ON TWO-SIDED ARTINIAN QUOTIENT RINGS

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Djabali [1] has proved that, if R is a right and left noetherian ring with an identity and if the proper prime ideals of R are maximal, then R has a right and left artinian two-sided quotient ring. Robson [5, Theorem 2.11] and Small [6, Theorem 2.13] have proved independently that, if R is a commutative noetherian ring, then R has an artinian quotient ring if and only if the prime ideals of R that belong to the zero ideal are all minimal. We shall generalise these results by proving the

THEOREM. *Let R be a right and left noetherian ring with a regular element. Then R has a right and left artinian two-sided quotient ring if and only if each prime ideal of* R *consisting of zero-divisors is minimal.*

1. Small's Theorem. Let S be a non-empty subset of a ring R. Then the *right annihilator* of S in R, denoted by $r(S)$, is $\{r \in R : Sr = 0\}$. The *left annihilator*, $l(S)$, is $\{r \in R : rS = 0\}$. If $c \in R$ we shall denote the right and left annihilators of $\{c\}$ by $r(c)$ and $l(c)$, respectively.

An element c of R is *right regular* if $r(c) = 0$ and is *left regular* if $l(c) = 0$. If $c \in R$ and $l(c) = r(c) = 0$, then c is called *regular*. For any ideal I of R we set

$$\mathscr{C}'(I) = \{r \in R : [r+I] \text{ is a right regular element of } R/I\}$$

and

$$'\mathscr{C}(I) = \{r \in R : [r+I] \text{ is a left regular element of } R/I\}.$$

In addition, we set $\mathscr{C}(I) = \mathscr{C}'(I) \cap {}'\mathscr{C}(I)$. In this notation $\mathscr{C}(0)$ denotes the set of regular elements of R. If $r \in R$ and $r \notin \mathscr{C}(0)$, then r is called a *zero-divisor* of R.

From now on " ring " will mean " ring with a regular element ".

A ring Q with an identity is said to be the *right quotient ring* of a ring R if

(i) $R \subseteq Q$,

(ii) every regular element of R has an inverse in Q, and

(iii) every element of Q has the form rc^{-1} with $r \in R$ and $c \in \mathscr{C}(0)$.

There is an analogous definition of the left quotient ring of R. If a ring R has a right quotient ring Q_1 and a left quotient ring Q_2, then $Q_1 = Q_2$ and we call Q_1 the *two-sided quotient ring* of R.

Let M be a multiplicatively closed set of elements of a ring R. Let I be any ideal of R. We shall say that R satisfies the *right Ore condition with respect to M modulo I* if, for given $r \in R$, $m \in M$, there exist $r_1 \in R$, $m_1 \in M$ such that $rm_1 - mr_1 \in I$. In the special case when $I = 0$, we shall say simply that R satisfies the *right Ore condition with respect to M*. There are analogous left-handed definitions.

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THEOREM 1.1. (See [4, p. 118].) *A necessary and sufficient condition for a ring R to have a right (left) quotient ring is that R satisfies the right (left) Ore condition with respect to* $\mathscr{C}(0)$.

A ring R will be called *right artinian* if R has an identity and R satisfies the minimum condition for right ideals. There is an analogous definition for left artinian.

The sum of all the nilpotent ideals of a ring R is called the *nilpotent radical* of R. A prime ideal P of R is *minimal* if $P \supseteq P'$, with P' a prime ideal of R, implies $P = P'$. If R is a right noetherian ring (i.e., if R satisfies the maximum condition for right ideals), then the family of proper ideals $\{r(T) : T \text{ is a nonzero ideal of } R\}$ has maximal members and these ideals are called the *maximal right annihilators* of R. There is an analogous definition of the *maximal left annihilators* of R. The next result can easily be checked.

LEMMA 1.2. *Let R be a right noetherian ring. Let P be a maximal right annihilator of R. Then*

(i) *P is a prime ideal of R, and*
(ii) *P consists of zero-divisors of R.*

We recall the following results.

THEOREM 1.3. (See [3, Theorem 1.5].) *Let R be a right noetherian ring and let N be the nilpotent radical of R. Then $\mathscr{C}'(0) \subseteq \mathscr{C}(N)$. When $r \in R$, $c \in \mathscr{C}'(0)$, there exist $r_1 \in R$, $c_1 \in \mathscr{C}(N)$ with $rc_1 = cr_1$.*

COROLLARY 1.4. *If $\mathscr{C}'(0) = \mathscr{C}(N)$, then R satisfies the right Ore condition with respect to $\mathscr{C}(N)$.*

THEOREM 1.5. (See [6, Theorems 2.11 and 2.12].) *Let R be a right noetherian ring with nilpotent radical N. Then R has a right artinian right quotient ring if and only if $\mathscr{C}(N) \subseteq \mathscr{C}(0)$.*

THEOREM 1.6. (See [3, 1.11].) *Let R be a right noetherian ring with nilpotent radical N. Let P be a prime ideal of R. Then P is minimal if and only if P does not meet $\mathscr{C}(N)$.*

Combining Theorems 1.5 and 1.6, we have

COROLLARY 1.7. *Let R be a right noetherian ring which has a right artinian right quotient ring. Let P be a prime ideal of R consisting of zero-divisors. Then P is a minimal prime ideal.*

LEMMA 1.8. *Let R be a right and left noetherian ring with nilpotent radical N. Suppose that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. Let $J = \{a \in R : ac = 0 \text{ for some } c \in \mathscr{C}(N)\}$. Then*

(i) *J is an ideal of R and $\mathscr{C}(N) \subseteq \mathscr{C}(J)$,*
(ii) *there exists $c \in \mathscr{C}(N)$ such that $Jc = 0$.*

Proof. (i) See [2, Chapter 5, Notes on Chapters 3, 4, 5, Propositions 1 and 3].

(ii) Choose $c \in \mathscr{C}(N)$ such that $l(c)$ is maximal in the family of left ideals $\{l(e) : e \in \mathscr{C}(N)\}$. If $a \in J$, then there exists $d \in \mathscr{C}(N)$ such that $ad = 0$. Since R satisfies the right Ore condition with respect to $\mathscr{C}(N)$, it follows that there exist $d_1 \in \mathscr{C}(N)$, $c_1 \in R$ such that $cd_1 = dc_1$. Then $acd_1 = 0$. By the maximality of $l(c)$, $l(c) \subseteq l(cd_1)$ implies that $l(c) = l(cd_1)$. Therefore $ac = 0$. It follows that $Jc = 0$.

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2. Artinian quotient rings. Throughout this section we shall suppose that R is a right and left noetherian ring (with a regular element).

LEMMA 2.1. *Let N be the nilpotent radical of R. Suppose that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. Suppose also that each prime ideal of R consisting of zero-divisors is minimal. Then R has a right and left artinian two-sided quotient ring.*

Proof. Let $J = \{a \in R : ac = 0 \text{ for some } c \in \mathscr{C}(N)\}$. By Lemma 1.8, J is an ideal of R and $Jc = 0$ for some $c \in \mathscr{C}(N)$. If $J \neq 0$, then $r(J)$ is contained in some maximal right annihilator P of R. By Lemma 1.2(i), P is a prime ideal of R. Since P meets $\mathscr{C}(N)$, Theorem 1.6 shows that P is not minimal. Hence, by hypothesis, P does not consist of zero-divisors. This contradicts Lemma 1.2(ii). It follows that $J = 0$ and, by Lemma 1.8, that $\mathscr{C}(N) \subseteq \mathscr{C}(0)$. Finally Theorem 1.5 shows that the right and left noetherian ring R has a right and left artinian two-sided quotient ring.

THEOREM 2.2. *Let R be a right and left noetherian ring. Then R has a right and left artinian two-sided quotient ring if and only if each prime ideal of R consisting of zero-divisors is minimal.*

Proof. The necessity is proved by Corollary 1.7.

Conversely, suppose that each prime ideal consisting of zero-divisors is minimal. Let N be the nilpotent radical of R. By Levitzki's Theorem, N is nilpotent. Therefore there exists a positive integer s such that $N^{s-1} \neq 0$, $N^s = 0$.

By Lemma 2.1, to prove that R has a right and left artinian two-sided quotient ring it is sufficient to prove that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. By Theorem 1.3, R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo N. Suppose that $1 \leq k \leq s-1$ and that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo N^k. We shall prove that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo N^{k+1}. The proof will be given in a series of lemmas.

LEMMA 2.3. *Let $K = \{r \in R : rc \in N^{k+1} \text{ for some } c \in \mathscr{C}(N)\}$. Then K is an ideal of R.*

Proof. Note first that $K \subseteq N$. Let $r_1, r_2 \in K$. Then there exist $c_1, c_2 \in \mathscr{C}(N)$ such that $r_1 c_1 \in N^{k+1}$, $r_2 c_2 \in N^{k+1}$. Moreover, there exist $d \in \mathscr{C}(N)$, $a \in R$, $b \in N^k$ such that $c_1 d - c_2 a = b$. Therefore

$$(r_1 - r_2)c_1 d = r_1 c_1 d - r_2(c_2 a + b) = (r_1 c_1)d - (r_2 c_2)a - r_2 b \in N^{k+1}$$

since $r_2 \in N$. Let $r \in K$, $x \in R$. Then clearly $xr \in K$. On the other hand, there exists $c \in \mathscr{C}(N)$ such that $rc \in N^{k+1}$. In addition there exist $e_1 \in \mathscr{C}(N)$, $x_1 \in R$ such that $xe_1 - cx_1 \in N^k$. Then $rxe_1 \in N^{k+1}$. This implies that $rx \in K$. It follows that K is an ideal of R.

LEMMA 2.4. $\mathscr{C}(0) \subseteq \mathscr{C}(K)$.

Proof. By Theorem 1.3, $\mathscr{C}(0) \subseteq \mathscr{C}(N)$. Hence $\mathscr{C}(0) \subseteq {}'\mathscr{C}(K)$. Suppose that $r \in R$, $c \in \mathscr{C}(0)$ and $cr \in K$. For each positive integer t we set $L_t = \{r \in R : c^t r \in K\}$. Since R is right noetherian, the ascending chain of right ideals $L_1 \subseteq L_2 \subseteq \dots$ must terminate. That is, there exists n such that $L_n = L_{n+1}$. By Theorem 1.3, there exist $c_1 \in \mathscr{C}(N)$, $r_1 \in R$ such that $c^n r_1 = rc_1$.

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Now $cr \in K$ implies $c^{n+1}r_1 \in K$. By the choice of n, $c^n r_1 \in K$ and hence $rc_1 \in K$. It follows that $r \in K$. In this way, $\mathscr{C}(0) \subseteq \mathscr{C}'(K)$. Hence $\mathscr{C}(0) \subseteq \mathscr{C}(K)$.

LEMMA 2.5. *Let* $T = \{r \in R : cr \in K$ *for some* $c \in \mathscr{C}(N)\}$. *Then* T *is an ideal of* R *and there exists* $c \in \mathscr{C}(N)$ *such that* $cT \subseteq K$.

Proof. Since $K \subseteq N$, the ideal N/K is the nilpotent radical of the ring R/K. In addition, it is clear that $\mathscr{C}(N) \subseteq {}'\mathscr{C}(K)$. By Theorem 1.3, ${}'\mathscr{C}(K) = \mathscr{C}(N)$. Then Corollary 1.4 shows that R/K satisfies the left Ore condition with respect to $\mathscr{C}(N/K)$. The result now follows by Lemma 1.8.

Note. The proof of Lemma 2.5 uses the left hand versions of Theorem 1.3 and Lemma 1.4.

LEMMA 2.6. $T = K$.

Proof. Note first that $K \subseteq T$. Let $V = \{r \in R : rT \subseteq K\}$. Then V is an ideal of R and $K \subseteq V$. If $T \neq K$, then V/K is contained in a maximal left annihilator P' of R/K. By Lemma 1.2(i), P' is a prime ideal of R/K. Therefore $P' = P/K$ for some prime ideal P of R with $K \subseteq V \subseteq P$. By Lemma 2.5, P meets $\mathscr{C}(N)$ and, by Theorem 1.6, P is not minimal. By hypothesis, P meets $\mathscr{C}(0)$. Hence, by Lemma 2.4, P meets $\mathscr{C}(K)$. That is, P' does not consist of zero-divisors of R/K. This contradicts Lemma 1.2(ii). Hence $T = K$.

COROLLARY 2.7. $\mathscr{C}(N) \subseteq \mathscr{C}(K)$.

Proof. Let $r \in R$, $c \in \mathscr{C}(N)$. If $rc \in K$, then clearly $r \in K$. On the other hand, if $cr \in K$, then $r \in T = K$.

LEMMA 2.8. *R satisfies the right Ore condition with respect to* $\mathscr{C}(N)$ *modulo* N^{k+1}.

Proof. Recall that N/K is the nilpotent radical of R/K. By Theorem 1.3 and Corollary 2.7, $\mathscr{C}(N) = \mathscr{C}(K)$. Then Theorem 1.3 also shows that R/K satisfies the right Ore condition with respect to $\mathscr{C}(N/K)$. In other words, R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo K. It follows easily that R satisfies the right Ore condition with respect to $\mathscr{C}(N)$ modulo N^{k+1}.

We recall that $N^s = 0$. Therefore, by induction, R satisfies the right Ore condition with respect to $\mathscr{C}(N)$. As we remarked earlier, this allows us to conclude that R has a right and left artinian two-sided quotient ring. This completes the proof of Theorem 2.2.

It might be conjectured that, if R satisfies the condition that

$$\text{each prime ideal which does not meet } \mathscr{C}'(0) \text{ is minimal,} \qquad (*)$$

then R has a right and left artinian two-sided quotient ring. The following example of Small [7] shows that this conjecture is false.

Example. Let Z denote the ring of integers. Let p be a prime in Z. Let S be the ring of all two-by-two " matrices " of the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

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with $a \in Z$, $b \in Z/(p)$ and $c \in Z/(p)$. Addition in S is defined component-wise and multiplication is given by

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ ba'+cb' & cc' \end{pmatrix}$$

where Z acts on $Z/(p)$ in the usual way. Then S has the following properties (see [7]).

(i) S is a right and left noetherian ring with an identity.

(ii) If N is the nilpotent radical of S, then

$$N = \left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in Z/(p)\right\}.$$

(iii) S has a two-sided quotient ring Q but Q is neither right nor left artinian.

It is not hard to prove that S has the further property:

(iv) $\mathscr{C}(N) = \mathscr{C}'(0) = \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a \neq 0 \text{ and } c \neq 0\right\}$.

Combining (iv) with Theorem 1.6 we have immediately:

(v) S satisfies (*).

Finally, let

$$r = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in S \quad \text{and} \quad c = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{C}(N).$$

Then there does not exist $c' \in \mathscr{C}(N)$ such that $c'r \in Sc$. Using this fact it is not hard to prove that

(vi) S satisfies the right Ore condition with respect to $\mathscr{C}(N)$ but S does not satisfy the left Ore condition with respect to $\mathscr{C}(N)$.

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