Dear Editor,

On concavity of the mean function and stochastic ordering
for reflected processes with stationary increments

1. Introduction

In Kella (1992) it was shown that the expected value of a reflected Lévy process (that is, a process having stationary and independent increments which is continuous in probability) having no negative jumps and starting at the origin, as well as the reflection map are concave (non-decreasing) functions. In order to obtain the proof, a necessary and sufficient condition for concavity was established and was applied together with some martingale results. In this extension to that paper we show that in fact, this result holds for more general processes, that is, for processes having stationary (not necessarily independent) increments and for which no assumptions on non-negative jumps are made. One such process is the virtual waiting time in a single-server queue to which the input process is a compound stationary point process. That is, the arrivals occur according to a stationary point process and the service times form a discrete-time stationary process which is independent of the arrival process.

The method of proof is different and simpler than in Kella (1992), and there is no need for background on processes having stationary increments.

For discussion and further references of stochastic monotonicity see (among many others) Stoyan (1983) or Chapter 8 of Ross (1983).

2. Preliminaries and notation

Let $X = \{X(t) \mid t \geqq 0\}$ be a right-continuous stochastic process having the property that $X_s \equiv \{X(s+t) - X(s) \mid t \geqq 0\}$ is distributed like X. This will be our definition of a process having stationary increments. The usual definition is that for every $0 = t_0 < t_1 < \cdots < t_n$, the joint distribution of $\{X(t_i + s) - X(t_{i-1} + s) \mid 1 \leqq i \leqq n\}$ does not depend on s. For right-continuous processes these two definitions coincide. Note that necessarily $\boldsymbol{P}[X(0) = 0] = \boldsymbol{P}[X_s(0) = 0] = \boldsymbol{P}[X(s) - X(s) = 0] = 1$.

For $a < b$, denote by $L(a, b) = -\inf_{a \leqq u \leqq b} X(u)$ and observe that, by right continuity of X, $L(a, b)$ is a random variable (measurable). Also note that $L(a, b)$ is non-decreasing in b and non-increasing in a (for $a < b$). For what follows we assume that for some $\varepsilon > 0$, $\boldsymbol{E}L(0, \varepsilon) < \infty$ and $\boldsymbol{E}\,|X(t)| < \infty$ for every $0 \leqq t \leqq \varepsilon$.

Lemma 2.1. For every $t \geqq 0$, $\boldsymbol{E}L(0, t) < \infty$ (thus X is bounded below on finite intervals), $\boldsymbol{E}\,|X(t)| < \infty$ and $\boldsymbol{E}X(t) = \xi t$, where $\xi = \boldsymbol{E}X(1)$.

Proof. From the stationary increments property it follows that $\boldsymbol{E}|X(t)| \leqq n\boldsymbol{E}|X(t/n)|$ for every positive integer n, which implies that $\boldsymbol{E}|X(t)| < \infty$ for every $t \geqq 0$. It is clear that $\boldsymbol{E}X(t)$ is an additive function which is bounded below (by $-\boldsymbol{E}L(0, \varepsilon)$) on the interval $[0, \varepsilon]$. An additive function which is bounded below on some set of positive Lebesgue measure must be linear (see Theorem 1.1.7 on page 4 of Bingham et al. (1989) or Theorem 8 on p. 17 of Aczél and Dhombres (1989)). Hence, $\boldsymbol{E}X(t) = \xi t$ for every $t \geqq 0$. Finally if we note that for every $s, t \geqq 0$, $L(s, t+s) - X(s)$ has the same distribution as $L(t)$ (so that $\boldsymbol{E}L(s, t+s) = \boldsymbol{E}L(t) + \xi s$), it follows that for every positive integer n,

$$\boldsymbol{E}L(0, t) \leqq \sum_{k=1}^{n} \boldsymbol{E}L\left(\frac{k-1}{n}t, \frac{k}{n}t\right) = n\boldsymbol{E}L(0, t/n) + \frac{n-1}{2}\xi t, \tag{2.1}$$

which implies that $\boldsymbol{E}L(0, t) < \infty$ for every $t \geqq 0$.

In the light of Lemma 2.1 we denote $W(t) = X(t) + L(0, t)$, $w(t) = \boldsymbol{E}W(t)$, and $l(t) = \boldsymbol{E}L(0, t)$ and observe that W is a well-defined (finite-valued) process and that w and l are finite-valued functions such that $w(t) = \xi t + l(t)$.

3. The main result

We now state and prove the main result.

Theorem 3.1. W is stochastically increasing, and w and l are concave (necessarily non-decreasing) functions.

Proof. The main idea is to derive Equation (3.1) below, by making use of a method which was employed in the derivation of Equation (43), p. 33 in Freedman (1971). That is, to compare $W(t+s)$ and $W_s(t)$. There the goal was to show that the processes $Y = M - B$ and $|B|$ have the same Markov transition kernel (hence the same distribution), where B is a standard Brownian motion and M is its running maximum. Here, the processes involved and the focus are, of course, different.

Let $L_s(a, b) = -\inf_{a \leqq u \leqq b} X_s(u)$, where we recall that $X_s(t) = X(t+s) - X(s)$, and let $W_s(t) = X_s(t) + L_s(0, t)$. From the stationary increments property we have that the processes W_s, X_s, and L_s have the same distribution as W, X, and L, respectively. Since $L_s(0, t) = L(s, s+t) + X(s)$ we have that $W_s(t) = X(s+t) + L(s, s+t)$. Using the notation $a \vee b = \max(a, b)$ and observing that $L(0, s+t) = L(0, s) \vee L(s, s+t)$ we can write $W(s+t) = X(s+t) + L(0, s) \vee L(s, s+t)$. Therefore it follows that

$$W(s+t) - W_s(t) = [L(0, s) - L(s, s+t)]^+, \tag{3.1}$$

where $a^+ = a \vee 0$.

Equation (3.1) immediately implies that $W(s+t) \geqq W_s(t)$ and since $W_s(t)$ and $W(t)$ are identically distributed, stochastic monotonicity follows. To obtain concavity of w, first observe that the right side of (3.1) is non-increasing in t and therefore, taking expected values (recalling that $L(0, t) < \infty$ for every $t \geqq 0$) gives that $w(s+t) - w(t)$ is a non-increasing function of t. By itself, this property does not necessarily imply con-

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cavity, as there are additive functions which are not continuous (hence not concave) for which this property is clearly satisfied. However, as $w(\cdot)$ is non-negative, concavity does in fact follow (for instance, combine Theorem C on p. 215 and Exercise N on p. 224 of Roberts and Varberg (1973)). The concavity of l follows from $w(t) = \xi t + l(t)$.

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Yours sincerely,

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