Delta|$. (This result holds in a much wider context; see [31].) In particular, if R/I_Δ satisfies Serre's condition (S_2), then $\lambda_{d,d}(R/I_\Delta) = 1$, while the converse is not true.

5.2 Lyubeznik table is a topological invariant

Recall that if $R = \Bbbk[C]$ is simplicial, then an order ideal Δ of L is essentially a simplicial complex, and hence it has the geometric realization $|\Delta|$. It is natural to ask how Lyubeznik numbers of R/I_Δ depend on $|\Delta|$. The next theorem shows that Lyubeznik numbers are not only algebraic invariants but also topological invariants.

THEOREM 5.3. *Let $R = \Bbbk[C]$ be a simplicial normal semigroup ring that is Gorenstein, and let $I_\Delta \subset R$ be a radical monomial ideal. Then, $\lambda_{p,i}(R/I_\Delta)$ depends only on the homeomorphism class of $|\Delta|$ and* $\mathrm{char}(\Bbbk)$.

Bearing in mind Theorem 5.2, it suffices to show that

$$\dim_{\Bbbk}[\mathrm{Ext}_R^{n-p}(\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R), \omega_R)]_0$$

depends only on the topology of $|\Delta|$ and $\mathrm{char}(\Bbbk)$. For this statement, the assumption that R is simplicial and Gorenstein is irrelevant. (If R is not simplicial, then Δ is essentially a CW complex.) In [19, Theorem 2.10], Okazaki and the second author showed that the invariant that is (essentially) equal to

$$\mathrm{depth}_R(\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R)) = \min\{j \mid \mathrm{Ext}_R^{n-j}(\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R), \omega_R) \neq 0\}$$

depends only on $|\Delta|$ and $\mathrm{char}(\Bbbk)$ for each i. Our proof here uses similar arguments to the aforementioned result. To do so, we have to recall some previous work of the second author in [28].

Recall that $P = \mathbb{R}_{\geqslant 0}C$ is a polyhedral cone associated with the semigroup ring $R = \Bbbk[C]$. We have a hyperplane $H \subset \mathbb{R}^n$ such that $B := H \cap P$ is an $(n-1)$-polytope (an $(n-1)$-simplex, if R is simplicial). For $F \in L$, set $|F|$ to be the relative interior of the face $F \cap H$ of B. We can regard an order ideal $\Delta \subseteq L$ as a CW complex (a simplicial complex, if R is simplicial) whose geometric realization is $|\Delta| := \bigcup_{F \in \Delta} |F| \subseteq B$.

For $F \in L$,

$$U_F := \bigcup_{F' \in L,\, F' \supset F} |F'|$$

is an open set of B. Note that $\{U_F \mid \{0\} \neq F \in L\}$ is an open covering of B. In [28], from $M \in \mathrm{Sq}\, R$, we constructed a sheaf M^+ on B. More precisely, the assignment

$$\Gamma(U_F, M^+) = M_{\mathbf{c}\,(F)}$$

for each $F \neq \{0\}$ and the map

$$\varphi^M_{F,G} : \Gamma(U_G, M^+) = M_{\mathbf{c}\,(G)} \longrightarrow M_{\mathbf{c}\,(F)} = \Gamma(U_F, M^+)$$

for $F, G \neq \{0\}$ with $F \supset G$ (equivalently, $U_G \supset U_F$) define a sheaf. Note that M_0 is irrelevant to M^+.

For example, $(R/I_\Delta)^+ \cong j_* \underline{\Bbbk}_{|\Delta|}$, where $\underline{\Bbbk}_{|\Delta|}$ is the constant sheaf on $|\Delta|$ with coefficients in $\Bbbk$, and j is the embedding map $|\Delta| \hookrightarrow B$. Similarly, we have that $(\omega_R)^+ \cong h_! \underline{\Bbbk}_{B^\circ}$, where $\underline{\Bbbk}_{B^\circ}$ is the constant sheaf on the relative interior B° of B, and h is the embedding map $B^\circ \hookrightarrow B$. Note that $(\omega_R)^+$ is the orientation sheaf of B with coefficients in $\Bbbk$.

Let $\Delta \subseteq L$ be an order ideal, and set $X := |\Delta| \subseteq B$. For $M \in \mathrm{Sq}\, R$, M is an R/I_Δ-module (i.e., $\mathrm{ann}(M) \supset I_\Delta$) if and only if $\mathrm{Supp}(M^+) := \{x \in B \mid (M^+)_x \neq 0\} \subseteq X$. In this case, we have

$$H^i(B; M^+) \cong H^i(X; M^+|_X)$$

for all i. Here, $M^+|_X$ is the restriction of the sheaf M^+ to the closed set $X \subseteq B$. Combining this fact with [28, Theorem 3.3], we have the following.

THEOREM 5.4. (Cf. [28, Theorem 3.3]) *With the above situation, we have*

$$H^i(X; M^+|_X) \cong [H^{i+1}_{\mathfrak{m}}(M)]_0 \quad \textit{for all } i \geqslant 1,$$

and an exact sequence

$$0 \longrightarrow [H^0_{\mathfrak{m}}(M)]_0 \longrightarrow M_0 \longrightarrow H^0(X; M^+|_X) \longrightarrow [H^1_{\mathfrak{m}}(M)]_0 \longrightarrow 0. \tag{5.1}$$

In particular, $[H^{i+1}_{\mathfrak{m}}(R/I_\Delta)]_0 \cong \widetilde{H}^i(X; \Bbbk)$ *for all* $i \geqslant 0$*, where* $\widetilde{H}^i(X; \Bbbk)$ *denotes the* i*th reduced cohomology of* X *with coefficients in* $\Bbbk$*.*

Recall that X admits Verdier's dualizing complex $\mathcal{D}^\bullet_X$ with coefficients in $\Bbbk$. For example, $\mathcal{D}^\bullet_B$ is quasi-isomorphic to $(\omega_R)^+[n-1]$. The first half of (1) of the next theorem is a restatement of [28, Theorem 4.2], and the rest is that of [30, Lemma 5.11].

Theorem 5.5. [28, Theorem 4.2] and [30, Lemma 5.11] *With the above notation, we have the following.*

(1) $\operatorname{Supp}(\operatorname{Ext}^{n-i}_R(M, \omega_R)^+) \subseteq X$ *and*

$$\operatorname{Ext}^{n-i}_R(M, \omega_R)^+|_X \cong \mathcal{E}xt^{1-i}(M^+|_X, \mathcal{D}^\bullet_X).$$

Moreover, for $i \geqslant 2$*, we have*

$$[\operatorname{Ext}^{n-i}_R(M, \omega_R)^+]_0 \cong \operatorname{Ext}^{1-i}(M^+|_X, \mathcal{D}^\bullet_X).$$

(2) *Via the isomorphisms in (1), for* $i \geqslant 2$*, the natural map*

$$\operatorname{Ext}^{1-i}(M^+|_X, \mathcal{D}^\bullet_X) \longrightarrow \Gamma(X; \mathcal{E}xt^{1-i}(M^+|_X, \mathcal{D}^\bullet_X))$$

coincides with the middle map

$$[\operatorname{Ext}^{n-i}_R(M, \omega_R)]_0 \longrightarrow \Gamma(X; \operatorname{Ext}^{n-i}_R(M, \omega_R)^+|_X)$$

of the sequence (5.1) for $\operatorname{Ext}^{n-i}_R(M, \omega_R) \in \operatorname{Sq} R$*.*

The Proof of Theorem 5.3. We show that the dimension of

$$[\operatorname{Ext}^{n-p}_R(\operatorname{Ext}^{n-i}_R(R/I_\Delta, \omega_R), \omega_R)]_0 \ (\cong [H^p_{\mathfrak{m}}(\operatorname{Ext}^{n-i}_R(R/I_\Delta, \omega_R))^*]_0)$$

depends only on X and $\operatorname{char}(\Bbbk)$. If $p \geqslant 2$, then we have

$$[H^p_{\mathfrak{m}}(\operatorname{Ext}^{n-i}_R(R/I_\Delta, \omega_R))]_0 \cong H^{p-1}(X; \mathcal{E}xt^{1-i}(\underline{\Bbbk}_X, \mathcal{D}^\bullet_X))$$

by Theorems 5.4 and 5.5 (1). The right-hand side of the equation clearly depends only on X and $\operatorname{char}(\Bbbk)$ for each p, i. Next, we consider the case $p = 0, 1$. By Theorem 5.4, $H^0_{\mathfrak{m}}(\operatorname{Ext}^{n-i}_R(R/I_\Delta, \omega_R))$ and $H^1_{\mathfrak{m}}(\operatorname{Ext}^{n-i}_R(R/I_\Delta, \omega_R))$

are the kernel and the cokernel of the map

$$[\mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R)]_0 \longrightarrow \Gamma(X; \mathrm{Ext}_R^{n-i}(R/I_\Delta, \omega_R)^+|_X),$$

respectively. If $i \geqslant 2$, the above map is equivalent to the natural map

$$\mathrm{Ext}^{1-i}(\underline{\Bbbk}_X, \mathcal{D}_X^\bullet) \longrightarrow \Gamma(X; \mathcal{E}xt^{1-i}(\underline{\Bbbk}_X, \mathcal{D}_X^\bullet))$$

by Theorem 5.5 (2), and the dimensions of its kernel and cokernel are invariants of X.

It remains to show the case ($p = 0, 1$ and) $i = 0, 1$. Clearly, $\mathrm{Ext}_R^n(R/I_\Delta, \omega_R) \neq 0$ if and only if $\mathrm{Ext}_R^n(R/I_\Delta, \omega_R) = \Bbbk$, if and only if $I_\Delta = \mathfrak{m}$, if and only if $X = \emptyset$. Hence, $\lambda_{0,0}(R/I_\Delta) \neq 0$ if and only if $\lambda_{0,0}(R/I_\Delta) = 1$, if and only if $X = \emptyset$. On the other hand, it is easy to check that $\lambda_{1,1}(R/I_\Delta)$ is always trivial; that is,

$$\lambda_{1,1}(R/I_\Delta) = \begin{cases} 1, & \text{if } \dim(R/I_\Delta) = 1 (\text{i.e., } \dim |\Delta| = 0), \\ 0, & \text{otherwise.} \end{cases}$$

(The same is true for the local ring case using the spectral sequence argument as in the proof of Theorem 2.1 or adapting the techniques used in [24].) Hence, the remaining case is only $\lambda_{0,1}(R/I_\Delta)$, but the following fact holds.

Claim. If $R = \Bbbk[C]$ is a simplicial normal semigroup ring that is Gorenstein, then we have

$$\lambda_{0,1}(R/I_\Delta) = \begin{cases} c - 1, & \text{if } \dim(R/I_\Delta) \geqslant 2 \ (\text{i.e.}, \dim |\Delta| \geqslant 1), \\ 0, & \text{otherwise,} \end{cases}$$

where c is the number of connected components of $|\Delta'| := |\Delta| \setminus \{\text{isolated points}\}$.

Let us prove the claim. We may assume that $\dim(R/I_\Delta) > 0$. If $\dim(R/I_\Delta) = 1$, then R/I_Δ is Cohen–Macaulay, and the assertion is clear. Therefore, we may assume that $\dim(R/I_\Delta) \geqslant 2$. First, we consider the case when I_Δ does not have one-dimensional associated primes; equivalently, $|\Delta|$ does not admit isolated points (i.e., $|\Delta| = |\Delta'|$). Then, we have

$$\dim_R(\mathrm{Ext}_R^{n-1}(R/I_\Delta, \omega_R)) < 1.$$

Since $\mathrm{Ext}_R^{n-1}(R/I_\Delta, \omega_R)$ is a squarefree module, we have

$$\mathrm{Ext}_R^{n-1}(R/I_\Delta, \omega_R) = [\mathrm{Ext}_R^{n-1}(R/I_\Delta, \omega_R)]_0.$$

https://doi.org/10.1017/nmj.2017.10 Published online by Cambridge University Press

We also have

$$[\mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R)]_0 \cong [H^1_{\mathfrak{m}}(R/I_\Delta)]_0 \cong \widetilde{H}^0(X;\Bbbk) \cong \Bbbk^{c-1},$$

where the second isomorphism follows from the last statement of Theorem 5.4. Hence,

$$\begin{aligned}\lambda_{0,1}(R/I_\Delta) &= \dim_\Bbbk[\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R),\omega_R)]_0\\ &= \dim_\Bbbk[\mathrm{Ext}_R^n(\Bbbk^{c-1},\omega_R)]_0 = c-1,\end{aligned}$$

and we are done.

Therefore, we now consider the case where I_Δ admits one-dimensional associated primes. Set $I := I_{\Delta'}$. Then, there is a monomial ideal J of R with $I_\Delta = I \cap J$ and $\dim R/J = 1$. Note that $I + J = \mathfrak{m}$. The short exact sequence $0 \to R/I_\Delta \to R/I \oplus R/J \to R/\mathfrak{m}\ (\cong \Bbbk) \to 0$ yields the exact sequence

$$\begin{aligned}&0 \longrightarrow \mathrm{Ext}_R^{n-1}(R/I,\omega_R) \oplus \mathrm{Ext}_R^{n-1}(R/J,\omega_R) \longrightarrow \mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R)\\ &\longrightarrow \Bbbk \longrightarrow 0.\end{aligned} \tag{5.2}$$

Since Lyubeznik numbers of type $\lambda_{1,1}(-)$ are always trivial, we have

$$[\mathrm{Ext}_R^{n-1}(\mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R),\omega_R)]_0 = [\mathrm{Ext}_R^{n-1}(\mathrm{Ext}_R^{n-1}(R/I,\omega_R),\omega_R)]_0 = 0$$

and $[\mathrm{Ext}_R^{n-1}(\mathrm{Ext}_R^{n-1}(R/J,\omega_R),\omega_R)]_0 = \Bbbk$. It is also clear that

$$\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/J,\omega_R),\omega_R) = 0.$$

Thus, applying $\mathrm{Ext}_R^\bullet(-,\omega_R)$ to (5.2), we obtain

$$\begin{aligned}0 &\longrightarrow [\mathrm{Ext}_R^{n-1}(\mathrm{Ext}_R^{n-1}(R/J,\omega_R),\omega_R)]_0\ (\cong \Bbbk) \longrightarrow [\mathrm{Ext}_R^n(\Bbbk,\omega_R)]_0\ (\cong \Bbbk)\\ &\longrightarrow [\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R),\omega_R)]_0\\ &\longrightarrow [\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/I,\omega_R),\omega_R)]_0 \longrightarrow 0.\end{aligned}$$

Since $[\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/I,\omega_R),\omega_R)]_0 \cong \Bbbk^{c-1}$, as we have shown above, it follows that

$$[\mathrm{Ext}_R^n(\mathrm{Ext}_R^{n-1}(R/I_\Delta,\omega_R),\omega_R)]_0 \cong \Bbbk^{c-1},$$

and we are done. ∎

EXAMPLE 5.6. This example concerns the final step of the proof of Theorem 5.3. Let $R = \Bbbk[x_1, \ldots, x_7]$ be a polynomial ring, and consider the monomial ideal

$$I_\Delta = (x_2, x_3, x_4, x_5, x_6, x_7) \cap (x_1, x_4, x_5, x_6, x_7) \\ \cap (x_1, x_2, x_3, x_6, x_7) \cap (x_1, x_2, x_3, x_4, x_5).$$

Then, $|\Delta|$ consists of one isolated point and three segments; see Figure 1. Therefore, $|\Delta'|$, which is $|\Delta| \setminus \{v_1\}$, consists of three segments. We have $\lambda_{0,1}(R/I_\Delta) = 3 - 1 = 2$.

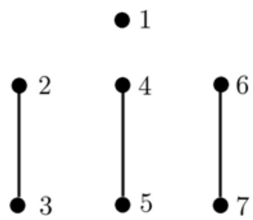

Figure 1.
Δ in Example 5.6.

REFERENCES

[1] J. Àlvarez Montaner and A. Vahidi, *Lyubeznik numbers of monomial ideals*, Trans. Amer. Math. Soc. **366** (2014), 1829–1855.

[2] J. Àlvarez Montaner, *Lyubeznik table of sequentially Cohen–Macaulay rings*, Comm. Algebra **43** (2015), 3695–3704.

[3] J. E. Björk, *Rings of Differential Operators*, North-Holland Mathematics Library, Amsterdam, 1979.

[4] M. Blickle, *Lyubeznik's numbers for cohomologically isolated singularities*, J. Algebra **308** (2007), 118–123.

[5] M. Blickle and R. Bondu, *Local cohomology multiplicities in terms of étale cohomology*, Ann. Inst. Fourier **55** (2005), 2239–2256.

[6] R. García López and C. Sabbah, *Topological computation of local cohomology multiplicities*, Collect. Math. **49** (1998), 317–324.

[7] D. Helm and E. Miller, *Bass numbers of semigroup-graded local cohomology*, Pacific J. Math. **209** (2003), 41–66.

[8] J. Herzog and T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.

[9] J. Herzog, Y. Takayama and N. Terai, *On the radical of a monomial ideal*, Arch. Math. **85** (2005), 397–408.

[10] C. Huneke and R. Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765–779.

[11] S. Jacques and M. Katzman, *The Betti numbers of forests*, preprint, 2005, available at arXiv:0501226.

[12] K. I. Kawasaki, *On the Lyubeznik number of local cohomology modules*, Bull. Nara Univ. Ed. Natur. Sci. **49** (2000), 5–7.

[13] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), 41–55.

[14] G. Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic $p > 0$*, J. Reine Angew. Math. **491** (1997), 65–130.

[15] G. Lyubeznik, *On some local cohomology invariants of local rings*, Math. Z. **254** (2006), 627–640.

[16] W. Mahmood and P. Schenzel, *On invariants and endomorphism rings of certain local cohomology modules*, J. Algebra **372** (2012), 56–67.

[17] E. Miller, *The Alexander duality functors and local duality with monomial support*, J. Algebra **231** (2000), 180–234.

[18] M. Mustaţă, *Local cohomology at monomial ideals*, J. Symbolic Comput. **29** (2000), 709–720.

[19] R. Okazaki and K. Yanagawa, *Linearity defects of face rings*, J. Algebra **314** (2007), 362–382.

[20] T. Römer, *On minimal graded free resolutions*, Ph.D. Thesis, Essen, 2001.

[21] P. Schenzel, *On the structure of the endomorphism ring of a certain local cohomology module*, J. Algebra **344** (2011), 229–245.

[22] R. P. Stanley, *Combinatorics and Commutative Algebra*, 2nd ed., Progress in Mathematics, **41**, Birkhäuser, Boston, MA, 1996.

[23] N. Terai, *Local cohomology modules with respect to monomial ideals*, preprint, 1999.

[24] U. Walther, *On the Lyubeznik numbers of a local ring*, Proc. Amer. Math. Soc. **129** (2001), 1631–1634.

[25] K. Yanagawa, *Alexander duality for Stanley–Reisner rings and squarefree $\mathbb{N}^n$-graded modules*, J. Algebra **225** (2000), 630–645.

[26] K. Yanagawa, *Bass numbers of local cohomology modules with supports in monomial ideals*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 45–60.

[27] K. Yanagawa, *Sheaves on finite posets and modules over normal semigroup rings*, J. Pure Appl. Algebra **161** (2001), 341–366.

[28] K. Yanagawa, *Stanley–Reisner rings, sheaves, and Poincaré–Verdier duality*, Math. Res. Lett. **10** (2003), 635–650.

[29] K. Yanagawa, *Derived category of squarefree modules and local cohomology with monomial ideal support*, J. Math. Soc. Japan **56** (2004), 289–308.

[30] K. Yanagawa, *Notes on C-graded modules over an affine semigroup ring $K[C]$*, Comm. Algebra **36** (2008), 3122–3146.

[31] W. Zhang, *On the highest Lyubeznik number of a local ring*, Compos. Math. **143** (2007), 82–88.

Josep Àlvarez Montaner
Departament de Matemàtiques
Universitat Politècnica de Catalunya
Av. Diagonal 647
Barcelona 08028
Spain
Josep.Alvarez@upc.edu

Kohji Yanagawa
Department of Mathematics
Kansai University
Suita 564-8680
Japan
yanagawa@kansai-u.ac.jp