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RESEARCH ARTICLE

Fiber bundles associated with Anosov representations

Daniele Alessandrini[1], Sara Maloni[2], Nicolas Tholozan[3] and Anna Wienhard[4]

[1]Department of Mathematical Sciences, Stevens Institute of Technology; E-mail: daniele.alessandrini@gmail.com.
[2]Department of Mathematics, University of Virginia; E-mail: sm4cw@virginia.edu (corresponding author).
[3]Département de Mathématiques et Applications, École Normale Supérieure – PSL; E-mail: nicolas.tholozan@ens.fr.
[4]Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany;
E-mail: anna.wienhard@mis.mpg.de.

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Abstract

Anosov representations of hyperbolic groups form a rich class of representations that are closely related to geometric structures on closed manifolds. Any Anosov representation $\rho : \Gamma \to G$ admits cocompact domains of discontinuity in flag varieties G/Q [GW12, KLP18] endowing the compact quotient manifolds M_ρ with a $(G, G/Q)$–structure. In general, the topology of M_ρ can be quite complicated.

In this article, we will focus on the special case when Γ is a the fundamental group of a closed (real or complex) hyperbolic manifold N and ρ is a deformation of a (twisted) lattice embedding $\Gamma \to \mathrm{Isom}^\circ(\mathbb{H}_\mathbb{K}) \to G$ through Anosov representations. In this case, we prove that M_ρ is a smooth fiber bundle over N, and we describe the structure group of this bundle and compute its invariants. This theorem applies in particular to most representations in higher rank Teichmüller spaces, as well as convex divisible representations, AdS-quasi-Fuchsian representations and $\mathbb{H}_{p,q}$–convex cocompact representations.

Even when $M_\rho \to N$ is a fiber bundle, it is often very difficult to determine the fiber. In the second part of the paper, we focus on the special case when N is a surface, ρ a quasi-Hitchin representation into $\mathrm{Sp}(4, \mathbb{C})$, and M_ρ carries a $(\mathrm{Sp}(4, \mathbb{C}), \mathrm{Lag}(\mathbb{C}^4))$–structure. We show that in this case the fiber is homeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

This fiber bundle $M_\rho \to N$ is of particular interest in the context of possible generalizations of Bers' double uniformization theorem in the context of higher rank Teichmüller spaces, since for Hitchin-representations it contains two copies of the locally symmetric space associated to $\rho(\Gamma)$. Our result uses the classification of smooth 4–manifolds, the study of the $\mathrm{SL}(2, \mathbb{C})$–orbits of $\mathrm{Lag}(\mathbb{C}^4)$ and the identification of $\mathrm{Lag}(\mathbb{C}^4)$ with the space of (unlabelled) regular ideal hyperbolic tetrahedra and their degenerations.

Contents

1. Introduction

A (G, X)–manifold is a topological manifold that is locally modelled on a G–homogeneous space X. This means that the manifold is equipped with local charts with values in a model space X and transition functions with values in a Lie group G acting transitively on X. The theory of (G, X)–manifolds plays an important role in Thurston's geometrization program, but extends far beyond it. In particular, (G, X)–structures capture many geometric structures beyond Riemannian metrics – for example, projective or affine structures on manifolds.

The simplest examples of (G, X)–manifolds are quotients of X by a discrete subgroup of G acting freely and properly discontinuously – the *complete* (G, X)–manifolds. A larger class of examples arises more generally from quotients of an open domain Ω of X. These are sometimes called *Kleinian* (G, X)–manifolds.

The terminology *Kleinian* comes from the theory of *Kleinian groups*. A Kleinian group Γ is a (non-elementary) discrete subgroup of isometries of the hyperbolic 3–space $\mathbb{H}^3$. Its action on the boundary at infinity $\partial_\infty \mathbb{H}^3 \simeq \mathbb{CP}^1$ has a minimal invariant *limit set* Λ_Γ and is properly discontinuous on the complement Ω_Γ, called the domain of discontinuity. The quotient $\Gamma\backslash\Omega_\Gamma$ is (at least when Γ is torsion-free) a Riemann surface equipped with a *complex projective structure*. The group Γ is *convex-cocompact* when the action of Γ on $\mathbb{H}^3 \sqcup \Omega_\Gamma$ is cocompact. The quotient then gives a conformal compactification of the hyperbolic 3–manifold $\Gamma\backslash\mathbb{H}^3$. Convex-cocompact Kleinian groups and the corresponding hyperbolic 3–manifolds play a central role in Thurston's hyperbolization theorem.

In recent years, the theory of convex-cocompact subgroups in rank one Lie groups has been generalized to the higher rank setting through the theory of Anosov representations, introduced by Labourie [Lab06] and, more generally, by Guichard–Wienhard [GW12]. A strong connection between Anosov representations and (G, X)–manifolds has been established through the construction of domains of discontinuity by Guichard–Wienhard [GW12] and then by Kapovich–Leeb–Porti [KLP18].

Let G be a semisimple Lie group and P a parabolic subgroup of G. Informally, a *P–Anosov representation* of a Gromov hyperbolic group Γ into G is a homomorphism $\rho : \Gamma \to G$ that admits a

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ρ–equivariant continuous embedding of the boundary at infinity $\partial_\infty\Gamma$ of Γ into the flag variety G/P, which preserves the dynamics of the action of Γ on its boundary (see Definition 2.1). Anosov representations are quasi-isometric embeddings and are stable under small perturbations. Guichard–Wienhard proved in [GW12] that, for some parabolic subgroup Q (possibly different from P), a P–Anosov representation $\rho : \Gamma \to G$ defines a properly discontinuous and cocompact action of Γ on an open subset $\Omega_\rho \subset G/Q$, which is the complement of a disjoint union of Schubert subvarieties parametrized by $\partial_\infty\Gamma$ (see Definition 2.4). The precise parabolic subgroups Q for which this construction works have been systematically described by Kapovich–Leeb–Porti [KLP18]. We call the domains of discontinuity Ω_ρ obtained by their constructions *flag domains of discontinuity*.

This associates to an Anosov representation $\rho\colon\ \Gamma \to G$ of a torsion-free group Γ a closed manifold $M_\rho = \Gamma\backslash\Omega_\rho$ equipped with a Kleinian $(G, G/Q)$–structure. Even though the topological type of M_ρ only depends on the connected component of ρ in the space of P–Anosov representations [GW12], it is very difficult to determine the topology of M_ρ. Even in the case of convex-cocompact subgroups of a hyperbolic space of dimension ≥ 4, the topology of M_ρ is still very mysterious, and many wild phenomena can occur (see Section 1.4 for an example with a surface group), suggesting that a systematic answer is impossible. Nonetheless, important classes of examples of Anosov representations arise from deformations of uniform lattices in Lie groups of real rank 1 into a higher rank Lie group. For such representations, the topology of M_ρ becomes more tractable.

1.1. Part 1: Deformations of rank 1 lattices

In the first part of the paper, we prove a general fibration theorem for the quotients of flag domains of discontinuity associated to Anosov deformations of a rank 1 lattice into a higher rank Lie group.

Let H be a connected semisimple Lie group of real rank 1 with finite center, and Γ a torsion-free uniform lattice in H. Denote by $\rho_0\colon\ \Gamma \to H$ the identity representation. Let G be a connected semisimple Lie group with finite center, and $\iota\colon\ H \to G$ a faithful representation. Then $\iota \circ \rho_0$ is a P–Anosov representation of Γ in G for some parabolic subgroup P: we will call $\iota \circ \rho_0$ an ι–lattice representation. The set $\mathrm{Anosov}_P(\Gamma, G)$ of P–Anosov representations of Γ into G is an open subset of $\mathrm{Hom}(\Gamma, G)$ containing the ι–lattice representation $\iota \circ \rho_0$. We call a representation $\rho : \Gamma \to G$ a *P–Anosov deformation of* $\iota \circ \rho_0$ if ρ belongs to the connected component of $\iota \circ \rho_0$ in $\mathrm{Anosov}_P(\Gamma, G)$.

Let us now fix any parabolic subgroup Q of G such that P–Anosov representations in G admit a cocompact flag domain of discontinuity in G/Q. For $\rho \in \mathrm{Anosov}_P(\Gamma, G)$, we denote the domain by $\Omega_\rho \subset G/Q$ and the closed quotient manifold by $M_\rho = \rho(\Gamma)\backslash\Omega_\rho$. Finally, we denote by S_H the symmetric space of H. Our first main result is the following:

Theorem A (see Theorem 3.1). *For $H, \Gamma, G, \iota, P, Q$ as above, let $\rho : \Gamma \to G$ be a P–Anosov deformation of $\iota \circ \rho_0$. Then Ω_ρ admits a smooth Γ–equivariant fibration onto S_H. In particular, M_ρ is a smooth fiber bundle over the negatively curved locally symmetric space $\Gamma\backslash S_H$, and Ω_ρ deformation retracts to a closed manifold of dimension* $\dim(G/Q) - \dim(S_H)$.

This theorem was obtained independently by Davalo [Dav24] with different methods (see Section 1.3). It comes with companion theorems (see Theorem 3.3 and Corollary 3.4 below) that describe the structure group and the invariants of the fiber bundle.

Let us emphasize that Theorem A applies to every (!) cocompact flag domain of discontinuity. Even for a given representation, and a given flag variety, there can be many different cocompact flag domains of discontinuity (see [Ste18]). The theorem further applies when we consider the representation $\rho : \Gamma \to G \to G'$ as an Ansosov representation into a larger Lie group, and all the flag domains of discontinuity constructed in flag varieties of G'. See [GW12] and [GGKW17] for examples of how one can play around with such embeddings into larger groups.

An important source of applications of Theorem A is when an entire connected component of the representation variety $\mathrm{Hom}(\Gamma, G)$ consists of Anosov representations. For fundamental groups of closed surfaces with negative Euler characteristic (which we call *surface groups* from now on), such

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components are called higher rank Teichmüller components, and most of them contain representations that factor through a compact extension of $\mathrm{PSL}(2,\mathbb{R})$. Therefore, Theorem A gives a positive answer to [Wie18, Conjecture 13] in most cases, as well as a complete answer to a conjecture by Dumas–Sanders [DS20, Conjecture 1.1]. More precisely, we have the following.

Corollary B. *Let $\Gamma = \pi_1(\Sigma)$ be a surface group, and $\mathcal{C} \subset \mathrm{Hom}(\Gamma, G)$ be a higher rank Teichmüller component that contains a twisted Fuchsian representation. Then for every representation $\rho \in \mathcal{C}$, every parabolic subgroup Q and every cocompact flag domain of discontinuity $\Omega_\rho \subset G/Q$, the quotient manifold $M = \rho(\pi_1(\Sigma))\backslash\Omega$ is homeomorphic to a fiber bundle $M \to \Sigma$. In particular, the composition $\pi_1(M) \to \pi_1(\Sigma) \xrightarrow{\rho} G$ is the holonomy of a Kleinian $(G, G/Q)$–structure on M.*

Let us mention a list of interesting examples for which Theorem A and Corollary B apply.

1. Hitchin components. Let $\Gamma = \pi_1(\Sigma)$ be a surface group and G a split real simple linear group. The group $H = \mathrm{SL}(2,\mathbb{R})$ admits a *principal representation* $\iota_0 : H \to G$. Given a Fuchsian representation $\rho_0 : \Gamma \to H$, the composition $\iota_0 \circ \rho_0$ is called a principal Fuchsian representation in G. The representations of the connected component of $\iota_0 \circ \rho_0$ in $\mathrm{Hom}(\Gamma, G)$ are called *Hitchin representations*. Hitchin representations are Anosov with respect to the minimal parabolic subgroup $P_{min} < G$ or, equivalently, with respect to any parabolic subgroup $P < G$ [Lab06, FG06]. They are thus P–Anosov deformations of the lattice representation $\iota_0 \circ \rho_0$, and Theorem A applies.
2. P–quasi-Hitchin representations. Theorem A also applies to deformations of Hitchin representations into complex Lie groups. For this, we embed G into its complexification $G_\mathbb{C}$, and we consider the principal Fuchsian representation $\iota_0 \circ \rho_0 : \Gamma \to G < G_\mathbb{C}$ as taking values in $G_\mathbb{C}$. This representation is Anosov with respect to any parabolic subgroup $P_\mathbb{C} < G_\mathbb{C}$. However, not every continuous deformation of $\iota_0 \circ \rho_0$ will be Anosov. We define the set of P–quasi-Hitchin representation to be the connected component of the space of P–Anosov representations in $G_\mathbb{C}$ containing the principal Fuchsian representations. When $G = \mathrm{PSL}(2,\mathbb{R})$, this is precisely the set of quasi-Fuchsian representations. Note that, in higher rank, this set might depend on the choice of parabolic subgroup P.

   The geometry of P_{min}–quasi-Hitchin representations in $\mathrm{PSL}(n,\mathbb{C})$ has been studied by Dumas–Sanders [DS20]. In particular, they proved that flag domains of discontinuity Ω_ρ satisfy a Poincaré duality of rank $\dim(G/Q) - 2$ and computed the cohomology of M_ρ. They conjectured that M_ρ admits a fibration over the surface Σ. Theorem A applies in this situation and thus gives a positive answer to their conjecture.
3. Positive representations. The Hitchin component is one example of a higher rank Teichmüller component. Other examples are formed by maximal representations, and more generally by spaces of Θ–positive representations introduced in [GW18, GW]. Here, Θ is a subset of the set of simple roots Δ. Hitchin representations are Δ–positive representations. Maximal representations into Hermitian groups of tube type are $\{\alpha\}$–positive for a specific choice of $\alpha \in \Delta$. There are two further families of Lie groups that admit Θ–positive structures and Θ–positive representations. When G is a Lie group carrying a Θ–positive structure, then there is a special simple three-dimensional Θ–principal subgroup in G; see [GW]. Contrary to the principal subgroup of a split real Lie group, this H might have a compact centralizer, so there is a compact extension H of the Θ–principal subgroup that embeds into G. We choose a discrete and faithful representation $\rho_0 : \pi_1(\Sigma) \to H$ and call $\iota \circ \rho_0 : \pi_1(\Sigma) \to G$ a twisted Θ–principal embedding. This representation is P_Θ–Anosov, where P_Θ is the parabolic subgroup determined by Θ. In fact, any deformation of $\iota \circ \rho_0$ is P_Θ–Anosov [GLW, BGL+24], and thus, Theorem A applies.

   Note that there are cases where not every Θ–positive representation arises from a deformation of a principal or Θ–principal Fuchsian representation. In particular, when $G = \mathrm{Sp}(4,\mathbb{R}), \mathrm{SO}(2,3), \mathrm{SO}(n,n+1)$, there are connected components of the space of Θ–positive representations where every representation is Zariski-dense. In particular, Theorem A does not apply to these components. When $G = \mathrm{Sp}(4,\mathbb{R}), \mathrm{SO}(2,3)$, it has been proven by other means that the quotient manifolds M_ρ are fiber bundles over Σ; see the discussion in Section 1.3.

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4. P–quasi-positive representations. Similarly to the discussion of quasi-Hitchin representations, when G admits a Θ–positive structure, we can embed G into its complexification $G_{\mathbb{C}}$, and any Θ–positive representation $\rho : \pi_1(\Sigma) \to G < G_{\mathbb{C}}$ will be P–Anosov for a set of parabolic subgroups determined by Θ. Thus, we can define the set of P–quasi-positive representations as the connected components of the space of P–Anosov representations into $G_{\mathbb{C}}$ containing a Θ–positive representation into G. Theorem A then applies to the components of P–quasi-positive representations that contain a twisted Θ–principal embedding.

   Applications of Theorem A are not limited to representations of surface groups. There are notably interesting examples of Anosov deformations of fundamental groups of higher dimensional hyperbolic manifolds.
5. Convex divisible representations. The fundamental group $\pi_1(M)$ of a closed hyperbolic manifold of dimension $d \geq 3$ has a natural discrete and faithful embedding $\rho_0 : \pi_1(M) \to \mathrm{SO}(d,1) \simeq \mathrm{Isom}(\mathbb{H}^d)$. In many examples, ρ_0 can be deformed into $\mathrm{SL}(d+1,\mathbb{R})$. All such continuous deformations are Anosov (see Section 2.4 for details), so Theorem A applies to all such deformations, as well as further small deformations in $\mathrm{SL}(d+1,\mathbb{C})$.
6. $\mathbb{H}^{p,q}$-convex-cocompact representations. Similarly, the representation ρ_0 can often be continuously deformed into $\mathrm{SO}(d,d')$, $d' \geq 2$, and every such deformation remains Anosov (see Section 2.4 for details). Theorem A thus applies to all such deformations, as well as further small deformations into $\mathrm{SO}(d+d',\mathbb{C})$.

Even if we know that M_ρ is a fiber bundle over the locally symmetric space $\Gamma\backslash S_H$, it seems difficult in general to determine precisely the topology of the fiber. Explicit descriptions of the fibers have been given in some cases; see Section 1.3. In fact, the main reason why such a general result as Theorem A has been previously overlooked seems to be that, in interesting low dimensional situations, there are explicit and natural H–equivariant fibrations from Ω to S_H which are not smooth and whose fibers are not manifolds.

In the proof of Theorem A, the assumption that ρ is a P–Anosov deformation of a rank one lattice is used crucially in order to reduce to the 'Fuchsian' case. Indeed, Guichard–Wienhard proved that the topology of M_ρ is invariant under continuous deformation of ρ in $\mathrm{Anosov}_P(\Gamma, G)$. We can thus assume without loss of generality that $\rho = \iota \circ \rho_0$. In that case, the domain Ω_ρ is H–invariant, and our main theorem follows from the following general result:

Lemma C (See Lemma 3.5). *Let X be a smooth manifold with a proper action of a semisimple Lie group H. Then there exists a smooth H–equivariant fibration from X to the symmetric space S_H.*

Though this fairly general lemma sounds like a classical result, it seems to have been overlooked by people in the field. To prove it, we fix an arbitrary torsion-free uniform lattice $\Gamma \subset H$, choose a smooth Γ–equivariant map from X to S_H, and then take a barycentric average of f under some action of H.

A more precise version of Theorem A (see Theorem 3.3 and Corollary 3.4) shows that M_ρ is a fiber bundle over $\Gamma\backslash S_H$ associated to an explicit principal K–bundle, where K is a maximal compact subgroup of H. In order to complete the description of M_ρ, the only missing element is the topology of the fiber. The topology of the fiber has been determined in some cases; see Section 1.3. In the second part of the paper, we determine the fiber in a special low-dimensional case.

1.2. Part 2: Symplectic quasi-Hitchin representations

In the second part of the paper, we focus on *P–quasi-Hitchin representations* into $\mathrm{PSp}(4,\mathbb{C})$, where P is the stabilizer of a line in $\mathbb{CP}^3$. Let Γ be the fundamental group of a closed surface Σ of genus $g \geq 2$, embedded as a uniform lattice in $H = \mathrm{PSL}(2,\mathbb{R})$ via a Fuchsian representation ρ_0. Let $\iota_0\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSp}(4,\mathbb{C})$ be the principal representation. We see $\iota_0 \circ \rho_0$ as a P–Anosov representation, and we consider P–quasi-Hitchin representations (i.e., P–Anosov deformations of $\iota_0 \circ \rho_0$).

Guichard and Wienhard [GW12] show that P–quasi-Hitchin representations $\rho\colon\ \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{C})$ admit cocompact domains of discontinuity Ω_ρ in the space $\mathrm{Lag}(\mathbb{C}^4)$ of Lagrangian subspaces of

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$\mathbb{C}^4$, of complex dimension 3. We write as before $M_\rho = \rho(\Gamma)\backslash\Omega_\rho$. By topological invariance M_ρ is diffeomorphic to $M_{\iota_0\circ\rho_0}$, and Theorem A tells us that this manifold is a smooth fiber bundle over the hyperbolic surface $\Sigma = \Gamma\backslash\mathbb{H}^2$.

We prove the following theorem:

Theorem D. *Let ρ be a P–quasi-Hitchin representation of a surface group $\Gamma = \pi_1(\Sigma)$ into* $\mathrm{PSp}(4,\mathbb{C})$*, and let Ω_ρ be its flag domain of discontinuity in the space of complex Lagrangians. Then $M_\rho = \rho(\Gamma)\backslash\Omega_\rho$ is a smooth fiber bundle over Σ with fiber homeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.*

Remark 1.1. Theorem D does not say anything about the diffeomorphism type of the fiber. This question is still open, and it is particularly interesting. The question whether $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ admits exotic smooth structures is open, and it is an important question in the study of smooth 4–manifolds.

The domain of discontinuity $\Omega_\rho \subset \mathrm{Lag}(\mathbb{C}^{2n})$ is of particular interest in the context of potential generalizations of Bers' double uniformization for higher rank Teichmüller spaces. In the case when $n = 1$ and ρ is a Fuchsian representation, Ω_ρ is the disjoint union of the upper and the lower half disc; if ρ is a quasi-Fuchsian representation it is precisely the complement of the limit set, and thus consists of two connected components, whose quotients give rise to the two conformal structures associated to a quasi-Fuchsian representation. For general n and ρ a Hitchin representation into $\mathrm{PSp}(2n,\mathbb{R})$, the domain of discontinuity Ω_ρ contains two copies of the symmetric space associated to $\mathrm{PSp}(2n,\mathbb{R})$, a copy of the Siegel upper half space, and a copy of the Siegel lower half space, which are exchanged by the complex conjugation. However, it also contains other strata, (e.g., all the pseudo-Riemannian symmetric spaces $\mathrm{PSp}(2n,\mathbb{R})/\mathrm{PSU}(p,q)$, $p+q=n$), which are permuted by the complex conjugation. (For a more detailed discussion, see [Wie16].) The fact that for $\mathrm{PSp}(4,\mathbb{C})$ the fiber is $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ appears to be quite interesting in this respect.

In order to prove Theorem D, we actually have to take quite a bit of a detour. We first give a natural geometric construction of an H–equivariant continuous fibration π from $\Omega_{\iota_0\circ\rho_0}$ to $\mathbb{H}^2$. The map π is not smooth, and its fiber F is singular. Nevertheless, the fiber F is homotopy equivalent to the fiber F' of a smooth equivariant fibration since both are retractions of $\Omega_{\iota_0\circ\rho_0}$. By carefully studying F, we can determine its second cohomology and the intersection form on it. Finally, using the classification of smooth 4–manifolds due to Whitehead, Milnor, Milnor–Hausemoller, Freedman, Serre and Donaldson, we deduce the homeomorphism type of F' (which has the same second homology group) and prove the theorem.

1.3. Related works and perspectives

The topology of flag domains of discontinuity and their quotient manifolds M_ρ for Anosov representations ρ have been studied before in special examples, mainly for Anosov representations of a surface group $\pi_1(\Sigma)$. We review these results here.

In [GW08], Guichard–Wienhard constructed flag domains of discontinuity in $\mathbb{RP}^3$ for Hitchin representations into $\mathrm{PSL}(4,\mathbb{R})$ and $\mathrm{PSp}(4,\mathbb{R})$. These domains of discontinuity have two connected components Ω_1 and Ω_2. They showed that the quotient manifold $\pi_1(\Sigma)\backslash\Omega_1$ is homeomorphic to the unit tangent bundle T^1S of the surface and in fact gives rise to convex foliated projective structures of T^1S. The quotient manifold $\pi_1(S)\backslash\Omega_2$ is a quotient of T^1S by $\mathbb{Z}/3\mathbb{Z}$. They also show that deformations of quasi-Fuchsian representations (in $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PO}(3,1)$) into $\mathrm{PSL}(4,\mathbb{R})$ give rise to projective structures on T^1S.

The study of Hitchin representations in $\mathrm{PSL}(4,\mathbb{R})$ and $\mathrm{PSp}(4,\mathbb{R})$ and their domains of discontinuity in $\mathbb{RP}^3$ can be carried out also for lattices in $\mathrm{PSL}(2,\mathbb{R})$ that have torsion; see Alessandrini–Lee–Schaffhauser [ALS23]. There, they show that in this case, the quotient $\mathbb{RP}^3$-manifolds are homeomorphic to certain Seifert-fibered 3–manifolds that depend on the lattice.

In [GW12], determining part of the cohomology of the flag domains of discontinuity played a key role in showing that the action of $\rho(\Gamma)$ on Ω_ρ is cocompact. They describe several explicit examples of such flag domains of discontinuity – among them, some where M_ρ are in fact compact Clifford–Klein forms. For maximal representations in the symplectic group, and for the domain of discontinuity in $\mathbb{RP}^{2n-1}$, they announced that M_ρ is a fiber bundle over S with fiber $\mathrm{O}(n)/\mathrm{O}(n-2)$. This in particular also applies to the components of the space of maximal representations into $\mathrm{PSp}(4,\mathbb{R})$ where all representations are Zariski-dense. This result lead them to conjecture that the quotient manifold M_ρ is a compact fibre bundle over Σ for all higher Teichmüller spaces; see [Wie18, Conjecture 13].

When ρ is a quasi-Hitchin representation into a complex group G, Dumas–Sanders [DS20] computed the cohomology ring of Ω_ρ and M_ρ for all choices of parabolic subgroups and balanced ideals. They found that the cohomology of M_ρ is the tensor product of the cohomology of Σ with the cohomology of Ω_ρ and that, under their hypothesis, Ω_ρ is a Poincaré duality space. They remarked that this is compatible with M_ρ being a fiber bundle on Σ, and they stated a conjecture [DS20, Conjecture 1.1] that is a special case of our Theorem A. Interestingly, in their conjecture, they stated that M_ρ is a continuous fiber bundle over Σ because in some examples available at the time, the known fibrations were only continuous, but not smooth. They verified their conjecture in the special case when $G = \mathrm{SL}(3,\mathbb{C})$ and G/Q is the full flag variety.

When $G = \mathrm{SL}(2n,\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, ι is the principal representation and G/Q is $\mathbb{KP}^{2n-1}$, Alessandrini–Davalo–Li [ADL24] proved that M is a fiber bundle over Σ with structure group $\mathrm{SO}(2)$, described the topology of the fiber, and computed the Euler class of the underline $\mathrm{SO}(2)$-bundle. They used Higgs bundles, as described in the survey paper [Ale19]. In a paper in preparation, Alessandrini–Li [AL] extend some of these results to the case when $G = \mathrm{SL}(n,\mathbb{K})$ and G/Q is a partial flag manifold parametrizing flags consisting of lines and hyperplanes, and when $G = \mathrm{SL}(4n+3,\mathbb{R})$, $G/Q = \mathbb{S}^{4n+2}$, and M is the manifold constructed by Stecker–Treib [ST18].

In [CTT19], Collier–Tholozan–Toulisse studied the case where $\rho : \pi_1(\Sigma) \to \mathrm{SO}(2, n+1)$ is a maximal representation of a closed surface group. Such representations admit a flag domain of discontinuity Ω_ρ in the space of totally isotropic planes in $\mathbb{R}^{2,n+1}$. The authors prove that such maximal representations come with an equivariant spacelike embedding of $\mathbb{H}^2$ into the pseudo-hyperbolic space $\mathbb{H}^{2,n}$ and that the domain Ω_ρ fibers ρ–equivariantly over this spacelike disk, and deduce that M_ρ is a homogeneous fiber bundle over Σ with fiber a Stiefel manifold. The topological invariants of this fiber bundle turn out to depend on the connected component of ρ as the set of maximal representations. In particular, for $n = 3$, one obtains circle bundles over Σ whose Euler class varies with the connected component of maximal representations. Interestingly, there are connected components of the set of maximal representations into $\mathrm{SO}(2,3)$ that do not contain a representation factoring through $\mathrm{PSL}(2,\mathbb{R})$. For these representations, the fibration of M_ρ over Σ is not given by Theorem A.

More generally, let Γ be a $\mathbb{H}^{p,q}$-convex-cocompact subgroup of $\mathrm{SO}(p, q+1)$ and $\rho : \Gamma \to \mathrm{SO}(p, q+1)$. Then Γ admits a Guichard–Wienhard domain of discontinuity Ω_ρ in $\mathrm{SO}(p, q+1)/Q$, where Q is the stabilizer of a maximal isotropic subspace of $\mathbb{R}^{p,q+1}$. If $\partial_\infty\Gamma$ is a $p-1$-sphere, Seppi, Smith and Toulisse recently proved that Γ is virtually the fundamental group of an aspherical p-manifold M and that $\Gamma\backslash\Omega_\rho$ is a smooth fiber bundle over M [SST23]. Interestingly here, the group Γ need not be isomorphic to a rank one lattice (see [MST23]).

Finally, let us mention that Theorem A was obtained independently by Davalo [Dav24] with a different method. With the notations above, Davalo associates to a point in $\Omega_{\iota\circ\rho_0}$ a Buseman function on the symmetric space of G and shows that the restriction of this Buseman function to the symmetric space of H admits a unique critical point. The equivariant fibration derived from this construction is somewhat more explicit than the one in our proof of Theorem A, and its fibers can be expressed as real projective algebraic varieties. Nevertheless, their topology remains difficult to grasp, even in some low-dimensional cases such as the one studied in the second part of the paper.

1.4. Wild Kleinian groups

A crucial hypothesis in our Theorem A is that the representation is a deformation of an ι–lattice representation. We report on an example, by Gromov–Lawson–Thurston [GLT88], showing that this hypothesis is indeed necessary. They show that one can obtain wild convex-cocompact embeddings of a surface group $\Gamma = \pi_1(\Sigma)$ into $\mathrm{Isom}(\mathbb{H}^4)$ from a 'twisted necklace' of 2–spheres in $\partial_\infty \mathbb{H}^4$. They construct such convex-cocompact representations for which M_ρ is a nontrivial circle bundle over Σ. Again, by topological invariance, such ρ cannot be deformed to a Fuchsian representation within the domain of convex-cocompact representations. Such examples were also obtained independently by Kapovich [Kap89].

Gromov–Lawson–Thurston also point out that, starting from a knotted necklace, one obtains a convex-cocompact representation whose limit set is a *wild knot*. The associated conformal 3–manifold M_ρ is then obtained by gluing a circle bundle over a surface with boundary with one or several knot complements. These examples do not fiber over the surface Σ and their domains of discontinuity have infinitely generated fundamental group, showing that Theorem A cannot be true in general for Anosov representations which are not Fuchsian deformations.

For more examples of convex-cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ with 'wild' limit set (e.g., Antoine's necklace of Alexander's horned sphere), we refer to the survey of Kapovich [Kap08].

Outline of the paper

In Section 2, we review the required background on Anosov representations and their domains of discontinuity. Section 3 is dedicated to the proof of Theorem A. These form the first part of the paper.

The second part of the paper focuses on quasi-Hitchin representations in $\mathrm{Sp}(4, \mathbb{C})$. In Section 4, we describe the action of $\mathrm{PSL}(2, \mathbb{C})$ on the $\mathrm{Lag}(\mathbb{C}^4)$ and identify the Lagrangian Grassmannian to the space of (possibly degenerate) regular ideal tetrahedra in $\mathbb{H}^3$. Using this point of view, we construct a $\mathrm{PSL}(2, \mathbb{R})$–equivariant 'projection' from $\mathrm{Lag}(\mathbb{C}^4)$ to $\overline{\mathbb{H}^2}$ that we study more closely in Section 5. In Section 6, we carefully study the topology of the fiber F of this projection. In particular, we compute the intersection form on its second cohomology group and conclude the proof of Theorem D using the topological classification of simply connected 4–manifolds.

Part I
Topology of the quotient of the domain of discontinuity

2. Anosov representations

In this section, we recall the notion of Anosov representation, originally introduced in [Lab06, GW12], and we discuss several interesting examples. We then review the construction of their flag domains of discontinuity, based on [GW12, KLP18].

2.1. Definition and properties

There are several equivalent definitions of Anosov representations in literature; see [Lab06, GW12, KLP17, GGKW17, BPS19, KP22]. Here, we will describe the one that is more suitable for our aims. Let G be a connected semisimple Lie group with finite center and P a parabolic subgroup of G that is conjugate to its opposite parabolic subgroup P^{op}. Two points p and q in G/P are called *transverse* if there exists $g \in G$ such that $g\,\mathrm{Stab}_G(p)g^{-1} = P$ and $g\,\mathrm{Stab}_G(q)g^{-1} = P^{op}$.

Let now Γ be a finitely generated hyperbolic group with Gromov boundary $\partial_\infty\Gamma$.

Definition 2.1. A representation $\rho\colon\ \Gamma \to G$ is *P–Anosov* if there exists a continuous, ρ–equivariant map

$$\xi = \xi_\rho\colon\ \partial_\infty\Gamma \longrightarrow G/P$$

that is

∘ *transverse* (i.e., $\xi_\rho(x)$ and $\xi_\rho(y)$ are transverse for all $x \neq y \in \partial_\infty\Gamma$);

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◦ *strongly dynamics preserving* (i.e., for any sequence $(\gamma_n)_{n\in\mathbb{N}} \in \Gamma^{\mathbb{N}}$ with $\gamma_n \underset{n\to+\infty}{\longrightarrow} \gamma_+ \in \partial_\infty\Gamma$ and $\gamma_n^{-1} \underset{n\to+\infty}{\longrightarrow} \gamma_- \in \partial_\infty\Gamma$,

$$\rho(\gamma_n)\cdot p \underset{n\to+\infty}{\longrightarrow} \xi_\rho(\gamma_+)$$

for all $p \in G/P$ transverse to $\xi_\rho(\gamma_-)$).

A subgroup Γ of G is called Anosov if it is hyperbolic and the inclusion $\Gamma \hookrightarrow G$ is Anosov with respect to some proper parabolic subgroup P of G.

We denote by $\mathrm{Anosov}_P(\Gamma, G)$ the subset of $\mathrm{Hom}(\Gamma, G)$ consisting of P–Anosov representations. Note that P–Anosov representations are discrete and have finite kernel. In this paper, we will only work with groups Γ that are torsion-free. For such groups, P–Anosov representations are thus discrete and faithful.

One of the most important properties of Anosov representations is their structural stablility (i.e., $\mathrm{Anosov}_P(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$). Structural stability gives a way to construct several Anosov representations as small deformations of a fixed Anosov representation. This is a major source of examples, as we will discuss in Section 2.2.

Another important property of P–Anosov representations is that they admit cocompact domains of discontinuity in boundaries of G (i.e., in homogeneous spaces G/Q, where Q is a proper parabolic subgroup of G), possibly different from P. We will discuss this property in Section 2.5.

2.2. *Construction of Anosov representations via deformation*

Let us fix a connected semisimple Lie group H of real rank 1 with finite center, and let $K \subset H$ be its maximal compact subgroup. The symmetric space $S_H = H/K$ has strictly negative sectional curvature and is thus Gromov hyperbolic. Recall that a *uniform lattice* $\Gamma < H$ is a discrete cocompact subgroup of H. Any such lattice is quasi-isometric to S_H and is thus a hyperbolic group. Moreover, H has a unique conjugacy class of parabolic subgroups P_H. By Guichard–Wienhard [GW12, Thm 5.15], Γ is a P_H–Anosov subgroup of H. We will always assume that Γ is torsion-free, which is always virtually true by Selberg's lemma.

Remark 2.2. Note that the Anosov subgroups of a real rank 1 Lie group H are precisely its quasi-isometrically embedded (equivalently: quasi-convex, or convex-cocompact) subgroups.

An important case is when H is a compact extension[1] of $\mathrm{PSL}(2,\mathbb{R})$ (i.e., H admits a surjective morphism to $\mathrm{PSL}(2,\mathbb{R})$ with compact kernel). In that case, S_H is the hyperbolic plane $\mathbb{H}^2$, and a torsion-free cocompact lattice Γ in H is a *surface group* (i.e., $\Gamma = \pi_1(\Sigma)$, where Σ is a closed orientable surface of genus $g \geq 2$). A representation $\rho_0 : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ is called *Fuchsian* if it is discrete and faithful (in which case $\rho_0(\pi_1(\Sigma))\backslash\mathbb{H}^2$ is a closed hyperbolic surface diffeomorphic to Σ). Similarly, a discrete and faithful representation into a compact extension H of $\mathrm{PSL}(2,\mathbb{R})$ will be called a *twisted Fuchsian* representation. It is the case if and only if its projection to $\mathrm{PSL}(2,\mathbb{R})$ is Fuchsian.

Other interesting cases arise when H is (a compact extension of) $\mathrm{PO}_0(1,n)$ or $\mathrm{PU}(1,n)$, in which cases the symmetric space S_H is respectively the real hyperbolic space $\mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$ and the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. The group Γ is then the fundamental group of a closed real hyperbolic or complex hyperbolic manifold. The other Lie groups of real rank 1 (namely, $\mathrm{Sp}(1,n)$ and F_4^{-20}) are slightly less interesting for this paper since their lattices are superrigid (see below). Still, our Theorem A applies also to them.

Let us fix a uniform torsion-free lattice $\Gamma \subset H$ and an embedding $\iota : H \to G$, where G is a connected semisimple Lie group G with finite center. By Guichard–Wienhard [GW12, Prop. 4.7], the representation $\iota \circ \rho_0 : \Gamma \to G$ is P–Anosov for certain parabolic subgroups P of G (depending only on G, H and ι). We will call such a representation an *ι–lattice representation* of Γ in G.

[1] For example, H can be $\mathrm{SL}(2,\mathbb{R})$, or $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(n)$.

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When H is $\mathrm{PSL}(2,\mathbb{R})$, the representation $\iota \circ \rho_0 : \pi_1(\Sigma) \to G$ will be called an *ι–Fuchsian representation* in G. Similarly, when H is a compact extension of $\mathrm{PSL}(2,\mathbb{R})$, $\iota \circ \rho_0$ will be called a *twisted ι–Fuchsian representation* in G.

Using the property of structural stability, we can deform the representation $\iota \circ \rho_0$, obtaining an open subset of $\mathrm{Hom}(\Gamma, G)$ entirely consisting of P–Anosov representations of Γ in G. In the following, we denote by $\mathrm{Anosov}_{P,\iota,\rho_0}(\Gamma, G)$ the connected component of $\mathrm{Anosov}_P(\Gamma, G)$ that contains the representation $\iota \circ \rho_0$. We will say that a representation of Γ is a *P–Anosov deformation of a lattice representation* if it belongs to one of the connected components $\mathrm{Anosov}_{P,\iota,\rho_0}(\Gamma, G)$. In the special case when H is a compact extension of $\mathrm{PSL}(2,\mathbb{R})$, such representations will be called *P–Anosov deformations of a twisted ι–Fuchsian representation*.

For this paper, we are particularly interested in P–Anosov deformations of lattice representations because our Theorem A applies to this special class of Anosov representations. It is a special class, but it is also rather general since most known Anosov representations fall in this class. This is mainly because this is the easiest way to construct Anosov representations.

Now, we want to describe the most important examples of such representations. The main source of examples of interesting deformations of lattice representations come from the case when H is a compact extension of $PSL(2,\mathbb{R})$ (i.e., the case of surface groups). These examples are discussed in Section 2.3. The other important source of examples is the case of uniform lattices in $\mathrm{PO}_0(d,1)$ (i.e., fundamental groups of closed hyperbolic d–manifolds). These examples are discussed in Section 2.4.

Lattices in $\mathrm{PU}(d,1)$ exhibit more rigid behaviour; see [Kli11] and references therein. Still, some of them admit interesting Zariski dense deformations into higher rank Lie groups, but very few examples are known. When $H \simeq \mathrm{Sp}(d,1)$ or F_4^{-20}, by a theorem of Corlette [Cor92], Γ is superrigid. In particular, there are no nontrivial deformations of $\iota \circ \rho_0 : \Gamma \to G$.

Remark 2.3. All the arguments in this Section 2.2 are more general than the way we presented them. The hypothesis that Γ is torsion-free is not really needed, and we can also replace the assumption that Γ is a uniform lattice in H with the more general assumption that Γ is a convex-cocompact subgroup of H. Also in this higher generality, embeddings of H in other groups G allow to construct open subsets of Anosov representations in G. In our discussion, however, we restricted our attention to torsion-free uniform lattices for additional clarity and because our Theorem A works in this special case.

2.3. Anosov representations of surface groups

The case of surface groups is the one that is best understood. When H is a compact extension of $\mathrm{PSL}(2,\mathbb{R})$, Lie theory gives a classification of all representations $\iota : H \to G$, for a simple group G. Most of the time, twisted ι–Fuchsian representations into G admit small deformations with Zariski dense image.

In special cases, for particular twisted ι–Fuchsian representations into G, all deformations (not just small ones) are Anosov. This phenomenon gives rise to the so-called *higher rank Teichmüller components*, defined as connected components of the representation variety $\mathrm{Hom}(\pi_1(S), G)/G$ that consist entirely of discrete and faithful representations. They generalize many aspects of classical Teichmüller spaces, which can be seen as connected components of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$. An interesting feature for us is that most higher Teichmüller components consist of deformations of twisted ι–Fuchsian representations.

There are four families of higher rank Teichmüller spaces (see Guichard–Wienhard [GW]). The first family consists of *Hitchin representations*, introduced by Hitchin [Hit87]. In fact, Labourie's original motivation for defining Anosov representations in [Lab06] was showing that Hitchin representations form higher rank Teichmüller components. Hitchin representations are defined when G is a split real simple Lie group. Then G admits a special conjugacy class of representations $\iota_0 : \mathrm{SL}(2,\mathbb{R}) \to G$ called the *principal representation*. For this choice, the representation $\iota_0 \circ \rho_0$ is called a *principal Fuchsian representation* in G, and it is Anosov with respect to the minimal parabolic subgroups P_{min}. Hitchin components are the connected components containing a *principal Fuchsian representation*. In particular, any Hitchin representation is a deformation of a principal Fuchsian representation.

The second family consists of *Maximal representations*, which are defined when G is a real simple Lie group of Hermitian type [BIW10]. Maximal representations are in general only Anosov with respect to a particular maximal parabolic subgroup. Most maximal representations are deformations of twisted ι–Fuchsian representations, but for G locally isomorphic to $\mathrm{Sp}(4,\mathbb{R})$, there exist connected components in the space of maximal representations where every representation is Zariski dense [Got01, GW10, BGPG12].

The other two families of higher Teichmüller components arise from the notion of Θ–positivity introduced in [GW18, GW, GLW], which leads to the notion of positive representations. Hitchin representations and maximal representations are positive representations, but there are two further families of Lie groups admitting positive representations, Lie groups locally isomorphic to $\mathrm{SO}(p,q)$, as well as an exceptional family. With a positive structure comes again a special representation $\mathrm{SL}(2,\mathbb{R}) \to G$. This representation might have a compact centralizer, so there is a compact extension H of $\mathrm{SL}(2,\mathbb{R})$ that embeds into G; this is the representation ι. From the classification of the connected components of positive representations, we see that deformations of twisted ι–Fuchsian representations account for most of the connected components; exceptions occur only for $\mathrm{SO}(p,p+1)$. See [GW18], Guichard–Labourie–Wienhard [GLW], Collier [Col15], Aparicio-Arroyo–Bradlow–Collier–Garcia-Prada–Gothen–Oliveira [AABC+19], Bradlow–Collier–Garcia-Prada–Gothen–Oliveira [BCGP+], Beyrer–Pozzetti [BP], Beyrer–Guichard–Labourie–Pozzetti–Wienhard [BGL+24] for more details on positive representations.

Given a representation $\rho\colon \pi_1(\Sigma) \to G$ in a higher Teichmüller space, we can embed G into its complexification $G_{\mathbb{C}}$. If ρ is Anosov with respect to a parabolic subgroup P, the composition will be Anosov with respect to the parabolic $P_{\mathbb{C}} < G_{\mathbb{C}}$. In the complex group not every deformation will be discrete and faithful, but we can consider the space of Anosov representations $\mathrm{Anosov}_{P_{\mathbb{C}}}(\pi_1(\Sigma), G_{\mathbb{C}})$ and the connected component of this space containing $\rho : \pi_1(\Sigma) \to G < G_{\mathbb{C}}$. This generalizes the notion of quasi-Fuchsian representation into $\mathrm{PSL}(2,\mathbb{C})$ to this higher rank setting. Of particular interest to us will be the connected component of $\mathrm{Anosov}_{P_{\mathbb{C}}}(\pi_1(\Sigma), G_{\mathbb{C}})$ which contains the principal Fuchsian representation $\iota_0 \circ \rho_0$. We call this set the *quasi-Hitchin space* and representations therein *quasi-Hitchin representations*. Theorem A applies in particular to quasi-Hitchin representations, and Theorem D focuses on quasi-Hitchin representations for $G_{\mathbb{C}} = \mathrm{Sp}(4,\mathbb{C})$

2.4. Fundamental groups of hyperbolic manifolds

As mentioned in the introduction, applications of Theorem A are not limited to representations of surface groups. There are indeed interesting classes of Anosov deformations of fundamental groups of closed hyperbolic manifolds of higher dimension.

Let $\Gamma = \pi_1(M)$ be the fundamental group of a closed orientable hyperbolic manifold of dimension $d \geq 3$. Then Γ identifies with a uniform torsion-free lattice in $\mathrm{SO}_0(d,1) \simeq \mathrm{Isom}_+(\mathbb{H}^d)$ via a representation $\rho_0 : \Gamma \to \mathrm{SO}_0(p,1)$. One can construct in every dimension many examples where M contains an embedded totally geodesic hypersurface. There is then a general 'bending' procedure that allows one to deform the representation ρ_0 into a higher rank Lie group. We mention two particularly interesting examples.

Representations dividing a convex set. In a series of four papers, Benoist developed the theory of *divisible convex sets* – that is, proper convex subsets of a real projective space admitting a cocompact action of a discrete group of projective transformations. In particular, taking $\iota : \mathrm{SO}_0(d,1) \to \mathrm{SL}(d+1,\mathbb{R})$ to be the standard representation, Benoist proved in [Ben05] that any continous deformation $\rho : \Gamma \to \mathrm{SL}(d+1,\mathbb{R})$ of $\iota \circ \rho_0$ is discrete, faithful and acts properly discontinuously and cocompactly on an open strictly convex domain Ω_ρ of the projective space $\mathbb{RP}^d$. It follows (see [GW12, Proposition 6.1]) that any continuous deformation of $\iota \circ \rho_0$ is $P_{1,d}$-Anosov.

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The domain $\Omega_\rho \subset \mathbb{RP}^d$ is a flag domain of discontinuity whose quotient is easily described: Ω_ρ is contractible and $\rho(\Gamma)\backslash\Omega_\rho$ is diffeomorphic to M. However, it is not the only flag domain of discontinuity one can associate to a convex divisible representation. There are many flag domains of discontinuity in other flag varieties of $\mathrm{SL}(d+1,\mathbb{R})$, as well as flag domains of discontinuities in G'/Q' when we embed $\mathrm{SL}(d+1,\mathbb{R})$ into a larger Lie group G'. The topology of these domains of discontinuity can be more complicated. Theorem A applies to all these domains of discontinuity, as long as ρ is in the same connected component (of Anosov representations) as $\iota \circ \rho_0$.

AdS quasi-Fuchsian and $\mathbb{H}^{\mathbf{p},\mathbf{q}}$-convex-cocompact representations. Similarly, consider the embedding $\iota : \mathrm{SO}_0(d,1) \to \mathrm{SO}(d,2)$. Barbot and Mérigot proved in [BM12] and [Bar15] that any continuous deformation ρ of $\iota \circ \rho_0$ into $\mathrm{SO}(d,2)$ is P-Anosov, where P is the stabilizer of an isotropic line. Moreover, ρ acts properly discontinuously on a convex domain Ω_ρ contained in the *anti-de Sitter space* of dimension $d+1$, and the quotient $\rho(\Gamma)\backslash\Omega_\rho$ is a *globally hyperbolic Cauchy compact anti-de Sitter spacetime*. Barbot and Mérigot call these representations *AdS-quasi-Fuchsian*. Our main theorem applies to these representations.

Recently, these results have been generalized to deformations of $\iota\circ\rho_0$, for $\iota : \mathrm{SO}_0(d,1) \to \mathrm{SO}(d,d')$ the standard embedding: Beyrer and Kassel proved in [BK23] that any continuous deformation into $\mathrm{SO}(d,d')$ is P-Anosov and $\mathbb{H}^{d,d'-1}$-*convex-cocompact* in the sense of [DGK18]. Our theorem applies to these deformations, which again exist in many examples. Such groups have in particular a cocompact flag domain of discontinuity in the space of maximal totally isotropic subspaces of $\mathbb{R}^{d,d'}$. The topology of this domain is described in the recent paper [SST23].[2]

Complex deformations. Note finally that whenever $\iota\circ\rho_0$ admits deformations into a real linear algebraic group G, it also admits deformations into its complexification $G_\mathbb{C}$ which are not real. For instance, the above examples admit respectively Anosov deformations in $\mathrm{SL}(d+1,\mathbb{C})$ and $\mathrm{SO}(d+d',\mathbb{C})$. Theorem A applies to such complex deformations as well.

2.5. *Domains of discontinuity*

A P–Anosov representation $\rho\colon\ \Gamma \to G$ acts on all homogeneous spaces G/Q, where Q is a proper parabolic subgroup. The theory of *domains of discontinuity*, introduced by Guichard–Wienhard [GW12] and further developed by Kapovich–Leeb–Porti [KLP18], gives conditions for the existence of a ρ–invariant open subset $\Omega \subset G/Q$ where the action is properly discontinuous and/or cocompact. We sketch very briefly this construction here and refer the reader to [KLP18] for details.

The action of P on G/Q has finitely many orbits which are labelled by elements of $W_P\backslash W/W_Q$, where W is the Weyl group of G and W_P, W_Q are the subgroups corresponding to P and Q. A subset I of $W_P\backslash W/W_Q$ corresponds to a P–invariant subset K_I of G/Q (consisting of the union of the orbits labelled by elements of I). The set K_I is closed if and only if I is an *ideal* for the Bruhat order on W.

Given $x = gP \in G/P$, set $K_I(x) = gK_I$ (this is well-defined since K_I is P–invariant). The I–*thickening* of a subset $A \subset G/P$ is the set $K_I(A) = \bigcup_{x\in A} K_I(x)$. Finally, an ideal is called *balanced* if $I \cap -I = \emptyset$ and $I \cup -I = W_P\backslash W/W_Q$.

Now, let Γ be a hyperbolic group, $\rho\colon\ \Gamma \to G$ a P–Anosov representation and $\xi_\rho\colon\ \partial_\infty\Gamma \to G/P$ the associated boundary map.

Theorem 2.4 (Kapovich–Leeb–Porti, [KLP18]). *If $I \subset W_P\backslash W/W_Q$ is a balanced ideal, then Γ acts properly discontinuously and cocompactly on the domain*

$$\Omega_{\rho,I} = (G/Q)\ \backslash\ K_I(\xi_\rho(\partial_\infty\Gamma)).$$

Remark 2.5. If the ideal satisfies $I \cup -I = W_P\backslash W/W_Q$, the construction still gives rise to a domain of discontinuity, but then the action of Γ on $\Omega_{\rho,I}$ is not necessarily cocompact.

[2]The works of Beyrer–Kassel [BK23] and Seppi–Smith–Toulisse [SST23] actually apply to a larger class of groups including some that are not isomorphic to hyperbolic lattices, such as those constructed in [LM19] and [MST23].

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Remark 2.6. The domain $\Omega_{\rho,I}$ could a priori be empty, but this requires the topological dimension of $\partial_\infty\Gamma$ to equal the codimension of K_I, which is quite exceptional. In particular, one can always find a nonempty cocompact flag domain of discontinuity after embedding G into a larger group G' (see [GW12, Remark 1.10]).

When Γ is a uniform lattice in a rank 1 subgroup H of G, these domains of discontinuity are actually preserved by the group H:

Lemma 2.7. *Assume Γ is a uniform lattice in a rank 1 Lie group H and $\rho = \iota \circ \rho_0$, where $\rho_0 : \Gamma \to H$ is the inclusion and ι is an embedding of H into G. Then H preserves $\Omega_{\rho,I}$ and acts properly on it.*

Proof. In this situation, the boundary map ξ_ρ is actually H-equivariant. Hence, its image is H-invariant and so is the domain $\Omega_{\rho,I}$ (which only depends on the image of ξ_ρ). Now, the properness of the action of H follows from the proper discontinuity of the action of Γ as a consequence of a general topological fact (Proposition 2.8 below). □

Proposition 2.8. *Let X be a manifold equipped with a smooth action of H and Γ a uniform lattice in H. Then H acts properly on X if and only if Γ acts properly discontinuously. In that case, the action of Γ is cocompact if and only if the action of H is.*

Proof. Now, let C be a compact subset of H such that $\Gamma C = H$. Let D be any compact subset of $\Omega_{\rho,I}$.

Assume Γ is properly discontinuous. The set

$$\{h \in H \mid h \cdot D \cap D \neq \emptyset\}$$

is contained in FC, where

$$F = \{\gamma \in \Gamma \mid \gamma \cdot (C \cdot D) \cap D \neq \emptyset\}.$$

Since $C \cdot D$ is compact, F is finite by proper discontinuity of Γ. Hence, FC is compact, proving the properness of the action of H.

Conversely, if H is properly discontinuous, the set

$$\{\gamma \in \Gamma \mid h \cdot D \cap D \neq \emptyset\}$$

is the intersection of Γ with a compact subset of H, which is thus finite since Γ is discrete.

Assume some compact subset D of X satisfies $\Gamma \cdot D = X$. Then obviously $H \cdot D = X$. Conversely, assume $H \cdot D = X$. Then $\Gamma \cdot (C \cdot D) = H$ and $C \cdot D$ is compact. This proves that cocompactness of Γ and H are equivalent. □

Remark 2.9. The construction of domains of discontinuity was further generalized by Stecker–Treib [ST18], who extended it to the case where Q is an *oriented parabolic subgroup* (i.e., a subgroup of G lying between a parabolic subgroup and its identity component). The corresponding homogeneous space G/Q is called an *oriented flag variety*. Stecker–Treib [ST18] give conditions for the existence of (possibly cocompact) domains of discontinuity on G/Q. This generalization is interesting because some new cocompact domains of discontinuity arise that are not lifts of domains of discontinuity in the corresponding unoriented flag varieties. We refer the reader to Kapovich–Leeb–Porti [KLP18] and Stecker–Treib [ST18] for more details. The previous lemma and all the results of Part I apply to these domains as well.

In order to illustrate the theory, let us now describe the example that will be studied in detail in Part II of this paper. Consider the case where the group G is $\mathrm{Sp}(2n, \mathbb{K})$, where $\mathbb{K}$ can be $\mathbb{R}$ of $\mathbb{C}$, and the parabolic subgroup P is the stabilizer of a point in $\mathbb{KP}^{2n-1}$. In this case, $G/P = \mathbb{KP}^{2n-1}$. Every P–Anosov representation $\rho\colon \Gamma \to \mathrm{Sp}(2n, \mathbb{K})$ has an associated ρ–equivariant map

$$\xi_\rho\colon \partial_\infty\Gamma \to \mathbb{KP}^{2n-1}.$$

We now consider as second parabolic subgroup Q the stabilizer of a Lagrangian subspace in $\mathbb{K}^{2n}$. Then G/Q is the *Lagrangian Grassmannian* $\mathrm{Lag}(\mathbb{K}^{2n})$ (i.e., the space of all the Lagrangian subspaces of $\mathbb{K}^{2n}$). The action of P on G/Q has only two orbits: a closed orbit consisting of Lagrangian subspaces containing the line fixed by P, and its complement which is open. In this case, $W_P\backslash W/W_Q$ has only two elements and admits a unique nontrivial ideal I, for which K_I is the closed P–orbit. This ideal is balanced.

For each line $\ell \in \mathbb{KP}^{2n-1}$, we have

$$K_\ell = K_I(\ell) = \{W \in \mathrm{Lag}(\mathbb{K}^{2n}) \mid \ell \subset W\} \subset \mathrm{Lag}(\mathbb{K}^{2n}),$$

and we define the subset

$$K_{\rho,I} = K_I(\xi(\partial_\infty\Gamma)) = \bigcup_{t\in\partial_\infty\Gamma} K_{\xi(t)} \subset \mathrm{Lag}(\mathbb{K}^{2n}).$$

Guichard and Wienhard [GW12] showed that the complement

$$\Omega_{\rho,I} = \mathrm{Lag}(\mathbb{K}^{2n}) \setminus K_{\rho,I}$$

is a cocompact domain of discontinuity for ρ. Recall that Γ acts freely since it is torsion-free, so we have that

$$M_{\rho,I} = \rho(\Gamma)\backslash\Omega_{\rho,I}$$

is a closed manifold endowed with a geometric structure modelled on the parabolic geometry $(G, G/Q) = \big(\mathrm{Sp}(2n,\mathbb{K}), \mathrm{Lag}(\mathbb{K}^{2n})\big)$. Determining the topology of this quotient manifold (and more general such constructions) is a main focus of this paper.

2.6. Deformations

Consider one of our spaces $\mathcal{A} = \mathrm{Anosov}_{P,\iota,\rho_0}(\Gamma, G)$, defined in Section 2.2. Let Q be a parabolic subgroup and I a balanced ideal of $W_P\backslash W/W_Q$. For every representation $\rho \in \mathcal{A}$, we obtain a closed manifold $M_{\rho,I} = \rho(\Gamma)\backslash\Omega_{\rho,I}$ endowed with a geometric structure locally modelled on the parabolic geometry $(G, G/Q)$, and whose holonomy factors through[3] the representation ρ.

Theorem 2.10 (Guichard–Wienhard, [GW12]). *Let ρ be a P–Anosov representation of a hyperbolic group Γ into a semisimple Lie group G and let ρ' be a P–Anosov deformation of ρ. Then for any parabolic subgroup Q of G and any balanced ideal I of $W_P\backslash W/W_Q$, there exists a smooth (ρ,ρ')–equivariant diffeomorphism from $\Omega_{\rho,I}$ to $\Omega_{\rho',I}$. In particular, $M_{\rho,I}$ and $M_{\rho',I}$ are diffeomorphic.*

Remark 2.11. The theorem also applies to the quotients of domains of discontinuity constructed by Stecker–Treib in oriented flag varieties. More generally, it essentially follows from Ehresmann's fibration theorem that a smooth family of closed (G, X)–manifolds is locally topologically trivial.

For a $\rho \in \mathcal{A} = \mathrm{Anosov}_{P,\iota,\rho_0}(\Gamma, G)$, the topology of $M_{\rho,I}$ does not depend on ρ; hence, we can denote this smooth manifold by $M_{\rho_0,\iota,I}$. Thus, the space $\mathcal{A}$ can be seen as a deformation space for a family of $(G, G/Q)$–structures on a the fixed closed manifold $M_{\rho_0,\iota,I}$. This is particularly interesting for higher rank Teichmüller spaces because it gives a nice geometric interpretation of these spaces. It is also interesting for the theory of geometric structures on manifolds because it gives several interesting examples of closed manifolds with a large deformation space of geometric structures.

[3]If Ω_ρ is not simply connected, then Γ is only a quotient of $\pi_1(M_\rho)$.

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3. Topology of the quotient

3.1. General statement

We can now rephrase Theorem A, which describes the topology of $M_{\rho_0,\iota,I}$ constructed from an Anosov deformation of an ι–lattice representation.

Let us fix a connected semisimple Lie group H of real rank 1 with finite center, a uniform torsion-free lattice $\Gamma \subset H$ and a representation ι of H into some connected semisimple Lie group G with finite center. Denote by ρ_0 the inclusion of Γ into H, let P be a parabolic subgroup of G such that $\iota \circ \rho_0$ is P–Anosov, and let ρ be a P–Anosov deformation of $\iota \circ \rho_0$. Finally, let S_H denote the symmetric space of H.

Theorem 3.1. *For every parabolic subgroup Q of G and every balanced ideal I of $W_P\backslash W/W_Q$, the domain $\Omega_{\rho,I}$ is a smooth Γ–equivariant fiber bundle over the symmetric space S_H, with fiber a closed manifold $\mathfrak{F}$. In particular, $\Omega_{\rho,I}$ deformation retracts to $\mathfrak{F}$ and the manifold $M_{\rho_0,\iota,I} = \Gamma\backslash\Omega_{\rho,I}$ is a fiber bundle over the locally symmetric space $\Gamma\backslash S_H$ with fiber $\mathfrak{F}$.*

In fact, one can say a bit more on the structure of this bundle. For this, let us recall the notion of fiber bundle *associated* to a principal bundle. Let K be a Lie group, $T \to B$ a principal K-bundle and $\mathfrak{F}$ a smooth manifold equipped with a smooth action of K.

Definition 3.2. Let $\mathfrak{E}$ denote the quotient of $T \times \mathfrak{F}$ by the diagonal action of K. Then the projection $T \times \mathfrak{F} \to T$ factors to a smooth fibration $\mathfrak{E} \to B$ with fibers diffeomorphic to $\mathfrak{F}$. This fiber bundle is called the *$\mathfrak{F}$-bundle associated to T.*

Theorem 3.3. *In the setting of Theorem 3.1, the manifold $\mathfrak{F}$ admits a smooth action of the compact subgroup K, and the manifold $M_{\rho_0,\iota,I}$ is the $\mathfrak{F}$–bundle over $\Gamma\backslash S_H$ associated to the principal K–bundle*

$$\Gamma\backslash H \to \Gamma\backslash S_H .$$

In the previous theorem, the bundle $\Gamma\backslash H \to \Gamma\backslash S_H$ must be thought of as an explicit object that depends only on the lattice Γ. For example, when $H = PO_0(n,1)$, $\Gamma\backslash S_H$ is a closed hyperbolic manifold, Γ is its fundamental group, and the bundle $\Gamma\backslash H \to \Gamma\backslash S_H$ is its frame bundle.

In the special case when H is a compact extension of $\mathrm{PSL}(2,\mathbb{R})$, these theorems take an even more explicit form. In this case, $S_H = \mathbb{H}^2$ is the hyperbolic plane, the group $\Gamma = \pi_1(\Sigma)$ is a surface group, and $\Gamma\backslash\mathbb{H}^2$ is the surface Σ. The principal bundle

$$\Gamma(\Gamma)\backslash H \to \Sigma$$

depends on the extension H. For example, when $H = \mathrm{PSL}(2,\mathbb{R})$, this bundle is a circle bundle isomorphic to the unit tangent bundle of Σ (i.e., a circle bundle with Euler class $2g-2$). When $H = \mathrm{SL}(2,\mathbb{R})$, this bundle is the double cover of the unit tangent bundle of Σ (i.e., a circle bundle with Euler class $g-1$). For all the interesting groups H, it is possible to understand this bundle explicitly. We will now restate the previous theorems in the case when $H = \mathrm{SL}(2,\mathbb{R})$.

Corollary 3.4. *Let $H = \mathrm{SL}(2,\mathbb{R})$, P be a parabolic subgroup of G, Q another parabolic subgroup and I a balanced ideal of $W_Q\backslash W/W_P$. Let ρ be a P–Anosov deformation of an ι–Fuchsian representation of a surface group $\pi_1(\Sigma)$. Then*

- *$\Omega_{\rho,I}$ retracts to a closed submanifold $\mathfrak{F}$ of codimension 2 carrying a smooth circle action.*
- *The quotient $\rho(\pi_1(\Sigma))\backslash\Omega_{\rho,I}$ is diffeomorphic to a fiber bundle over Σ with fiber $\mathfrak{F}$. This is the $\mathfrak{F}$–bundle associated to the principal circle bundle of Euler class $g-1$ over Σ.*

One of the main applications of this corollary is for (quasi)-Hitchin representations. Recall that Hitchin representations are deformations of $\iota_0 \circ \rho_0$ where $\rho_0 : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ is a Fuchsian representation and ι_0 is the principal representation of $\mathrm{SL}(2,\mathbb{R})$ into a real split semisimple Lie group G, and quasi-Hitchin representations are their $P_{\min}$–Anosov deformations into its complexification $G_{\mathbb{C}}$.

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We can also apply Theorems 3.1 and 3.3 to the positive representations that are Anosov deformations of twisted ι–Fuchsian representations. As discussed in Section 2.3, almost all the positive representations in the classical groups are of this type, with the only exception of the exceptional components in $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}(p,p+1)$. In order to apply our results to positive representations, we need to consider the group $H = \mathrm{SL}(2,\mathbb{R}) \times C$ for a certain compact subgroup C. The statement is similar to Corollary 3.4, except that the structure group of the bundle is now $\mathrm{SO}(2) \times C$. The invariants that characterize the bundle are the Euler class and the characteristic classes of the C component of ρ_0.

3.2. Proof of the theorems

A key hypothesis in Theorem 3.1 is the assumption that ρ is a P–Anosov deformation of an ι–Fuchsian representation $\iota \circ \rho_0$. Indeed, by Guichard–Wienhard's Theorem 2.10, the topology of $M_{\rho,I}$ does not change, and we only have to determine it for $\rho = \iota \circ \rho_0$. The key result for the proof is thus the following.

Lemma 3.5. *Let H be a semisimple Lie group with finite center, and $K \subset H$ be its maximal compact subgroup. Let X be a manifold with a proper action of H. Then there exists an H–equivariant smooth fibration*

$$p : X \to S_H,$$

where S_H denotes the symmetric space of H.

For the proof, we use the fact that the symmetric space $S_H = H/K$ has non-positive curvature. We need the notion of barycenter: Given a finite measure of compact support ν on S_H, we consider the function

$$b : S_H \to \mathbb{R}$$

defined by

$$b(y) = \int d(y,z)^2 \mathrm{d}\nu(z).$$

Since S_H has non-positive curvature, the squared distance function is smooth (as push-forward of the squared norm under the exponential map), the distance function is convex (see [BGS85, Thm 1.3]), and hence the squared distance function is strongly convex. This implies that the function b is strongly convex, and since it is also proper, it has a unique critical point, which is a global minimum. Moreover, the Hessian of b at the global minimum is positive definite. The *barycenter* $\mathrm{Bar}\{\nu\}$ of ν is defined to be the unique critical point of b.

Proof of Lemma 3.5. We choose a torsion-free uniform lattice Γ in H. This always exists. See, for example, Borel and Harish–Chandra [BHC62]. Then Γ acts freely and properly discontinuously on X (see Proposition 2.8); hence, the quotient $\Gamma\backslash X$ is a manifold. Then Γ is isomorphic to the quotient $\pi_1(\Gamma\backslash X)/\pi_1(X)$, and we have a homomorphism $\psi\colon \pi_1(\Gamma\backslash X) \to \Gamma = \pi_1(\Gamma\backslash S_H)$. Since S_H is contractible, there exists a map $\Gamma\backslash X \to \Gamma\backslash S_H$ inducing ψ (for the details, [Ale03, Prop. 13]). A priori this map is only continuous, but since smooth maps are dense in the space of continuous maps between compact manifolds, we can assume that the map is smooth. Lifting this map, we obtain a smooth Γ-equivariant map

$$f\colon X \to S_H.$$

This allows us to define, for all $x \in X$, a smooth map $F^x\colon H \to S_H$ by

$$F^x(g) = g \cdot f(g^{-1} \cdot x).$$

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Since f is Γ–equivariant, we have that $F^x(g\gamma) = F^x(g)$ for all $\gamma \in \Gamma$. Hence, F^x descends to a continuous map $\overline{F}^x : H/\Gamma \to S_H$.

Note that we have

$$\overline{F}^{h\cdot x}(g) = g \cdot f(g^{-1}h \cdot x) = h \cdot \overline{F}^x(h^{-1}g). \tag{1}$$

We can finally define a map $\bar{f}\colon\ X \to S_H$ as follows:

$$\bar{f}(x) = \mathrm{Bar}\{\overline{F}^x_* \mu\},$$

where μ is the Haar measure on H/Γ, $\overline{F}^x_* \mu$ its push-forward by $\overline{F}^x$, and Bar is the barycenter of a finite measure with compact support, as defined above. Note that the image of $\overline{F}^x$ is compact since H/Γ is compact; hence, $\overline{F}^x_* \mu$ does have compact support.

We claim that $\bar{f}$ is H–equivariant. Indeed, for $x \in X$ and $g \in H$, we have

$$\begin{aligned}
\bar{f}(gx) &= \mathrm{Bar}\{\overline{F}^{gx}_* \mu\} \\
&= \mathrm{Bar}\{g_* \overline{F}^x_* g^{-1}_* \mu\} \quad \text{by (1)} \\
&= \mathrm{Bar}\{g_* \overline{F}^x_* \mu\} \quad \text{by left invariance of the Haar measure} \\
&= g \cdot \mathrm{Bar}\{\overline{F}^x_* \mu\} \quad \text{by equivariance of the barycenter map} \\
&= g \cdot \bar{f}(x).
\end{aligned}$$

The rest follows from the two lemmas below. Lemma 3.6 guarantees that the map $\bar{f}$ is smooth, and Lemma 3.7 shows that it is an Ehresmann fibration. □

Lemma 3.6. *The map $\bar{f}$, constructed in the proof of Lemma 3.5, is smooth.*

Proof. For every $x \in X$, $\bar{f}(x)$ is the unique critical point of the function

$$b(y) = \int_{S_H} d(y,z)^2 \mathrm{d}\nu(z),$$

where the measure ν is the push-forward $\nu = \overline{F}^x_* \mu$. By the change-of-variable formula, this can be written as

$$b(y) = \int_{\Gamma\backslash H} d(y, \overline{F}^x(h))^2 \mathrm{d}\mu(h) = \int_{\Gamma\backslash H} d(y, hf(h^{-1}x))^2 \mathrm{d}\mu(h).$$

In this setting, the function b depends on the parameter x. To make this more explicit, we write it as $b(x, y)$, a smooth function of two variables $x \in X$ and $y \in S_H$. We consider the differential of b with respect to y:

$$\beta(x, y) = d_y b(x, y) : X \times S_H \to T^* S_H.$$

Let us fix an $x_0 \in X$ and the corresponding $y_0 = \bar{f}(x_0) \in S_H$. We choose local coordinates on a small neighborhood U of y_0 in S_H. This trivializes the cotangent bundle on U: $T^*U \simeq U \times \mathbb{R}^k$, where $k = \dim(S_H)$. Let $\pi_2 : T^*U \to \mathbb{R}^k$ denote the projection onto the second factor. Now we consider the composition

$$\pi_2 \circ \beta(x, y) = d_y b(x, y) : X \times U \to \mathbb{R}^k.$$

Now, for all x close enough to x_0, the pairs $(x, \bar{f}(x))$ are precisely the solutions to the equation

$$\pi_2 \circ \beta(x, y) = 0.$$

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We can now apply the implicit function theorem to the function $\pi_2 \circ \beta(x, y)$. Consider a tangent vector to $X \times U$ of the form $(0, u)$ at a point $(x, \bar{f}(x))$. Then we have

$$d\pi_2(0, u) : v \mapsto \mathrm{Hess}_y b_{\bar{f}(x)}(u, v),$$

where $\mathrm{Hess}_y b\bar{f}(x)(u, v)$ denotes the Hessian of the function $b(x, \cdot)$ at the point $\bar{f}(x)$ (recall that the Hessian of a function at critical point is well-defined independently of the local coordinate). Since $b(x, \cdot)$ is strongly convex, $\mathrm{Hess}_y b\bar{f}(x)$ is positive definite, and the differential of π_2 in the y direction is an isomorphism from $T_{\bar{f}(x)}U$ to $T^*_{\bar{f}(x)}U = \mathbb{R}^k$. Hence, π_2 is a submersion, and the set $\{(x, \bar{f}(x))\} = \pi_2^{-1}(0)$ is the graph of a smooth function. □

Lemma 3.7. *Let H be a Lie group, and let X, Y be manifolds equipped with smooth H-actions, where the action on Y is transitive. Denote by L the stabilizer in H of a point of Y, and identify Y with H/L. Then, every H–equivariant map $\phi : X \to Y$ is a smooth fiber bundle, associated to the principal L-bundle $H \mapsto H/L$ (in the sense of Definition 3.2).*

Proof. First note that, by homogeneity of Y, the map ϕ needs to be onto. We will construct a local trivialization around every point $y \in Y$. We choose the subgroup L as the stabilizer of y. Hence, L is acting on the fiber $F = \phi^{-1}(y)$. Let U be a neighborhood of y in Y that trivializes the bundle $H \to H/L$. The trivialization is a map $t : U \times L \to H$.

A trivialization of ϕ over U is given by the map

$$T : U \times F \ni (u, f) \to t(u, e)f \in X,$$

where e is the identity of H. Clearly, $\phi(T(u, f)) = u$, because $t(u, e)$ sends y to u. The map T is 1–1 because if $t(u, e)f = t(u', e)f'$, then $u = u'$ because $\phi(T(u, f)) = u$, and then by multiplying by $t(u, e)^{-1}$, we see that $f = f'$. We can also see that the map T is onto $\phi^{-1}(U)$ because given $x \in \phi^{-1}(U)$, let $f = t(\phi(x), e)^{-1}x \in F$, and then $x = T(\phi(x), f)$.

The construction above shows that every atlas for the bundle $H \to H/L$ induces an atlas for the bundle $\phi : X \to Y$. It is easy to check that the two atlases have the same transition functions, and hence, the two bundles are associated. □

The following Proposition 3.8 is similar to Theorems 3.1 and 3.3, but the difference is that it can be applied to domains of discontinuity that are not necessarily cocompact. Anyway, if the domain is not cocompact, our conclusion only holds for ι–lattice representations, but does not automatically extend to their deformations. After that, we will add the hypothesis that the domain of discontinuity is cocompact and prove the full Theorems 3.1 and 3.3 for their deformations.

Proposition 3.8. *Let H be a connected semisimple Lie group with finite center of real rank 1, $K \subset H$ a maximal compact subgroup, and $S_H = H/K$ the symmetric space for H. Let $\rho_0 : \Gamma \to H$ be the inclusion of a torsion-free uniform lattice in H. Let G be a connected semi-simple Lie group with finite center, and $\iota : H \to G$ be a representation. Let $P < G$ be a parabolic subgroup of G such that the representation $\rho = \iota \circ \rho_0$ is P–Anosov.*

Let Q be a parabolic subgroup of G and $\Omega_{\rho,I} \subset G/Q$ a domain of discontinuity for ρ constructed from a thickening of a (not necessarily balanced) ideal I. Let $M_{\rho,I} = \rho\backslash\Omega_{\rho,I}$ be the quotient manifold.

Then $M_{\rho,I}$ is diffeomorphic to a smooth fiber bundle over S_Γ. The fiber F of the bundle is homotopically equivalent to the domain Ω_ρ and carries a K–action that gives the bundle a structure of K–bundle. The bundle is isomorphic to the K–bundle associated to the K–principal bundle $\Gamma\backslash H \to \Gamma\backslash S_H$ via a change of fiber.

Proof. As we saw in Section 2.2, it was proved in Guichard–Wienhard [GW12, Prop. 4.7] that the representation ρ is P–Anosov for a certain family of parabolic subgroups described there.

Since Γ is a uniform lattice in H, the group H preserves the domain of discontinuity Ω_ρ and acts properly on it by Lemma 2.7. Applying Lemmas 3.5 and 3.7 with $X = \Omega_\rho$ and $Y = S_H$, we get a smooth

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H–equivariant fiber bundle map from Ω_ρ to S_H, which factors to a smooth fiber bundle map from M to S_Γ. □

Proof of Theorems 3.1 and 3.3. In Proposition 3.8, we have already proved the theorem for twisted ι–Fuchsian and for lattice representations. Now, since we are assuming that the domain of discontinuity is cocompact, it follows from Theorem 2.10 that the topology of M is constant in $\mathcal{A}$. □

Part II
Quasi-Hitchin representations into $\mathrm{Sp}(4,\mathbb{C})$

In the second part of the paper, we will focus on quasi-Hitchin representations into $G = \mathrm{Sp}(4,\mathbb{C})$. We fix the principal representation $\iota_0\colon \mathrm{SL}(2,\mathbb{R}) \to G$ and a Fuchsian representation $\rho_0\colon \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$. The principal Fuchsian representation $\iota_0 \circ \rho_0$ is Anosov with reference to every parabolic subgroup P of G. Recall that a P–quasi-Hitchin representation is a P–Anosov deformation of $\iota_0 \circ \rho_0$. Our aim is to determine the topology of the quotient manifolds of the cocompact domains of discontinuity for these representations.

The group $\mathrm{Sp}(4,\mathbb{C})$ has (up to conjugation) three different proper parabolic subgroups, so there are three flag varieties for us to consider: the projective space $\mathbb{CP}^3$, the Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{C}^4)$, and the full flag variety, which consists of full isotropic flags (i.e., pairs consisting of a line in $\mathbb{C}^4$ and a Lagrangian subspace of $\mathbb{C}^4$ containing that line). The principal Fuchsian representation $\iota_0 \circ \rho_0$ admits four cocompact domains of discontinuity constructed by a balanced thickening: one in the projective space $\mathbb{CP}^3$, one in the Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{C}^4)$ (whose construction is described in Section 2.5), and two in the isotropic full flag variety. The two domains of discontinuity in the isotropic full flags variety are in fact the pullback of the two domains in $\mathbb{CP}^3$ and in $\mathrm{Lag}(\mathbb{C}^4)$ under the natural projection from the full flag variety to the partial flag varieties; hence, they can be understood from a description of the latter two. The domain of discontinuity in $\mathbb{CP}^3$ was described in Alessandrini–Davalo–Li [ADL24, Corol. 10.2], where it is proved that the quotient manifold M is diffeomorphic to a fiber bundle over the surface Σ with fiber $\mathbb{S}^2 \times \mathbb{S}^2$.

The only cocompact domain of discontinuity that is not yet understood is the one in $\mathrm{Lag}(\mathbb{C}^4)$. The second part of this paper is devoted to the description of this domain and its quotient manifold. In fact, the domain of discontinuity in the Lagrangian Grassmannian is of particular interest because it contains two copies of the symmetric space associated to $\mathrm{Sp}(4,\mathbb{R})$: the Siegel upper half space and the Siegel lower half space; see [Wie16]. This is very reminiscent of the situation for quasi-Fuchsian representations, and we hope that a good understanding of the domain of discontinuity and its quotient manifold might help to shed some light on possible generalizations of the Bers' double uniformization theorem for quasi-Hitchin representations.

The construction of the domain of discontinuity in $\mathrm{Lag}(\mathbb{C}^4)$, described in Section 2.5, only uses the fact that the representation is Anosov with respect to P, where P is the stabilizer of a point in $\mathbb{CP}^3$. We will thus consider a representation ρ in the quasi-Hitchin space $\mathrm{QHit}_P(\Sigma, \mathrm{Sp}(4,\mathbb{C})) := \mathrm{Anosov}_{P,\iota_0,\rho_0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{C}))$.

There is a unique nontrivial ideal I, which allows us to define a cocompact domain of discontinuity $\Omega_{\rho,I}$, with quotient manifold $M_{\rho,I}$; see Section 2.5. Since, by Theorem 2.10, the topology of the quotient manifold does not depend on ρ, we can restrict our attention to the case when $\rho = \iota_0 \circ \rho_0$. With the representation fixed once for all, we will denote the domain of discontinuity and the quotient manifold simply by Ω and M, instead of $\Omega_{\rho,I}$ and $M_{\rho,I}$.

Our Theorem A gives smooth fibrations

$$p\colon \Omega \to \mathbb{H}^2 \quad \text{and} \quad \widehat{p}\colon M \to \Sigma.$$

In this second part of the paper, we will study the fiber $\mathfrak{F} = \mathfrak{F}_p$ of these maps and prove Theorem D, which states that $\mathfrak{F}$ is homeomorphic to the 4–manifold $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

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Our strategy will be to describe a new ρ–equivariant fibration

$$q\colon\ \Omega \to \mathbb{H}^2 \quad \text{and} \quad \widehat{q}\colon\ M \to \Sigma,$$

which is not smooth, but has a more geometric definition and for which the fiber $F = F_q$ is easier to understand. This new fiber F is not a manifold, but it is homotopically equivalent to the domain of discontinuity Ω; hence, F is homotopically equivalent to the smooth fiber $\mathfrak{F}$. We will see that F is simply connected, and hence $\mathfrak{F}$ is a simply connected 4–manifold. A smooth simply connected 4–manifold is determined up to homeomorphism by its homotopy type; hence, we can determine $\mathfrak{F}$ by computing the homotopy invariants of F.

In order to define the fibration q, we will study the action of $\mathrm{SL}(2,\mathbb{C})$ on the Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{C}^4)$ induced by the principal representation ι_0. In Section 4, we will discuss the $\mathrm{SL}(2,\mathbb{C})$–orbits in $\mathrm{Lag}(\mathbb{C}^4)$, and this discussion will allow us to define q at the end of the section. In Section 5, we will study the fiber F, and in Section 6, we will use our results on the topology of F to understand $\mathfrak{F}$.

4. Lagrangians as regular ideal tetrahedra

In this section, we will study the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^4)$ and its orbits. This will allow us to define the projection $q\colon\ \Omega \to \mathbb{H}^2$ explicitly.

4.1. Lagrangian subspaces in $\mathbb{C}^{2n}$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and let $V_{n,\mathbb{K}} = \mathbb{K}^{(2n-1)}[X,Y]$ be the space of homogeneous polynomials of degree $2n-1$ in the variables X and Y. We fix an explicit basis of $V_{n,\mathbb{K}}$ given by the polynomials $P_k = X^{2n-1-k}Y^k$ for $k = 0,\dots,2n-1$. In particular, $\dim(V_{n,\mathbb{K}}) = 2n$. We remark that $V_{n,\mathbb{K}} = \mathrm{Sym}^{2n-1}(V_{1,\mathbb{K}})$, or, in words, that $V_{n,\mathbb{K}}$ is the $(2n-1)$-st symmetric power of $V_{1,\mathbb{K}}$.

We endow $V_{1,\mathbb{K}}$ with the symplectic form $\omega_{1,\mathbb{K}}$ determined by $\omega_{1,\mathbb{K}}(X,Y) = 1$. This induces a symplectic form $\omega_{n,\mathbb{K}} = \mathrm{Sym}^{2n-1}\omega_{1,\mathbb{K}}$ on $V_{n,\mathbb{K}}$. Explicitly, this symplectic form is determined by the following formulae:

$$\begin{cases} \omega_{n,\mathbb{K}}(P_k,P_l) = 0 & \text{if } k+l \neq 2n-1 \\ \omega_{n,\mathbb{K}}(P_k,P_{2n-1-k}) = (-1)^k\ \frac{k!(2n-1-k)!}{(2n-1)!}. & \end{cases}$$

Here, we are mainly interested in the case $n = 2$, where these formulae can be written more explicitly:

$$\begin{cases} \omega_{2,\mathbb{K}}(X^3,Y^3) = 1\,, \\ \omega_{2,\mathbb{K}}(X^2Y,XY^2) = -\frac{1}{3}\,, \end{cases} \tag{2}$$

all other pairings being zero.

The group $\mathrm{Sp}(V_{1,\mathbb{K}},\omega_{1,\mathbb{K}}) \cong \mathrm{Sp}(2,\mathbb{K}) \cong \mathrm{SL}(2,\mathbb{K})$ acts, via symmetric power, on $V_{n,\mathbb{K}}$, and this action preserves the symplectic form $\omega_{n,\mathbb{K}}$. This defines a representation

$$\iota_0 : \mathrm{SL}(2,\mathbb{K}) \to \mathrm{Sp}(V_{n,\mathbb{K}},\omega_{n,\mathbb{K}}) \cong \mathrm{Sp}(2n,\mathbb{K})$$

that is irreducible: the principal representation in $\mathrm{Sp}(2n,\mathbb{K})$. This representation induces an action of $\mathrm{SL}(2,\mathbb{K})$ on the projective space $\mathbb{P}(\mathbb{K}^{(2n-1)}[X,Y]) \cong \mathbb{KP}^{2n-1}$. This action can be described explicitly by considering how $\mathrm{SL}(2,\mathbb{K})$ moves the $2n-1$ roots of a polynomial.

A vector subspace $L \subset V_{n,\mathbb{K}}$ is called *isotropic* if $L \subset L^{\perp_{\omega_{n,\mathbb{K}}}}$, where $L^{\perp_{\omega_{n,\mathbb{K}}}}$ is the orthogonal complement with respect to $\omega_{n,\mathbb{K}}$. An isotropic subspace $L \subset V_{n,\mathbb{K}}$ is maximal if it has dimension n, or equivalently if $L = L^{\perp_{\omega_{n,\mathbb{K}}}}$. In this case, L is called a *Lagrangian* subspace. Using the fact that $\omega_{n,\mathbb{K}}$ is skew-symmetric, we can see that all the subspaces of $V_{n,\mathbb{K}}$ of dimension one are isotropic, and hence, the space of 1–dimensional isotropic subspaces can be identified with the projective space $\mathbb{P}(V_{n,\mathbb{K}})$. We

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denote the space of n–dimensional Lagrangian subspaces of $V_{n,\mathbb{K}}$ by $\mathrm{Lag}(V_{n,\mathbb{K}})$, and we will call it the *Lagrangian Grassmannian*. We can think of it as a subspace of the Grassmannian of n–dimensional subspaces in $V_{n,\mathbb{K}}$. The representation ι_0 induces an action of $\mathrm{SL}(2,\mathbb{K})$ on $\mathrm{Lag}(V_{n,\mathbb{K}})$.

One last thing we want to recall about this Lagrangian Grassmanian is its topology. There are different ways to describe the topology, but the way we will mostly use in this paper is via the subspace topology inherited from the Grassmanian space, whose topology can be described using the Plücker map or Plücker coordinates. In our case, the Plücker map is an embedding of the Grassmanian of n–planes in $V_{n,\mathbb{K}}$ into the projectivization of the n-th exterior power of $V_{n,\mathbb{K}}$:

$$\mathrm{Gr}(n, V_{n,\mathbb{K}}) \to \mathbb{P}\big(\Lambda^n(V_{n,\mathbb{K}})\big)\,,$$

which realizes $\mathrm{Gr}(n, V_{n,\mathbb{K}})$ as an algebraic variety, since the image consists of the intersection of a number of quadrics defined by the Plücker relations. To write these relations, we need to be more precise. Given $W \in \mathrm{Gr}(n, V_{n,\mathbb{K}})$, choose a basis $W_1, W_2, \dots, W_n$ of W consisting of column vectors. Let $\widehat{W}$ be the $(2n)\times n$ matrix whose columns are $W_1, W_2, \dots, W_n$. For any ordered sequence $1 \le i_1 < i_2 < \cdots < i_n \le 2n$ of n integers, let $W_{i_1,\dots,i_n}$ be the determinant of the $n \times n$ matrix given by the rows $i_1, \dots, i_n$ of $\widehat{W}$. The numbers $W_{i_1,\dots,i_n}$ are projective coordinates for W, and they satisfy the following relations:

$$\sum_{l=1}^{n+1} W_{i_1,\dots,i_{n-1},j_l} W_{j_1,\dots,\hat{j}_l,\dots j_{n+1}} = 0\,,$$

for $1 \le i_1 < i_2 < \cdots < i_{n-1} \le 2n$ and $1 \le j_1 < j_2 < \cdots < j_{n+1} \le 2n$ and where $\hat{j}_l$ denotes the fact that the j_l term is omitted. For example, in the case $n = 2$, which will be the main focus of the article, we will have coordinates $W_{1,2}$, $W_{1,3}$, $W_{1,4}$, $W_{2,3}$, $W_{2,4}$ and $W_{3,4}$, with the relation

$$W_{1,2}W_{3,4} - W_{1,3}W_{2,4} + W_{1,4}W_{2,3} = 0.$$

The condition that the space W is Lagrangian gives one additional polynomial equation, so that $\mathrm{Lag}(\mathbb{C}^4)$ is a projective variety of complex dimension 3.

4.2. $\mathrm{SL}(2,\mathbb{C})$*–orbits of* $\mathrm{Lag}(\mathbb{C}^4)$

From now on, we focus on dimension 4. So, let $V_{\mathbb{K}} = \mathbb{K}^{(3)}[X, Y] \cong \mathbb{K}^4$ be the symplectic space of homogeneous polynomials of degree 3 in X and Y, equipped with the symplectic form $\omega_{\mathbb{K}} = \omega_{2,\mathbb{K}}$ defined above. We define the *Veronese embeddings*

$$\xi^1_{\mathbb{C}}\colon\ \mathbb{CP}^1 \to \mathbb{CP}^3 \quad \text{and} \quad \xi^2_{\mathbb{C}}\colon\ \mathbb{CP}^1 \to \mathrm{Lag}(\mathbb{C}^4)$$

by

$$\begin{aligned}
&\xi^1_{\mathbb{C}}([a:b]) := \langle (bX - aY)^3 \rangle \in \mathbb{CP}^3\\
&\xi^2_{\mathbb{C}}([a:b]) := \langle (bX - aY)^3, (dX - cY)(bX - aY)^2 \rangle \in \mathrm{Lag}(\mathbb{C}^4),
\end{aligned}$$

where $[c : d]$ is any point chosen in $\mathbb{CP}^1 \setminus \{[a : b]\}$. The Lagrangian $\xi^2_{\mathbb{C}}([a : b])$ does not depend on the choice of $[c : d]$. Let

$$\xi^1 = \xi^1_{\mathbb{R}} = \xi^1_{\mathbb{C}}\,|_{\mathbb{RP}^1} \qquad \text{and} \qquad \xi^2 = \xi^2_{\mathbb{R}} = \xi^2_{\mathbb{C}}\,|_{\mathbb{RP}^1}\ .$$

Similar to Section 2.5, for a line $\ell \in \mathbb{CP}^3$, we define

$$K_\ell = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \ell \subset W\} \subset \mathrm{Lag}(\mathbb{C}^4).$$

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We now introduce the set

$$K_{\mathbb{C}} = \bigcup_{t \in \mathbb{CP}^1} K_{\xi^1_{\mathbb{C}}(t)} \subset \mathrm{Lag}(\mathbb{C}^4).$$

More explicitly, we can write

$$K_{\mathbb{C}} = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \exists\, [a:b] \in \mathbb{CP}^1 \text{ s.t. } (bX - aY)^3 \in W\};$$

that is, $K_{\mathbb{C}}$ is the set of Lagrangian subspaces that contain a polynomial with a triple complex root.

Lemma 4.1. *$K_{\mathbb{C}}$ is the set of Lagrangians $W \in \mathrm{Lag}(\mathbb{C}^4)$ with a common root; that is,*

$$K_{\mathbb{C}} = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \exists\, [a:b] \in \mathbb{CP}^1,\ \forall p \in W,\ p(X,Y) = (bX - aY)q(X,Y)\}.$$

Proof. Let $W \in K_{\mathbb{C}}$. We know that it contains an element with a triple root. By acting with $\mathrm{SL}(2,\mathbb{C})$, we can assume that the triple root is zero – in other words, that $X^3 \in W$. Let $p \in W$. We can write $p = aX^3 + bX^2Y + cXY^2 + dY^3$. Since W is isotropic, we know that $\omega_{2,\mathbb{C}}(X^3, p) = 0$, and hence, by (2), we see that $d = 0$. Hence, zero is a common root of every element of W.

Conversely, assume that all elements of W have a common root. By acting with $\mathrm{SL}(2,\mathbb{C})$, we can assume that the common root is zero, and hence, all elements of W are of the form $p = aX^3+bX^2Y+cXY^2$. By (2),

$$\omega_{2,\mathbb{C}}(a_1X^3 + b_1X^2Y + c_1XY^2, a_2X^3 + b_2X^2Y + c_2XY^2) = \frac{1}{3}(c_1b_2 - b_1c_2).$$

Hence, all polynomials in W have the same ratio $\frac{b}{c}$ (which can be infinite if $c = 0$). Given a basis p_1, p_2 of W, we can multiply one of them by a scalar to make sure they have the same coefficients b, c. Then $p_1 - p_2$ is a multiple of X^3, and hence, W has an element with a triple root. □

We can now prove the following:

Proposition 4.2. *$K_{\mathbb{C}}$ is in bijection with $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Proof. We construct an explicit bijection

$$g\colon\ \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{\cong} K_{\mathbb{C}}$$

defined by

$$g(([a:b],[c:d])) = \begin{cases} \xi^2_{\mathbb{C}}([a:b]) & \text{if } [a:b] = [c:d] \\ \langle (bX-aY)^3, (dX-cY)^2(bX-aY)\rangle & \text{if } [a:b] \neq [c:d] \end{cases}$$

Easy calculations show that the map g is well-defined and bijective. □

Remark 4.3 (The space $K_{\mathbb{R}}$). The space

$$K_{\mathbb{R}} = \bigcup_{t \in \mathbb{RP}^1} K_{\xi^1_{\mathbb{R}}(t)} \subset \mathrm{Lag}(\mathbb{C}^4)$$

is precisely the space $K_{\rho,I}$ in Section 2.5; hence, $\Omega = \mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{R}}$. Note that

$$K_{\mathbb{R}} = \{W \in \mathrm{Lag}(\mathbb{C}^4) \mid \exists\, [a:b] \in \mathbb{RP}^1 \text{ s.t. } (bX - aY)^3 \in W\}$$

corresponds to the set of Lagrangian subspaces with a triple real root $[a : b] \in \mathbb{RP}^1$. We thus see that

$$K_{\mathbb{R}} := \cup_{t \in \mathbb{RP}^1} K_{\xi^1_{\mathbb{R}}(t)} \cong \mathbb{RP}^1 \times \mathbb{CP}^1 \cong \mathbb{RP}^1 \times \mathrm{Lag}(\mathbb{C}^2)\,,$$

or, more precisely, $K_{\mathbb{R}} = g(\mathbb{RP}^1 \times \mathrm{Lag}(\mathbb{C}^2))$.

Remark 4.4 (Generalisation to $\mathrm{Lag}(\mathbb{C}^{2n})$). The second factor $\mathbb{CP}^1$ in the maps above should be interpreted as $\mathrm{Lag}(\mathbb{C}^2)$. In fact, in more generality, we can prove that, for any dimension, $K_{\mathbb{C}} \cong \mathbb{CP}^1 \times \mathrm{Lag}(\mathbb{C}^{2(n-1)})$ and $K_{\mathbb{R}} \cong \mathbb{RP}^1 \times \mathrm{Lag}(\mathbb{C}^{2(n-1)})$.

Lemma 4.5. *Every Lagrangian W contains a polynomial*

$$p(X,Y) = (bX - aY)^2(dX - cY)$$

with a double root $[a : b] \in \mathbb{CP}^1$ and a single root $[c : d] \neq [a : b] \in \mathbb{CP}^1$.

Proof. If W has a polynomial with a triple root, then $W \in K_{\mathbb{C}}$, and we saw above that these Lagrangians also contain a polynomial with a double root and a single root. If W does not contain a polynomial with a triple root but has a polynomial with a double root, the third root must be distinct, and hence, we are done.

Assume now that W has a polynomial with three distinct roots. Acting with $\mathrm{SL}(2,\mathbb{C})$, we can assume that the three roots are $[-1 : 1], [0 : 1], [1 : 0]$, and hence that $XY(X + Y) = X^2Y + XY^2 \in W$. Let $p \in W$, $p = aX^3 + bX^2Y + cXY^2 + dY^3$. By (2), we have

$$\omega_{2,\mathbb{C}}(XY(X + Y), p) = \frac{1}{3}(b - c)\,,$$

which implies that $b = c$. Hence, $W = \langle XY(X + Y), aX^3 + dY^3\rangle$, and W is determined by $[a : d]$. The other elements of W are scalar multiples of the ones of the form

$$q = aX^3 + \beta X^2Y + \beta XY^2 + dY^3.$$

We want to find elements with a double root. We can find them using the discriminant. The *discriminant* Δ of a degree–3 polynomial $\alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3$ is defined by

$$\Delta(\alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3) := \beta^2\gamma^2 - 4\alpha\gamma^3 - 4\delta\beta^3 - 27\alpha^2\delta^2 + 18\alpha\beta\gamma\delta\,,$$

and polynomials with a double root correspond to the zeros of the discriminant. In our case, we have

$$\Delta(q) = \beta^4 - 4(a + d)\beta^3 + 18ad\beta^2 - 27a^2d^2.$$

A non-constant polynomial has always at least one solution over the complex numbers; hence, for every value of a and d, we can find a β such that q has a double root. □

The next statement summarizes the results of this subsection, describing the three orbits of the $\mathrm{SL}(2,\mathbb{C})$–action.

Theorem 4.6. *There are three $\mathrm{SL}(2,\mathbb{C})$–orbits in $\mathrm{Lag}(\mathbb{C}^4)$:*

- $\xi^2_{\mathbb{C}}(\mathbb{CP}^1) = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^3, X^2Y\rangle$ *is the only closed orbit, and it is in bijection with the diagonal* $\Delta \subset K_{\mathbb{C}} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.
- $K_{\mathbb{C}} \setminus \xi^2_{\mathbb{C}}(\mathbb{CP}^1) = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^3, XY^2\rangle$ *is not open nor closed, and it is in bijection with* $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$.
- $\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^2Y, X^3 + Y^3\rangle$ *is the only open orbit.*

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Proof. We have already discussed above the bijection $g\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{\cong} K_{\mathbb{C}}$. From the discussion above, you can see that

- $\xi^2_{\mathbb{C}}(\mathbb{CP}^1) = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^3, X^2Y\rangle$ corresponds to Lagrangians all of whose polynomials share a common double root;
- $K_{\mathbb{C}} \setminus \xi^2_{\mathbb{C}}(\mathbb{CP}^1) = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^3, XY^2\rangle$ corresponds to Lagrangians all of whose polynomials share a common single root.

Since $\xi^2_{\mathbb{C}}$ is an embedding, $\xi^2_{\mathbb{C}}(\mathbb{CP}^1) \cong \mathbb{CP}^1$ is closed.

To complete the proof, we need to show that $\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{C}}$ is one $\mathrm{SL}(2,\mathbb{C})$ orbit. Given $W \notin K_{\mathbb{C}}$, by Lemma 4.5, W contains a polynomial with a double root and a single root. Acting with a matrix in $\mathrm{SL}(2,\mathbb{C})$, we can assume that the roots are $[0:1]$ and $[1:0]$ (i.e., that $X^2Y \in W$). Let $p \in W$, $p = aX^3 + bX^2Y + cXY^2 + dY^3$. By (2), we have

$$\omega_{2,\mathbb{C}}(X^2Y, p) = -\frac{1}{3}c\,,$$

which implies that $c = 0$. Hence, $W = \left\langle X^2Y, aX^3 + dY^3\right\rangle$. Now we act again with a matrix $A \in \mathrm{SL}(2,\mathbb{C})$. We choose $\alpha = \sqrt[6]{\frac{d}{a}}$, one of the 6 complex roots, and we consider the matrix

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

When acting on $\mathbb{CP}^1$, A fixes $[0:1]$ and $[1:0]$ and hence sends X^2Y to one of its multiples. Moreover, A sends $aX^3 + dY^3$ to a multiple of $X^3 + Y^3$. Hence, $A \cdot W = \left\langle X^2Y, X^3 + Y^3\right\rangle$, and this proves that

$$\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^2Y, X^3 + Y^3\rangle.$$

□

4.3. The space of regular ideal tetrahedra

Recall that an ideal hyperbolic tetrahedron in $\mathbb{H}^3$ is called *regular* when all the dihedral angles are equal to each other (and equal to $\frac{\pi}{3}$). These tetrahedra can also be characterized by their volume or their cross-ratio, since a tetrahedron is regular if and only if it has maximal volume, if and only if the cross-ratio of its vertices is $\frac{1-\sqrt{3}i}{2}$. Recall that given 4 points z_1, z_2, z_3, z_4 in $\mathbb{CP}^1$, here seen as the boundary at infinity of $\mathbb{H}^3$, we define their *cross-ratio* as

$$[z_1, z_2, z_3, z_4] = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

Equivalently, $[z_1, z_2, z_3, z_4] = Az_4$, where $A \in \mathrm{PSL}(2,\mathbb{C})$ is defined by $Az_1 = \infty$, $Az_2 = 0$, and $Az_3 = 1$. When computing the cross-ratio of the vertices of a tetrahedron, we use the orientation of $\mathbb{H}^3$, and we choose the order of the vertices in a way that is compatible with the orientation. More precisely, if z_1 is a vertex, the other three vertices lie on the boundary of a copy of the hyperbolic plane. When watched from z_1, we require that the vertices z_2, z_3, z_4 rotate in the clockwise direction. Note that if you change the order of the points, in a way that is still compatible with the orientation of $\mathbb{H}^3$ as explained, then the cross-ratio z can become $1 - \frac{1}{z}$ or $\frac{1}{1-z}$. However, if $z_0 = \frac{1-\sqrt{3}i}{2}$, then $z_0 = 1 - \frac{1}{z_0} = \frac{1}{1-z_0}$, so the characterization of regular tetrahedra in terms of cross-ratio does not depend on the chosen order of the vertices when calculating the cross-ratio, as it should be. We will discuss more properties of regular ideal hyperbolic tetrahedra in Section 5.1.

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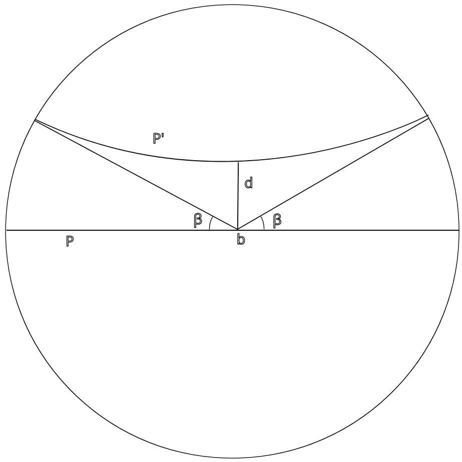

Figure 1. *The distance d between the two geodesics.*

We will denote by $\mathfrak{T}_{\mathbb{H}^3}$ the space of regular ideal hyperbolic tetrahedra in $\mathbb{H}^3$. In order to define a topology on $\mathfrak{T}_{\mathbb{H}^3}$, we embed it in the symmetric product

$$\mathrm{Sym}^4\left(\mathbb{CP}^1\right) = \left(\mathbb{CP}^1\right)^4/S_4.$$

The group $\mathrm{SL}(2,\mathbb{C})$ acts transitively on $\mathfrak{T}_{\mathbb{H}^3}$ via hyperbolic isometries. Moreover, the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ gives a way to extend this action to the whole symmetric product.

The homogeneous space $\mathfrak{T}_{\mathbb{H}^3}$ is open but not closed in the symmetric product. We will now describe its closure; this gives an $\mathrm{SL}(2,\mathbb{C})$–invariant compactification of $\mathfrak{T}_{\mathbb{H}^3}$.

Proposition 4.7. *The topological frontier of* $\mathfrak{T}_{\mathbb{H}^3}$ *in* $\mathrm{Sym}^4(\mathbb{CP}^1)$ *consists of all the points in the symmetric product where one element has multiplicity at least 3. Hence, the frontier is homeomorphic to* $\mathbb{CP}^1 \times \mathbb{CP}^1$*, where the first coordinate contains the element with high multiplicity.*

Proof. Given a regular ideal tetrahedron T, denote by $b \in \mathbb{H}^3$ its barycenter. The 4 lines from b to the 4 vertices meet at b, each pair of lines with the same angle, that we will denote here by α. Note that $\frac{\pi}{2} < \alpha < \pi$. Denote by $\beta = \frac{1}{2}(\pi - \alpha)$, where $0 < \beta < \frac{\pi}{4}$.

We construct the distance d as in Figure 1: Given a plane P passing through b, we construct the cone with angle β from P, and we consider the plane P' tangent to the cone at infinity. The distance d is defined as the distance between the two planes P and P'. The quantities α, β and d can be made explicit, but we do not need their explicit values here.

We claim that given a regular ideal tetrahedron with barycenter contained in the open half-space below P, then at most one of its vertices can be in the closed half-space above P', and the other three vertices must lie in the open half-space below P'. This is because of the way we have chosen the distance d between P and P'. Assume that a regular ideal tetrahedron has barycenter at O, in the open half-space below P. Assume by contradiction that at least two of its vertices are in the closed half-space above P'. Then the line between the two vertices lies entirely in the closed half-space above P', but it must be at distance d from O, a contradiction.

Now consider a sequence of regular ideal tetrahedra that converges to a point of the symmetric product that does not represent a regular ideal tetrahedron. The barycenters of the tetrahedra of the sequence are unbounded in $\mathbb{H}^3$; otherwise, a subsequence would converge to a regular ideal tetrahedron. Up to subsequences, we can assume that the barycenters of the tetrahedra converge to a point $x \in \mathbb{CP}^1$.

We consider a small ball in $\mathbb{CP}^1$ containing x. This small ball bounds a plane P'. Let P denote a plane whose boundary lies in the ball and whose distance from P' is at least d. For all the tetrahedra in our

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subsequence whose barycenter lies in the half-space bounded by P, we know from the considerations above that at least three of their vertices lie in the half-space bounded by P'.

This shows that for our subsequence, at least three of the vertices of the tetrahedra converge to the same point x, while the fourth vertices are free to go anywhere. Again up to subsequences, we can assume that the fourth vertices are also converging to some point in $\mathbb{CP}^1$.

This implies that the limit of our original sequence has an element with multiplicity at least 3. □

We will call the points of the frontier *degenerate tetrahedra*. We will denote the compactified space by $\mathfrak{T}_{\overline{\mathbb{H}^3}}$. This space carries a natural $\mathrm{SL}(2,\mathbb{C})$–action and can be described as

$$\mathfrak{T}_{\overline{\mathbb{H}^3}} = \mathfrak{T}_{\mathbb{H}^3} \cup \mathbb{CP}^1 \times \mathbb{CP}^1.$$

We can now state the main result for this section, which identifies the $\mathrm{SL}(2,\mathbb{C})$–action on $\mathrm{Lag}(\mathbb{C}^4)$ with the one on $\mathfrak{T}_{\overline{\mathbb{H}^3}}$. This will be a key step in the proof of Theorem D.

In order to do this, we will use Theorem 4.6, but we still need to identify the unique open $\mathrm{SL}(2,\mathbb{C})$–orbit in $\mathrm{Lag}(\mathbb{C}^4)$ with the space $\mathfrak{T}_{\mathbb{H}^3}$, and verify the the map between the two spaces is continuous.

Theorem 4.8. *The space* $\mathrm{Lag}(\mathbb{C}^4)$ *is* $\mathrm{SL}(2,\mathbb{C})$*–equivariantly homeomorphic to the space*

$$\mathfrak{T}_{\overline{\mathbb{H}^3}} := \mathfrak{T}_{\mathbb{H}^3} \cup \mathbb{CP}^1 \times \mathbb{CP}^1.$$

Proof. We will construct an equivariant homeomorphism between the two spaces. Since both spaces are compact Hausdorff, it will be enough to check that the equivariant map is a continuous bijection. As a first step, we will show that $\mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \cdot \langle X^2Y, X^3 + Y^3\rangle$ is in bijection with the space $\mathfrak{T}_{\mathbb{H}^3}$ of regular ideal hyperbolic tetrahedra in $\mathbb{H}^3$.

Consider the Lagrangian subspace $W = \langle X^2Y, X^3 + Y^3\rangle$. We will see that it contains exactly 4 'special' polynomials which have a double root. To find them, we use the discriminant as in the proof of Lemma 4.5. The elements of W are of the form

$$q = \alpha X^3 + \beta X^2Y + \alpha Y^3\,;$$

hence, the discriminant is

$$\Delta(q) = -\alpha(4\beta^3 + 27\alpha^3).$$

We have that $\Delta(q) = 0$ if and only if

1. $[\alpha : \beta] = [0 : 1]$;
2. $[\alpha : \beta] = \left[-\frac{\sqrt[3]{4}}{3} : 1\right]$;
3. $[\alpha : \beta] = \left[\frac{1}{3\sqrt[3]{2}} - i\frac{1}{\sqrt[3]{2}\sqrt{3}} : 1\right]$;
4. $[\alpha : \beta] = \left[\frac{1}{3\sqrt[3]{2}} + i\frac{1}{\sqrt[3]{2}\sqrt{3}} : 1\right]$.

The associated polynomials have double and single roots, respectively, given by

1. 0 and ∞;
2. $\sqrt[3]{2}$ and $-\frac{1}{\sqrt[3]{4}}$;
3. $\frac{-1-i\sqrt{3}}{\sqrt[3]{4}}$ and $\frac{1+i\sqrt{3}}{2\sqrt[3]{4}}$;
4. $\frac{-1+i\sqrt{3}}{\sqrt[3]{4}}$ and $\frac{1-i\sqrt{3}}{2\sqrt[3]{4}}$.

The vertices corresponding to the 4 double roots define an ideal hyperbolic tetrahedron $T = \left\{0, \sqrt[3]{2}, \frac{-1-i\sqrt{3}}{\sqrt[3]{4}}, \frac{-1+i\sqrt{3}}{\sqrt[3]{4}}\right\}$, and the vertices corresponding to the 4 single roots define a 'dual' ideal hyperbolic tetrahedron $T_{dual} = \left\{\infty, -\frac{1}{\sqrt[3]{4}}, \frac{1+i\sqrt{3}}{2\sqrt[3]{4}}, \frac{1-i\sqrt{3}}{2\sqrt[3]{4}}\right\}$. The tetrahedron T_{dual} is the image of T by the

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central symmetry centered at the barycenter of T. A simple calculation shows that the cross-ratio of the vertices of T and T_{dual} are equal to $\frac{1-\sqrt{3}i}{2}$:

$$\left[0, \sqrt[3]{2}, \frac{-1+i\sqrt{3}}{\sqrt[3]{4}}, \frac{-1-i\sqrt{3}}{\sqrt[3]{4}}\right] = \left[\infty, -\frac{1}{\sqrt[3]{4}}, \frac{1+i\sqrt{3}}{2\sqrt[3]{4}}, \frac{1-i\sqrt{3}}{2\sqrt[3]{4}}\right] = \frac{1-\sqrt{3}i}{2}.$$

Hence, T and T_{dual} are regular ideal hyperbolic tetrahedra, as we wanted to prove. Notice that in the computation of the cross-ratio for T, we changed the order of the vertices to make sure it is compatible with the orientation of $\mathbb{H}^3$.

Now we want to see that the bijections described above are actually homeomorphisms. Remember also that the topology of $\mathrm{Lag}(\mathbb{C}^4)$ can be described by considering $\mathrm{Lag}(\mathbb{C}^4) \subset \mathrm{Gr}(2, \mathbb{C}^4)$ and using the Plücker coordinates for the topology of $\mathrm{Gr}(2, \mathbb{C}^4)$.

To prove this result, we will extend the map g, defined above, to a map

$$g\colon \mathfrak{T}_{\mathbb{H}^3} \cup \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathrm{Lag}(\mathbb{C}^4)\,,$$

and check that the map is a homeomorphism.

Given a tetrahedron $T = \{v_1, \dots, v_4\} \in \mathfrak{T}_{\mathbb{H}^3}$, we define the *dual tetrahedron* $T_{dual} = \{v_1^{dual}, \dots, v_4^{dual}\}$ as the tetrahedron (also in $\mathfrak{T}_{\mathbb{H}^3}$) with vertices v_i^{dual} such that v_i, the barycenter b of T and v_i^{dual} lie on the same geodesic for $i = 1, \dots, 4$. Let v_i and v_j be two distinct vertices of T. If we let $v_i = [a_1 : b_1] \in \mathbb{CP}^1$, $v_i^{dual} = [c_1 : d_1] \in \mathbb{CP}^1$, $v_j = [a_2 : b_2] \in \mathbb{CP}^1$, and $v_j^{dual} = [c_2 : d_2] \in \mathbb{CP}^1$, then the Lagrangian subspace associated to T is

$$g(T) = W := \langle (b_1X - a_1Y)^2(d_1X - c_1Y), (b_2X - a_2Y)^2(d_2X - c_2Y)\rangle \in \mathrm{Lag}(\mathbb{C}^4).$$

Its Plücker coordinates are

- $W_{1,2} = -b_1^2d_1(b_2^2c_2 + 2a_2b_2d_2) + b_2^2d_2(b_1^2c_1 + 2a_1b_1d_1)$;
- $W_{1,3} = b_1^2d_1(a_2^2d_2 + 2a_2b_2c_2) - b_2^2d_2(a_1^2d_1 + 2a_1b_1c_1)$;
- $W_{1,4} = -b_1^2d_1a_2^2c_2 + b_2^2d_2a_1^2c_1$;
- $W_{2,3} = -(b_1^2c_1 + 2a_1b_1d_1)(a_2^2d_2 + 2a_2b_2c_2) + (b_2^2c_2 + 2a_2b_2d_2)(a_1^2d_1 + 2a_1b_1c_1)$;
- $W_{2,4} = a_2^2c_2(b_1^2c_1 + 2a_1b_1d_1) - a_1^2c_1(b_2^2c_2 + 2a_2b_2d_2)$;
- $W_{3,4} = -a_2^2c_2(a_1^2d_1 + 2a_1b_1c_1) + a_1^2c_1(a_2^2d_2 + 2a_2b_2c_2)$.

Similarly, given a point $([a : b], [c : d]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$, where $\Delta = \{([a : b], [a : b]) \in \mathbb{CP}^1 \times \mathbb{CP}^1\}$ is the diagonal, then the associated Lagrangian subspace is

$$U = \langle (bX - aY)^3, (bX - aY)(dX - cY)^2\rangle \in \mathrm{Lag}(\mathbb{C}^4)\,,$$

which has Plücker coordinates

- $U_{1,2} = 2b^3d(bc - ad)$;
- $U_{1,3} = b^2(3ad + bc)(bc - ad)$;
- $U_{1,4} = -ba(bc + ad)(bc - ad)$;
- $U_{2,3} = -3ba(bc + ad)(bc - ad)$;
- $U_{2,4} = a^2(ad + 3bc)(bc - ad)$;
- $U_{3,4} = -2a^3c(bc - ad)$.

Lastly, given a point $([a : b], [a : b]) \in \Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$, the associated Lagrangian subspace is

$$Z = \langle (bX - aY)^3, (bX - aY)^2(dX - cY)\rangle \in \mathrm{Lag}(\mathbb{C}^4)\,,$$

for $[c : d] \neq [a : b] \in \mathbb{CP}^1$, and Z has Plücker coordinates

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- $Z_{1,2} = -2b^4(bc - ad)$;
- $Z_{1,3} = 2ab^3(bc - ad)$;
- $Z_{1,4} = -a^2b^2(bc - ad)$;
- $Z_{2,3} = -3a^2b^2(bc - ad)$;
- $Z_{2,4} = 2a^3b(bc - ad)$;
- $Z_{3,4} = -a^4(bc - ad)$.

We saw in Proposition 4.7 that when a sequence of tetrahedra in $\mathfrak{T}_{\mathbb{H}^3}$ degenerates, their barycenters converge to a point in $\mathbb{CP}^1$. In that case, at least three vertices will converge to the same point in $\mathbb{CP}^1$, so tetrahedra can only degenerate to points in $\mathbb{CP}^1 \times \mathbb{CP}^1$, that is $\mathbb{CP}^1 \times \mathbb{CP}^1 = \partial\mathfrak{T}_{\mathbb{H}^3}$.

We can now see that $g|_{\mathbb{CP}^1\times\mathbb{CP}^1}$ and $g|_{\mathfrak{T}_{\mathbb{H}^3}}$ are continuous. For the first case, we only have to consider the expression of the Plücker coordinates and look at the case $([a_n : b_n], [c_n : d_n]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$ such that $([a_n : b_n], [c_n : d_n]) \to ([a : b], [a : b]) \in \Delta$. In particular, we can do the calculations in the case that $[a : b] = [0 : 1] \in \mathbb{CP}^1$. If we denote $U^n_{i,j}$ the Plücker coordinates associated to $F(([a_n : b_n], [c_n : d_n]))$, we can see that the only nonzero coordinate in the limit is $U^n_{1,2}$, as we wanted. For the second case $g|_{\mathfrak{T}_{\mathbb{H}^3}}$, again, we only have to consider the expression of the Plücker coordinates in term of the vertices of the tetrahedra. Hence, we are left with the discussion of converging sequences $\{T_n\}$ of tetrahedra in $\mathfrak{T}_{\mathbb{H}^3}$ such that $T_n \to T_\infty \in \mathbb{CP}^1 \times \mathbb{CP}^1$. We have two possibilities:

- $T_\infty = ([a : b], [c : d]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta$.
- $T_\infty = ([a : b], [a : b]) \in \Delta$.

In the first case, three vertices of the tetrahedron $\{T_n\}$ and all the dual vertices of the tetrahedron $\{T_n^{dual}\}$ converge to $[a : b] \in \mathbb{CP}^1$, while in the second case, all the four vertices of the tetrahedron $\{T_n\}$ and at least three dual vertices of the tetrahedron $\{T_n^{dual}\}$ converge to $[a : b] \in \mathbb{CP}^1$. In particular, in the first case we, choose vertices $v_i^n = [a_1^n : b_1^n] \in \mathbb{CP}^1$ and $v_j^n = [a_2^n : b_2^n] \in \mathbb{CP}^1$ of T_n with dual vertices $(v_i^n)^{dual} = [c_1^n : d_1^n] \in \mathbb{CP}^1$ and $(v_j^n)^{dual} = [c_2^n : d_2^n] \in \mathbb{CP}^1$, such that

- $[a_1^n : b_1^n] \to [a : b]$;
- $[c_1^n : d_1^n] \to [a : b]$;
- $[a_2^n : b_2^n] \to [c : d]$;
- $[c_2^n : d_2^n] \to [a : b]$.

We can also assume $[a : b] = [0 : 1]$ and $[c : d] = [1 : 0]$. If we denote $W^n_{i,j}$ the Plücker coordinates associated to $g(T_n)$, we can see that the only nonzero coordinate in the limit is $W^n_{1,3}$, as we wanted.

In the second case, we choose vertices $v_i^n = [a_1^n : b_1^n] \in \mathbb{CP}^1$ and $v_j^n = [a_2^n : b_2^n] \in \mathbb{CP}^1$ of T_n with dual vertices $(v_i^n)^{dual} = [c_1^n : d_1^n] \in \mathbb{CP}^1$ and $(v_j^n)^{dual} = [c_2^n : d_2^n] \in \mathbb{CP}^1$, such that

- $[a_1^n : b_1^n] \to [a : b]$;
- $[a_2^n : b_2^n] \to [a : b]$.

Again, we can assume $[a : b] = [0 : 1]$. Let's denote $W^n_{i,j}$ the Plücker coordinates associated to $F(T_n)$. We have two cases:

- At least one of the sequences $[c_1^n : d_1^n]$ or $[c_2^n : d_2^n]$ do not converge to $[a : b]$. Then we can see that the only nonzero coordinate in the limit is $W^n_{1,2}$, as we wanted.
- If both $[c_1^n : d_1^n], [c_2^n : d_2^n]$ converge to $[a : b]$, then we need to be more careful and analyze the rate of convergence, but after diving all coordinates by c_1^n or c_2^n, we can see that the only nonzero coordinate in the limit is $W^n_{1,2}$, as we wanted.

□

In the following, we will denote by

$$g^{-1} : \mathrm{Lag}(\mathbb{C}^4) \longrightarrow \mathfrak{T}_{\overline{\mathbb{H}^3}}$$

the $\mathrm{SL}(2, \mathbb{C})$–equivariant map defined in the previous theorem.

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4.4. The domain $\Omega \subset \mathrm{Lag}(\mathbb{C}^4)$

We will use the barycenters of the tetrahedra to describe interesting subsets of $\mathrm{Lag}(\mathbb{C}^4)$. The barycenter of a regular ideal tetrahedron is a standard notion in hyperbolic geometry. For a degenerate tetrahedron, determined by a pair $(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, we define the barycenter as its first coordinate, here x. Recall that, when seen as an element of the symmetric product, the first coordinate is the point of $\mathbb{CP}^1$ that appears with higher multiplicity.

We now introduce the map

$$\pi_\beta \colon \mathfrak{T}_{\overline{\mathbb{H}^3}} \to \overline{\mathbb{H}^3} := \mathbb{H}^3 \cup \mathbb{CP}^1$$

that sends a (possibly degenerate) tetrahedron to its barycenter. This map is continuous: this follows from the same arguments we used in the proof of Proposition 4.7. In the rest of the paper, we will use a special notation for the inverse images of sets of $\overline{\mathbb{H}^3}$ under this map: if $S \subset \overline{\mathbb{H}^3}$ is any subset, we will denote by $\mathfrak{T}_S$ the set

$$\mathfrak{T}_S := \pi_\beta^{-1}(S).$$

So, for example, we can denote the compactification of $\mathfrak{T}_{\mathbb{H}^3}$ as

$$\mathfrak{T}_{\overline{\mathbb{H}^3}} = \mathfrak{T}_{\mathbb{H}^3} \cup \mathfrak{T}_{\mathbb{CP}^1},$$

where $\mathfrak{T}_{\mathbb{CP}^1} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$ is the set of degenerate tetrahedra. By considering the map $Q := \pi_\beta \circ g^{-1}$ we obtain the following:

Corollary 4.9. *There is a continuous* $\mathrm{SL}(2, \mathbb{C})$*–equivariant projection*

$$Q\colon \mathrm{Lag}(\mathbb{C}^4) \to \overline{\mathbb{H}^3}.$$

This corollary will help us to understand the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^4)$ because we understand very well the action on $\overline{\mathbb{H}^3}$. We will now show an example of how this works. We restrict our attention to the action of $\mathrm{SL}(2, \mathbb{R}) < \mathrm{SL}(2, \mathbb{C})$. This smaller subgroup preserves a copy of $\mathbb{H}^2 \subset \mathbb{H}^3$ and of its boundary $\mathbb{RP}^1 \subset \mathbb{CP}^1$. We define the map

$$\pi_{\mathcal{P}}\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^2}$$

as the orthogonal projection into the hyperbolic plane $\mathcal{P}$ bounded by $\mathbb{RP}^1 \subset \mathbb{CP}^1$. This map is only $\mathrm{SL}(2, \mathbb{R})$–equivariant. By composition, we obtain a projection

$$\pi_{\mathcal{P}} \circ Q\colon \mathrm{Lag}(\mathbb{C}^4) \to \overline{\mathbb{H}^2}.$$

By Remark 4.3, the inverse image of $\mathbb{RP}^1 = \partial\mathbb{H}^2$ is the set $K_{\mathbb{R}}$, and the inverse image of $\mathbb{H}^2$ is the set

$$\Omega = \mathrm{Lag}(\mathbb{C}^4) \setminus K_{\mathbb{R}}.$$

Restricting the map $\pi_{\mathcal{P}} \circ Q$ to Ω, we obtain

$$q = \pi_{\mathcal{P}} \circ Q|_{\Omega}\colon \Omega \to \mathbb{H}^2,$$

an $\mathrm{SL}(2, \mathbb{R})$–equivariant map from Ω to $\mathbb{H}^2$, which is a fiber bundle by Lemma 3.7. Notice that the map q is proper, because it is the restriction of the map $\pi_{\mathcal{P}} \circ Q$ which is defined on a compact space.

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We will identify $\mathbb{H}^2$ with the hyperbolic plane $\mathcal{P} \subset \mathbb{H}^3$. We denote by $\mathcal{O} \in \mathbb{H}^2 \subset \mathbb{H}^3$ the point $\mathcal{O} = (0, 1) \in \mathbb{C} \times \mathbb{R}_{>0}$, and by F the fiber of q over this point:

$$F = q^{-1}(\mathcal{O}) \subset \Omega.$$

Since $q\colon \ \Omega \to \mathbb{H}^2$ is a fiber bundle over a contractible base, we conclude the following result:

Corollary 4.10. *The space Ω is homeomorphic to the product $F \times \mathbb{H}^2$; hence, Ω deformation retracts to F.*

In the following sections, we will describe the topology of F. Since F is homotopy equivalent to Ω, and Ω is homotopically equivalent to our smooth fiber $\mathfrak{F}$, the information we will find about F will allow us to determine $\mathfrak{F}$.

5. Description of the fiber from the tetrahedra

We consider the geodesic $\overline{\ell} := \pi_{\mathcal{P}}^{-1}(\mathcal{O})$, where $\pi_{\mathcal{P}}\colon \ \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^2}$ is the orthogonal projection, and $\mathcal{O} \in \mathbb{H}^2$ is the point introduced at the end of the previous section. The geodesic $\overline{\ell}$ joins the points at infinity i and $-i$. We denote $\ell := \overline{\ell} \cap \mathbb{H}^3$, so we have $\overline{\ell} = \ell \cup \{\pm i\}$. We denote by ℓ^+ the ray of ℓ from $\mathcal{O}$ to i, and by ℓ^- the ray from $\mathcal{O}$ to $-i$. In both cases, $\mathcal{O}$ is included, and $\pm i$ is not. Similarly, we denote by $\overline{\ell^+}$ and $\overline{\ell^-}$ the compactified rays that include $\pm i$.

We will identify $\overline{\ell}$ with the segment $[-\infty, \infty]$ via the homeomorphism

$$\eta\colon \ \overline{\ell} \to [-\infty, \infty]$$

defined by the following properties:

- $\eta(\mathcal{O}) = 0$,
- $\eta(\pm i) = \pm\infty$,
- for every $x \in \ell^+$, $\eta(x) = d_{\mathbb{H}^3}(\mathcal{O}, x)$, and
- for any $x \in \ell^-$, $\eta(x) = -d_{\mathbb{H}^3}(\mathcal{O}, x)$.

We define the space $\mathfrak{T}_{\overline{\ell}}$ consisting of (possibly degenerate) tetrahedra with (possibly degenerate) barycenter on the geodesic $\overline{\ell}$. This space is homeomorphic to the fiber F: recall that in Theorem 4.8, we constructed an explicit homeomorphism from the space of (unlabelled) regular ideal tetrahedra to the Lagrangian Grassmannian

$$g\colon \ \mathfrak{T}_{\overline{\mathbb{H}^3}} = \mathfrak{T}_{\mathbb{H}^3} \cup \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathrm{Lag}(\mathbb{C}^4).$$

Then, the fiber $F = q^{-1}(\mathcal{O}) \subset \Omega \subset \mathrm{Lag}(\mathbb{C}^4)$ for the projection $q = \pi_{\mathcal{P}} \circ Q|_{\Omega}\colon \ \Omega \to \mathbb{H}^2$ is exactly the image $g(\mathfrak{T}_{\overline{\ell}})$.

Our main aim in this section will be to describe the space $\mathfrak{T}_{\overline{\ell}}$. It will be useful to distinguish between three subsets: the open subset $\mathfrak{T}_{\ell}$ consisting of tetrahedra with barycenter in ℓ, and the closed subsets $\mathfrak{T}_i$ and $\mathfrak{T}_{-i}$ consisting of degenerate tetrahedra with barycenter in i or $-i$, respectively.

We have that

$$\mathfrak{T}_i = \{i\} \times \mathbb{CP}^1, \qquad \mathfrak{T}_{-i} = \{-i\} \times \mathbb{CP}^1.$$

The space $\mathfrak{T}_{\ell}$ will be described in Section 5.1. The shape of each of the three pieces, $\mathfrak{T}_{\ell}, \mathfrak{T}_i, \mathfrak{T}_{-i}$ is easy to understand. The most interesting thing is to describe how they are glued together, which is done in Section 5.2.

We consider the upper half space model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$ of hyperbolic space. In this model, the compactified hyperbolic space is $\overline{\mathbb{H}^3} = \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$, and its boundary is $\partial\mathbb{H}^3 = \mathbb{C} \times \{0\} \cup \{\infty\} = \mathbb{CP}^1$. In the following, with a slight abuse of notation, we will simply use complex numbers (or ∞) to denote points of $\partial\mathbb{H}^3 = \mathbb{CP}^1$. We identify $\mathbb{H}^2$ with the plane $\mathcal{P} = \mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{H}^3$ whose boundary is $\partial\mathbb{H}^2 = \mathbb{RP}^1 \subset \mathbb{CP}^1$. Note that $\mathrm{PSL}(2,\mathbb{R})$ acts preserving $\mathcal{P}$.

5.1. Tetrahedra with a fixed barycenter

For any $c \in \mathbb{H}^3$, let $\mathfrak{T}_c$ be the set of regular ideal unlabelled tetrahedra with barycenter c. All the spaces $\mathfrak{T}_c$ are homeomorphic to each other. The space $\mathfrak{T}_\ell$ is homeomorphic to $\mathfrak{T}_c \times \mathbb{R}$ for any choice of c. We will now describe $\mathfrak{T}_c$.

Note that for all $c \in \ell$, the space $\mathfrak{T}_c$ is homeomorphic to

$$\mathfrak{T}_c \cong (T^1(\mathbb{S}^2))/A_4 \cong \mathrm{SO}(3)/A_4 \cong T^{1,orb}(\mathbb{S}^2/A_4).$$

This is a Seifert fibered space — an orbifold–$\mathbb{S}^1$–bundle over the 2–orbifold $\mathbb{S}^2(2,3,3) = \mathbb{S}^2/A_4$ — and it corresponds to the space described by Martelli [Mar22] in the second line of Table 10.6 for $q = -2$. The structure of Seifert fibered manifold of $\mathfrak{T}_c$ can be described geometrically.

Consider the action of $\mathrm{SO}(2)$ on $\mathbb{H}^3$ via rotations that fix ℓ. Since we are assuming that $c \in \ell$, this induces an action of $\mathrm{SO}(2)$ on $\mathfrak{T}_c$. The orbits of this action are the fibers of the Seifert fibration. The three circles associated with the three singular fibers correspond to tetrahedra (with barycenter at the point c) with special symmetries:

(i) the circles associated with the order–3 cone points correspond to tetrahedra in $\mathfrak{T}_c$ with one vertex in $-i$ or i, respectively,
(ii) the circle associated with the order–2 cone point corresponds to tetrahedra in $\mathfrak{T}_c$ with two sides orthogonal to ℓ.

We want to decompose $\mathfrak{T}_c$ in two sets:

$$\mathfrak{T}_c = \mathfrak{T}_c^{\uparrow} \cup \mathfrak{T}_c^{\downarrow}.$$

As above, we denote by ℓ_c^+ the ray of ℓ from c to i, and by ℓ_c^- the ray from c to $-i$. In both cases, c is included, and $\pm i$ are not. Similarly, we denote by $\overline{\ell_c^+}$ and $\overline{\ell_c^-}$ the compactified rays that include $\pm i$. The boundary of the tetrahedra in the family in (ii) (with order–2 symmetry) will intersect ℓ in two points: one in ℓ_c^- (which we will denote A_c) and the other in ℓ_c^+. Let B_c be the point in ℓ_c^- between $-i$ and A_c and at (hyperbolic) distance 1 from A_c. Let C_c be circle in $\mathbb{CP}^1$ that bounds the plane in $\mathbb{H}^3$ orthogonal to ℓ and intersecting it at B_c and let D_c (resp. $\overline{D_c}$) be the open (resp. closed) disk in $\mathbb{CP}^1$ with boundary C_c and containing $-i$.

With this notation, we can define $\mathfrak{T}_c^{\uparrow}, \mathfrak{T}_c^{\downarrow}$ as follows:

- $\mathfrak{T}_c^{\uparrow}$ is the set of tetrahedra in $\mathfrak{T}_c$ such that all vertices are in $\mathbb{CP}^1 \setminus D_c$;
- $\mathfrak{T}_c^{\downarrow}$ is the set of tetrahedra in $\mathfrak{T}_c$ such that one of their vertices is in D_c.

The set $\mathfrak{T}_c^{\uparrow}$ is closed, and the set $\mathfrak{T}_c^{\downarrow}$ is open. They share a common boundary $\partial\mathfrak{T}_c^{\uparrow} = \partial\mathfrak{T}_c^{\downarrow}$, the set of tetrahedra in $\mathfrak{T}_c$ such that one of their vertices is in C_c. This decomposition is motivated by the following property:

Proposition 5.1. *For every $c \in \ell$ and for every $T \in \overline{\mathfrak{T}_c^{\downarrow}}$, the tetrahedron T has exactly one vertex in $\overline{D_c}$.*

Proof. The proof is elementary, but it requires several computations. We will break it into three claims and a main argument following the claims.

Claim 1: The (hyperbolic) distance between the barycenter and the faces of a regular tetrahedron $T \in \mathfrak{T}_{\mathbb{H}^3}$ is $\ln\sqrt{2}$. To see that, we consider the tetrahedron T with vertices $\{\infty, -1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}\}$. We can check that T is regular by calculating the cross-ratio $[\infty, -1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}]$. The barycenter of this tetrahedron is $(0, 0, \sqrt{2})$, which corresponds to the point of intersection between the geodesic passing through 0 and ∞ and the geodesic passing through -1 and orthogonal to the plane $\{x = \frac{1}{2}\}$ in $\mathbb{H}^3$ (which is the plane containing ∞, $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$). It is easy now to see that the (hyperbolic) distance between the barycenter and the face of T passing trough $\{-1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}\}$ is $\ln\sqrt{2}$.

Claim 2: Given a tetrahedron with with order–2 symmetry, the (hyperbolic) distance between its barycenter c and the point A_c is $\ln\left(\frac{1}{2}(\sqrt{6}+\sqrt{2})\right)$. To prove this, we consider the tetrahedron $T = \{1, -1, (2-\sqrt{3})i, -(2-\sqrt{3})i\}$. The barycenter of this tetrahedron is $(0, 0, \frac{1}{2}(\sqrt{6}-\sqrt{2}))$, and the distance between the barycenter and the geodesic between 1 and -1 is $\ln\left(\frac{1}{2}(\sqrt{6}+\sqrt{2})\right)$.

Claim 3: Consider two points $O, P \in \mathbb{H}^3$, and let d be their distance, the length of the segment OP. Let C be the plane passing through P and perpendicular to OP. Let I be a point of $\mathbb{CP}^1$ at the boundary of C. The angle θ between the segment OP and the ray OI depends only on d, and it is given by the function $F(d)$ defined by

$$\theta = F(d) = 2\arctan(e^{-d}).$$

This is actually a 2-dimensional question; all the computations can be made in the Poincaré disc model of the hyperbolic plane. We choose O as the center of the disc, and P, to be determined later, will lie on the positive real axis. Using the angle θ as a parameter, we write the point $I = \cos\theta + i\sin\theta$. We compute the geodesic passing through I and $\bar{I}$, and this determines the position of the point P. An elementary computation gives the formula.

Now we go back to the statement to prove. Let $c \in \ell$, and let $T \in \partial\mathfrak{T}_c^{\downarrow}$ be a tetrahedron with one vertex v_T on the circle C_c. Then the other three vertices of T (different from v_T) lie on a circle C_T, spanning a disc D_T in $\mathbb{CP}^1$. The plane bounded by C_T is perpendicular to the geodesic between v_T and c and intersecting it at the point at distance $\ln\sqrt{2}$ (by Claim 1) from c and farthest from v_T. Then, in order to prove the result above, we just have to check that the circle C_T does not intersect $\overline{D}_c$. This can be checked with a computation in the plane S containing ℓ and v_T. The plane S intersects C_c in two points, v_T and P_1. The distance between c and B_c is $d_1 = 1 + \ln\left(\frac{1}{2}(\sqrt{6}+\sqrt{2})\right)$, by Claim 2; hence, by Claim 3, the angle between v_T and P_1, seen from c, is $2F(d_1) \simeq 0.752\ldots$. The intersection between the plane S and C_T is two points; we denote the P_2 the one closest to P_1. The angle between v_T and P_2, seen from c, is $\pi - F(\ln\sqrt{2}) \simeq 1.91\ldots$, by Claim 1 and 3. Since this is larger than $2F(d_1)$, the circle C_T does not intersect $\overline{D}_c$. □

We can describe more precisely the topology of $\mathfrak{T}_c^{\uparrow}$ and $\mathfrak{T}_c^{\downarrow}$.

Proposition 5.2. $\mathfrak{T}_c^{\uparrow}$ *is homeomorphic to the complement of an open tubular neighborhood of a* $(2,3)$*–torus knot (or, equivalently, a trefoil knot);*

Proof. Since we know that $\mathfrak{T}_c \cong \mathrm{SO}(3)/A_4$, its Seifert structure is well known; see, for example, the second line of Table 10.6 in Martelli [Mar22] with $q = -2$. In order to prove the first claim, we need to understand the Seifert structure of the trefoil knot complement. This is described in Moser [Mos71]. We can easily see that it is the same as the one of $\mathfrak{T}_c^{\uparrow}$. □

Proposition 5.3. $\mathfrak{T}_c^{\downarrow}$ *is homeomorphic to an open solid torus* $\mathbb{D} \times \mathbb{S}^1$*, where* $\mathbb{D}$ *is an open disc in* $\mathbb{C}$*. More precisely, there are explicit coordinates*

$$\mathfrak{T}_c^{\downarrow} \ni T \longrightarrow (v_T, \theta_T) \in D_c \times \mathbb{R}/(\tfrac{2\pi}{3}\mathbb{Z}),$$

where D_c *is the disc defined before Proposition 5.1.*

Proof. We will describe explicit coordinates for $\mathfrak{T}_c^{\downarrow}$ as we will need them in the following section. We will parametrize $\mathfrak{T}_c^{\downarrow}$ with $D_c \times \mathbb{R}/(\frac{2\pi}{3}\mathbb{Z})$, where $\mathbb{R}/(\frac{2\pi}{3}\mathbb{Z})$ is a circle with a parameter that ranges

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between 0 and $\frac{2\pi}{3}$, and we recall that D_c is an open disc in $\mathbb{CP}^1$ containing the point $-i$. Removing the point $-i$, we obtain an anulus $\mathbb{A} = D_c \setminus \{-i\}$, which we will parametrize with polar coordinates; that is, we will write it as

$$\mathbb{A} = D_c \setminus \{-i\} = \mathbb{S}^1 \times (0, 1).$$

Geometrically, we will consider the family $\mathcal{F}$ of hyperbolic half-planes bounded by the line ℓ. These half-planes are orthogonal to the plane bounded by $\mathbb{RP}^1$. We denote by $\mathcal{F}_0$ the half-plane in $\mathcal{F}$ that contains the point $0 \in \mathbb{RP}^1$. Every $z \in \mathbb{A}$ lies in exactly one half-plane in $\mathcal{F}$, denoted by $\mathcal{F}_z$. Define $\alpha(z) \in \mathbb{S}^1$ as the angle between $\mathcal{F}_0$ and $\mathcal{F}_z$, in the clockwise direction, and $\rho(z) \in (0, 1)$ as the angle between the line ℓ and the line cz as seen from the point c. The angle $\rho(z)$ needs to be rescaled suitably so that it ranges between 0 and 1 when z moves in D_c. This describes the coordinates

$$\mathbb{A} \ni z \to (\alpha(z), \rho(z)) \in \mathbb{S}^1 \times (0, 1).$$

We first consider the family $\mathfrak{T}_c^{\downarrow,3}$ of the tetrahedra in $\mathfrak{T}_c^{\downarrow}$ with bottom vertex in $-i$. This is one of the families of tetrahedra with the order–3 symmetry. They form the core of the solid torus and will be parametrized by $\{-i\} \times \mathbb{R}/(\frac{2\pi}{3}\mathbb{Z})$ in the following way: if v is another vertex of a tetrahedron $T \in \mathfrak{T}_c^{\downarrow,3}$, we associate to T the angle θ_T between the half-plane $\mathcal{F}_0$ and $\mathcal{F}_v$, in the clockwise direction, reduced modulo $\frac{2\pi}{3}\mathbb{Z}$.

The complement of this family (i.e., the space $\mathfrak{T}_c^{\downarrow} \setminus \mathfrak{T}_c^{\downarrow,3}$) will be parametrized by $\mathbb{A} \times \mathbb{R}/(\frac{2\pi}{3}\mathbb{Z})$ as follows. If $T \in \overline{\mathfrak{T}_c^{\downarrow}} \setminus \mathfrak{T}_c^{\downarrow,3}$, let v_T be the unique vertex of T in $D_c \setminus \{-i\} = \mathbb{A}$. The point v_T will be the coordinate of T in $\mathbb{A}$. We now define the coordinate $\theta_T \in \mathbb{R}/(\frac{2\pi}{3}\mathbb{Z})$. Once v_T is fixed (and c is fixed), the other three vertices lie on a circle C_T. Let w_T be the point where C_T meets the half-plane $\mathcal{F}_{v_T}$. Define the angle $\beta(T)$ as the angle between the half-plane bounded by v_T, c and containing w_T and the half-plane bounded by v_T, c and containing another vertex of T, in the clockwise direction, modulo $\frac{2\pi}{3}\mathbb{Z}$. The angle θ_T is defined as $\theta_T = \beta(T) + \alpha(v_T)$. This gives a parametrization

$$\mathfrak{T}_c^{\downarrow} \setminus \mathfrak{T}_c^{\downarrow,3} \ni T \to (v_T, \theta_T) \in \mathbb{A} \times \mathbb{R}/(\tfrac{2\pi}{3}\mathbb{Z}).$$

When the parametrizations of the two parts $\mathfrak{T}_c^{\downarrow,3}$ and $\mathfrak{T}_c^{\downarrow} \setminus \mathfrak{T}_c^{\downarrow,3}$ are put together, they give a homeomorphism

$$\mathfrak{T}_c^{\downarrow} \ni T \to (v_T, \theta_T) \in D_c \times \mathbb{R}/(\tfrac{2\pi}{3}\mathbb{Z}).$$

The fact that the joint map is continuous follows from the geometric definition: let's consider a sequence $T_n \in \overline{\mathfrak{T}_c^{\downarrow}} \setminus \mathfrak{T}_c^{\downarrow,3}$ such that $T_n \to T \in \mathfrak{T}_c^{\downarrow,3}$. Up to a rotation by an isometry that stabilizes ℓ, we can assume that T corresponds to the parameters $(-i, 0)$ (i.e., that one of its vertices lies on the half-plane $\mathcal{F}_0$). Since $T_n \to T$, we know that $v_{T_n} \to -i$. We only have to prove that $\beta(T_n) + \alpha(v_{T_n}) \to 0$. To see this, notice that the half-plane bounded by v_{T_n}, c and containing w_{T_n} is getting closer and closer to $\mathcal{F}_{v_{T_n}}$. One of the vertices of T_n is getting closer and closer to the half-plane $\mathcal{F}_0$; hence, the half-plane bounded by v_{T_n}, c and containing this other vertex of T_n is getting closer and closer to $\mathcal{F}_0$. Hence, the angle $\beta(T_n)$ is getting closer and closer to $-\alpha(V_{T_n})$ (modulo $\frac{2\pi}{3}\mathbb{Z}$) and this implies that θ_{T_n} is approaching 0. □

5.2. Description of the construction

In this section, we are going to study the topology of the space $\mathfrak{T}_{\overline{\ell}}$ and prove that it is homeomorphic to a certain quotient $\mathfrak{T}_{\mathcal{O}} \times [-\infty, +\infty]/\sim$, where $\mathcal{O} = \ell \cap \mathcal{P} \in \ell$.

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In order to define the construction, we need to use the following maps:

- $\iota\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ is the reflection in the plane $\mathcal{P}$ with boundary $\mathbb{RP}^1 \subset \mathbb{CP}^1$.
- $L^+_\lambda\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ is the hyperbolic isometry of $\mathbb{H}^3$ with axis ℓ and translation length $\lambda \in \mathbb{R}_{>0}$ and attracting fixed point i;
- $L^-_\lambda\colon \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$ is the hyperbolic isometry of $\mathbb{H}^3$ with axis ℓ and translation length $\lambda \in \mathbb{R}_{>0}$ and attracting fixed point $-i$.

In the following, we will use the decomposition $\mathfrak{T}_c = \mathfrak{T}_c^\uparrow \cup \mathfrak{T}_c^\downarrow$, described before Proposition 5.1. We will also use the notations A_c, B_c, C_c, D_c introduced there.

The isometry L^+_λ can be used to move a tetrahedron in $\mathfrak{T}$.

Proposition 5.4. *The transformations $L^\pm_\lambda$ satisfy the following properties:*

1. $L^\pm_\lambda(\mathfrak{T}_c) = \mathfrak{T}_{L^\pm_\lambda(c)}$;
2. $L^\pm_\lambda(B_c) = B_{L^\pm_\lambda(c)}$.

Together with the L^+_λ, we will also need a companion map that we will denote by M_λ. This will be, for every $c \in \ell$, the map

$$M_\lambda : \overline{\mathfrak{T}_c^\downarrow} \to \mathfrak{T}^\downarrow_{L^+_\lambda(c)}$$

that moves the barycenter of the tetrahedra along ℓ according to L^+_λ, but does not move the bottom vertex in $\overline{D_c}$. In order to define M_λ, we use the coordinates on $\mathfrak{T}_c^\downarrow$ described in the proof of Proposition 5.3. The map M_λ is defined as follows: For every tetrahedron $T \in \overline{\mathfrak{T}_c^\downarrow}$, let $M_\lambda(T)$ be the tetrahedron in $\mathfrak{T}^\downarrow_{L^+_\lambda(c)}$ with barycenter in $L^+_\lambda(c)$, same 'bottom' vertex v_T and same angle θ_T. Notice that in the special case when $T \in \mathfrak{T}_c^{\downarrow,3}$, then $M_\lambda(T) = L^+_\lambda(T)$, but for tetrahedra outside $\mathfrak{T}_c^{\downarrow,3}$, M_λ and L^+_λ can be very different. By Proposition 5.3, the map M_λ is continuous on $\overline{\mathfrak{T}_c^\downarrow}$.

We will need the definition of a function $h\colon \mathbb{CP}^1 \to [-\infty, +\infty]$, called the *height* of z. If $z \in \{\pm i\}$, we define $h_i := +\infty$ and $h_{-i} := -\infty$. For every point $z \in \mathbb{CP}^1 \setminus \{\pm i\}$, consider the unique hyperbolic plane perpendicular to ℓ and containing z in its boundary, and denote by d the intersection of this plane with ℓ. Denote by $h_z := \eta(d) \in \mathbb{R}$.

For every tetrahedron in F, we denote by b_T its barycenter. For every $c \in \ell$ and for every tetrahedron in $\overline{\mathfrak{T}_c^\downarrow}$, we denote by v_T the unique vertex of T in $\overline{D_c}$ and by h_T the height h_{v_T}.

We can now state the main result of the section.

Theorem 5.5. *There is a continuous surjective map*

$$\Phi\colon\ \mathfrak{T}_\mathcal{O} \times [-\infty, +\infty] \to \mathfrak{T}_{\overline{\ell}}$$

such that

1. *For all $T \in \mathfrak{T}_\mathcal{O}$, and $s \in [-\infty, +\infty]$, $\Phi(T, -s) = \iota(\Phi(\iota(T), s))$;*
2. *For all $T \in \mathfrak{T}_\mathcal{O}$, $\Phi(T, 0) = T$;*
3. *For all $T \in \mathfrak{T}_\mathcal{O}$, and $s \in [0, +\infty)$, $\Phi(T, s) \in \{T \in \mathfrak{T} \mid b_T \in \ell^+\}$;*
4. *The restriction*

$$\Phi|_{\mathfrak{T}_\mathcal{O}\times(-\infty,+\infty)} :\ \mathfrak{T}_\mathcal{O} \times (-\infty, +\infty) \to \mathfrak{T}_\ell$$

   is a homeomorphism;
5. $\Phi(\mathfrak{T}_\mathcal{O}^\uparrow \times \{+\infty\}) = \{(+i, +i)\} \in \mathfrak{T}_i$;
6. *The restriction $\Phi_{+\infty} := \Phi|_{\mathfrak{T}_\mathcal{O}^\downarrow \times\{+\infty\}} :\ \mathfrak{T}_\mathcal{O}^\downarrow \times \{+\infty\} \to \mathfrak{T}_i \setminus \{(+i, +i)\}$ is surjective;*

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7. *Consider the function f defined by*

$$f\colon\ (-\infty, \eta(B_{\mathcal{O}})) \to (-\infty, +\infty)$$

$$f(v) = v + \frac{1}{\eta(B_{\mathcal{O}}) - v} = \frac{v^2 - \eta(B_{\mathcal{O}})v - 1}{v - \eta(B_{\mathcal{O}})}.$$

Then f is a strictly increasing homeomorphism. For every $z \in \mathbb{CP}^1$ with $h_z < \eta(B_{\mathcal{O}})$, the f-uplift of z is the point $z^f := L^+_\lambda(z)$, where $\lambda = f(h_z) - h_z$. When $z = -i$, $z^f := -i$. In this way, $h_{z^f} = f(h_z)$.

8. *When $z \in \mathbb{CP}^1 \setminus \{i\}$, the fiber of Φ at the point $(i, z) \in \mathfrak{T}_i$ is the circle*

$$\Phi^{-1}(i, z) = \{\ (T, +\infty) \mid T \in \mathfrak{T}^{\downarrow}_{\mathcal{O}},\ \ (v_T)^f = z\ \}.$$

consisting of all the tetrahedra with a fixed vertex $v_T \in D_c$.

Proof. For the proof we first construct

$$\Phi^+ = \Phi|_{\mathfrak{T}_{\mathcal{O}}\times[0,\infty)}\colon\ (\mathfrak{T}_{\mathcal{O}} \times [0, \infty)) \to \{T \in \mathfrak{T} \mid b_T \in \overline{\ell^+}\}$$

with the property that for all $T \in \mathfrak{T}_{\mathcal{O}}$, $\Phi^+(T, 0) = T$. Then, we will define

$$\Phi^- = \Phi|_{\mathfrak{T}_{\mathcal{O}}\times(-\infty,0]}\colon\ (\mathfrak{T}_{\mathcal{O}} \times (-\infty, 0]) \to \{T \in \mathfrak{T} \mid b_T \in \overline{\ell^-}\}$$

by the formula

$$\Phi^-(T, -s) = \iota(\Phi^+(\iota(T), s)).$$

The map Φ will be obtained by glueing Φ^+ and Φ^-.

We will now discuss the construction of Φ^+. First of all, we notice that property (7) is an easy computation. Moreover, f satisfies the properties

(a) $f(v) > v$.
(b) $\widehat{f}(v) := f(v) - v = \frac{1}{\eta(B_{\mathcal{O}})-v}$ is strictly increasing and tends to $+\infty$ when $v \to \eta(B_{\mathcal{O}})$.

We define Φ^+ as follows:

- If $(T, \infty) \in \mathfrak{T}^{\uparrow}_{\mathcal{O}} \times \{+\infty\}$, then $\Phi^+((T, \infty)) := (+i, +i)$.
- If $(T, \infty) \in \mathfrak{T}^{\downarrow}_{\mathcal{O}} \times \{+\infty\}$, then $\Phi^+((T, \infty)) := L^+_{f(h_T)-h_T}(v_T)$.
- If $(T, s) \in \mathfrak{T}^{\uparrow}_{\mathcal{O}} \times [0, +\infty)$, then $\Phi^+((T, s)) := L^+_s(T)$.
- If $(T, s) \in \mathfrak{T}^{\downarrow}_{\mathcal{O}} \times [0, +\infty)$, then

$$\Phi^+((T, s)) := \begin{cases} L^+_s(T) & \text{if } s \le f(h_T) - h_T \\ M_{s-f(h_T)+h_T} \circ L^+_{f(h_T)-h_T}(T) & \text{if } s \ge f(h_T) - h_T, \end{cases}$$

where the map M_λ is the one defined before the theorem. In order to better understand the map Φ^+, consider, for every $s \in [0, \infty]$, the map represented in Figure 2, defined by

$$\rho_s\colon\ (-\infty, \eta(B_{\mathcal{O}}) \to (-\infty, \eta(B_{\mathcal{O}}) + s)$$

$$\rho_s(v) = \min\{v + s, f(v)\} = \begin{cases} v + s & \text{if } v + s \le f(v) \\ f(v) & \text{if } v + s \ge f(v). \end{cases}.$$

Note that for any $T \in \mathfrak{T}^{\downarrow}_{\mathcal{O}}$ and $s \in [0, +\infty)$, $\Phi^+((T, s))$ is a tetrahedron T' with height $h_{T'} = \rho_s(h_T)$.

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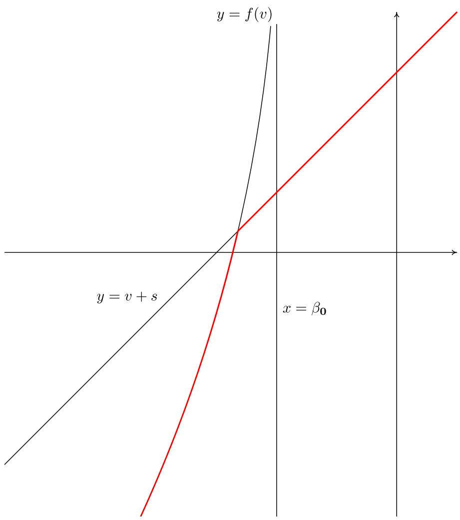

Figure 2. *The functions* $y = v + s$, $y = f(v)$ *and (in red)* $y = \mathfrak{T}_s(v) = \min\{v + s, f(v)\}$.

We can now check that for any $t \in \mathfrak{T}_{\mathcal{O}}$, we have that $\Phi^+(T, 0) = L_0^+(T) = T = \Phi^-(T, 0)$, so we can combine the maps Φ^+ and Φ^- into the map Φ we wanted. This calculation also shows that Φ satisfies properties (1) and (2). From the definition, we can also see that Φ^+ satisfies property (5).

Claim 5.6. The map Φ satisfies property (4); that is, the restriction

$$\Phi|_{\mathfrak{T}_{\mathcal{O}}\times(-\infty,+\infty)} \colon \mathfrak{T}_{\mathcal{O}} \times (-\infty, +\infty) \to \mathfrak{T}_\ell$$

is a homeomorphism.

Proof. For the surjectivity, let $T \in \mathfrak{T}_{c_s}$, where $c_s \in \ell^+$. Let $s = \eta(c_s) \in [0, +\infty)$.

- If $T \in \mathfrak{T}^{\uparrow}_{c_s}$, let $\widehat{T} = L_s^-(T) \in \mathfrak{T}^{\uparrow}_{\mathcal{O}}$. Then $\Phi^+(\widehat{T}, s) = T$.
- If $T \in \mathfrak{T}^{\downarrow}_{c_s} \setminus \mathfrak{T}^{\downarrow,3}_{c_s}$, let $h = \rho_s^{-1}(h_T)$. We need to use the parametrization of $\mathfrak{T}^{\downarrow}_{c_s}$ described in the definition of the map M_λ. Let $\widehat{T}$ be the (unique) tetrahedra in $\mathfrak{T}_{\mathcal{O}}$ such that $h_{\widehat{T}} = h$ and such that $\theta_{\widehat{T}} = \theta_T$. Then again we can see that $\Phi^+(\widehat{T}, s) = T$.
- If $T \in \mathfrak{T}^{\downarrow,3}_{c_s}$, let $\widehat{T} = L_s^-(T) \in \mathfrak{T}^{\uparrow}_{\mathcal{O}}$. Then $\Phi^+(\widehat{T}, s) = T$.

For the injectivity, let $T, T' \in \mathfrak{T}_{\mathcal{O}}$ and $s, s' \in [0, +\infty)$ such that $\Phi^+(T, s) = \Phi^+(T', s') = \widehat{T}$. Since the image is the same, the two image tetrahedra have the same barycenter, so $s = s'$. Also, since the map Φ^+ does not change the 'type' of the tetrahedra (that is, $\mathfrak{T}^{\uparrow}_c$ or $\mathfrak{T}^{\downarrow}_{c_s}$), we have two cases: either $T, T' \in \mathfrak{T}^{\uparrow}_{\mathcal{O}}$ or $T, T' \in \mathfrak{T}^{\downarrow}_{\mathcal{O}}$.

- In the first case, the fact that L_s^+ is an isometry implies that $T = T'$.
- In the second case, we have that $h_T = h_{T'} = \rho_s^{-1}(h_{\widehat{T}})$. Now we have two possibilities: either $s \le f(h_T) - h_T$ or $s \ge f(h_T) - h_T$.
  1. If $s \le f(h_T) - h_T$, then we use again the fact that L_s^+ is injective to see that $T = T'$.
  2. If $s \ge f(h_T) - h_T$, then we use again the fact that the map $M_{s-f(h_T)+h_T} L^+_{f(h_T)-h_T}$ is injective to conclude that $T = T'$.

Since all the maps we use are continuous, we only need to check the continuity at points in $\mathfrak{T}^{\uparrow}_{\mathcal{O}} \cap \overline{\mathfrak{T}^{\downarrow}_{\mathcal{O}}}$ – that is, at tetrahedra T such that $h_T = \eta(B_{\mathcal{O}})$. Since $f(\eta(B_{\mathcal{O}})) = +\infty$, then we are always in the case $s \le f(h_T) - h_T$, so $\Phi^+(T, s) = L_s^+(T)$. □

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Claim 5.7. The map Φ satisfies property (6) – that is, the restriction $\Phi_{+\infty} := \Phi|_{\mathfrak{T}_{\mathcal{O}}^{\downarrow}\times\{+\infty\}} \colon \mathfrak{T}_{\mathcal{O}}^{\downarrow}\times\{+\infty\} \to \mathfrak{T}_i \setminus \{(+i,+i)\}$ is surjective.

This also shows, together with Claim 5.6, that Φ is surjective.

Proof. Given a point $(i,z) \in \{+i\} \times (\mathbb{CP}^1 \setminus \{+i\})$, since the function f is a homeomorphism, we can find a point $v \in \mathbb{CP}^1$ such that $f(h_v) = h_z$ and such that $v^f = z$. Let $T \in \mathfrak{T}_{\mathcal{O}}$ be such that $v_T = v$. Then T is necessarily in $\mathfrak{T}_{\mathcal{O}}^{\downarrow}$. By definition of the map Φ^+, $\Phi^+(T) = (i,z)$. □

Now, we only have to prove the following:

Claim 5.8. The map Φ^+ is continuous.

Proof. The continuity on $\mathfrak{T}_{\mathcal{O}} \times [0,+\infty)$ was established in Claim 5.6. The continuity on $\mathfrak{T}_{\mathcal{O}} \times \{+\infty\}$ is clear from the definition. So, in order to check the continuity of Φ^+, it suffices to check the continuity for sequences $(T_n, s_n) \in \mathfrak{T}_{\mathcal{O}} \times [0,+\infty)$ such that $T_n \to T$ and $s_n \to +\infty$. We have three cases:

(1) If T is in the interior part of $\mathfrak{T}_{\mathcal{O}}^{\uparrow}$, then we can assume that all the T_n are also in $\mathfrak{T}_{\mathcal{O}}^{\uparrow}$. From the definition of the map,

$$\Phi^+(T_n, s_n) = L^+_{s_n}(T_n) \to (+i,+i) = \Phi^+(T,+\infty),$$

because all the T_n are in $\mathfrak{T}_{\mathcal{O}}^{\uparrow}$.

(2) If $T \in \mathfrak{T}_{\mathcal{O}}^{\downarrow}$, then we can assume that all the T_n are also in $\mathfrak{T}_{\mathcal{O}}^{\downarrow}$. Since $T_n \to T$, when n is big enough, we can assume that h_{T_n} is close enough to h_T. Hence, when n is big enough, we can assume that $s_n \geq f(h_{T_n}) - h_{T_n}$. From the definition, we have

$$\Phi^+(T_n, s_n) = M_{s_n - f(h_{T_n}) + h_{T_n}} \circ L^+_{f(h_{T_n}) - h_{T_n}}(T_n).$$

This is a tetrahedron T_n' with vertex $v_{T_n'}$ equal to $L^+_{f(h_{T_n})-h_{T_n}}(v_{T_n})$. Hence, the sequence $\Phi^+(T_n,s_n)$ converges to $\Phi^+(T,+\infty) = L^+_{f(h_T)-h_T}(v_T)$.

(3) Finally, we assume that $T \in \partial\mathfrak{T}_{\mathcal{O}}^{\uparrow}$. If a subsequence of T_n lies in $\mathfrak{T}_{\mathcal{O}}^{\uparrow}$, we can conclude that subsequence converges to (i,i) as in part (1). Now let's assume that all the T_ns are in $\mathfrak{T}_{\mathcal{O}}^{\downarrow}$, and write $T_n' = \Phi^+(T_n, s_n)$. Then, $h_{T_n} \to +\infty$ and also $f(h_{T_n}) - h_{T_n} \to +\infty$; hence, for big enough n, we can assume that s_n is as big as we want, and $f(h_{T_n}) - h_{T_n}$ is as big as we want. From this, we see that $h_{T_n'}$ becomes as big as we want; hence, $T_n' \to (+i,+i)$.

□

This concludes the proof of the theorem. □

On $\mathfrak{T}_{\mathcal{O}} \times [-\infty,+\infty]$, we consider the following equivalence relation: for $T, T' \in \mathfrak{T}_{\mathcal{O}}$ and $t, t' \in [-\infty,+\infty]$,

$$(T,t) \sim (T',t') \Leftrightarrow \begin{cases} T,T' \in \mathfrak{T}_{\mathcal{O}}^{\uparrow}, t = t' = +\infty, \text{ or} \\ T,T' \in \iota(\mathfrak{T}_{\mathcal{O}}^{\uparrow}), t = t' = -\infty, \text{ or} \\ T,T' \in \mathfrak{T}_{\mathcal{O}}^{\downarrow}, v_T = v_{T'}, t = t' = +\infty, \text{ or} \\ T,T' \in \iota(\mathfrak{T}_{\mathcal{O}}^{\downarrow}), v_{\iota(T)} = v_{\iota(T')}, t = t' = -\infty, \text{ or} \\ T = T', t = t'. \end{cases}$$

Corollary 5.9. *The map Φ from Theorem 5.5 descends to a homeomorphism*

$$\bar{\Phi} : \mathfrak{T}_{\mathcal{O}} \times [-\infty,+\infty]/\sim \longrightarrow \mathfrak{T}_{\overline{\ell}}.$$

Proof. The description of the fibers of the map Φ given in Theorem 5.5 guarantees that the map descends to the quotient. It is continuous and onto because Φ is continuous and onto. It is 1–1, again from the description of the fibers. It is a homeomorphism because it is a bijective continuous map from a compact space to a Hausdorff space. This concludes the proof. □

6. Topology of the fiber

In this section, we continue the study of the fiber $F = q^{-1}(\mathcal{O})$ of the projection $q\colon \Omega \longrightarrow \mathbb{H}^2$, where $\mathcal{O} = (0, 1) \in \mathbb{C} \times \mathbb{R}_{>0}$. In the end, we will use the study of the topology of F to describe the homeomorphism type of the smooth fiber $\mathfrak{F}$. We start by analyzing the structure of $\mathfrak{T}_{\overline{\ell}}$ in a bit more detail.

6.1. Singularities of the fiber F

The space $F \cong \mathfrak{T}_{\overline{\ell}}$ is not a manifold. We will show in this subsection that it has four singular points, and all the other points have neighborhoods homeomorphic to $\mathbb{R}^4$. The four singular points are $(+i, +i), (+i, -i) \in \mathfrak{T}_i$ and $(-i, -i), (-i, +i) \in \mathfrak{T}_{-i}$. Two of them, $(+i, -i)$ and $(-i, +i)$, are 'mild singularities' – they are orbifold points with isotropy group $\mathbb{Z}_3$. The other two singular points, $(+i, +i)$ and $(-i, -i)$, are more complicated singularities, and a small neighborhood of these points looks like the cone over a closed 3–manifold that is a Dehn filling of the trefoil knot. All such Dehn fillings are described in Moser [Mos71]. As a consequence, we will prove Corollary 6.3, stating that the fibration q is not a smooth map.

We already know, by part (4) of Theorem 5.5, that $\mathfrak{T}_\ell$ is a manifold. We will now describe a neighborhood of the points of $\mathfrak{T}_i$ and $\mathfrak{T}_{-i}$. We only need to discuss $\mathfrak{T}_i$ because we have the orientation reversing homeomorphism ι that exchanges $\mathfrak{T}_{-i}$ with $\mathfrak{T}_i$.

We first describe the 'mild' singular points and the manifold points.

Proposition 6.1. *Every point $\mathfrak{T}_i$, except from the two points $(+i, +i)$ and $(+i, -i)$, has a neighborhood in $\mathfrak{T}_{\overline{\ell}}$ that is homeomorphic to $\mathbb{R}^4$. The point $(+i, -i)$ has a neighborhood in $\mathfrak{T}_{\overline{\ell}}$ that is homeomorphic to $\mathbb{R}^4$ modded out by a linear action of $\mathbb{Z}_3$.*

Proof. A *labelled tetrahedron* is a tuple (T, v_1, v_2, v_3, v_4), where T is a regular ideal tetrahedron and $\{v_1, v_2, v_3, v_4\}$ is the set of vertices of T. We say that the labelling is *even* if when watching from the vertex v_1, the vertices v_2, v_3, v_4 appear in counter-clockwise cyclic order. The labelling is *odd* otherwise. An even labelled tetrahedron is determined by its baricenter b_T and the first two vertices v_1, v_2. The vertices v_3 and v_4 are determined by these data.

We denote by $\mathfrak{T}_{\mathcal{O}}^{even}$ the set of all even labelled tetrahedra with barycenter in $\mathcal{O}$. The group A_4 acts on $\mathfrak{T}_{\mathcal{O}}^{even}$ in the following way: if $\sigma \in A_4$, define

$$\sigma \cdot (T, v_1, v_2, v_3, v_4) = (T, v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, v_{\sigma(4)}).$$

We have a natural forgetful map

$$r : \mathfrak{T}_{\mathcal{O}}^{even} \ni (T, v_1, v_2, v_3, v_4) \longrightarrow T \in \mathfrak{T}_{\mathcal{O}}$$

that is 12 : 1. This map identifies $\mathfrak{T}_{\mathcal{O}}$ with a quotient:

$$\mathfrak{T}_{\mathcal{O}} = \mathfrak{T}_{\mathcal{O}}^{even}/A_4.$$

The space $\mathfrak{T}_{\mathcal{O}}^{even}$ is homeomorphic to $SO(3) \simeq \mathbb{RP}^3 \simeq T^1(\mathbb{S}^2)$. To explicitly see the homeomorphism between $\mathfrak{T}_{\mathcal{O}}^{even}$ and $T^1(\mathbb{S}^2)$, notice that we can see v_1 as a point of $\mathbb{S}^2$, identify the circle where the other three vertices lie with the tangent circle at v_1, and then see v_2 as a unit tangent vector to the point v_1. An interesting consequence is that $\mathfrak{T}_{\mathcal{O}} \simeq SO(3)/A_4$.

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We define the open subset $\mathfrak{T}_{\mathcal{O}}^{even,\downarrow}$ as

$$\mathfrak{T}_{\mathcal{O}}^{even,\downarrow} = \{\ (T, v_1, v_2, v_3, v_4) \in \mathfrak{T}_{\mathcal{O}}^{even}\ \mid\ v_1 \in D_{\mathcal{O}}\ \}.$$

The subset $\mathfrak{T}_{\mathcal{O}}^{even,\downarrow}$ is not preserved by the action of A_4. Only the subgroup of A_4 that fixes 1 acts there. This subgroup is isomorphic to $\mathbb{Z}_3$.

The restriction of r to $\mathfrak{T}_{\mathcal{O}}^{even,\downarrow}$ gives a $3:1$ map

$$r| : \mathfrak{T}_{\mathcal{O}}^{even,\downarrow} \longrightarrow \mathfrak{T}_{\mathcal{O}}^{\downarrow}$$

that identifies $\mathfrak{T}_{\mathcal{O}}^{\downarrow}$ with a quotient

$$\mathfrak{T}_{\mathcal{O}}^{\downarrow} = \mathfrak{T}_{\mathcal{O}}^{even,\downarrow}/\mathbb{Z}_3.$$

We now apply a construction called mapping cylinder to $\mathfrak{T}_{\mathcal{O}}^{even}$. The *mapping cylinder* of $p\colon T^1(\mathbb{S}^2) \to \mathbb{S}^2$ is the space

$$M_p = T^{\leq 1}(\mathbb{S}^2) = \Big((T^1(\mathbb{S}^2) \times [0,1]) \sqcup \mathbb{S}^2\Big)/\sim,$$

where $\sim$ is defined by $(y, 0) \sim p(y)$. Note that M_p corresponds to the unit disk bundle of $\mathbb{S}^2$, and its boundary is the unit tangent bundle $\partial M_p \cong T^1(\mathbb{S}^2) \cong \mathbb{RP}^3 \cong \mathrm{SO}(3)$. In particular, M_p is a manifold with boundary.

We now take its double; that is, we glue two copies of M_p along their boundary via the identity map:

$$M := M_p \sqcup_{id_\partial} M_p.$$

The space M is clearly a manifold, and it is not hard to see that it is indeed homeomorphic to the manifold $\mathbb{S}^2 \times \mathbb{S}^2$, even if we will not need this fact here.

Now, it is also clear from the definition that we have a map

$$\Psi : \mathfrak{T}_{\mathcal{O}}^{even} \times [-\infty, \infty] \longrightarrow M.$$

This map identifies M with the quotient

$$M \simeq \mathfrak{T}_{\mathcal{O}}^{even} \times [-\infty, \infty]/\sim,$$

where the equivalence relation $\sim$ identifies all the labelled tetrahedra in $\mathfrak{T}_{\mathcal{O}}^{even} \times \{\infty\}$ that have the same vertex v_1 and, similarly, identifies all the labelled tetrahedra in $\mathfrak{T}_{\mathcal{O}}^{even} \times \{-\infty\}$ that have the same vertex v_1.

Now, let's consider the following open subset of $\mathfrak{T}_{\mathcal{O}}^{even} \times [-\infty, \infty]$:

$$U = \mathfrak{T}_{\mathcal{O}}^{even,\downarrow} \times [-\infty, \infty] \subset \mathfrak{T}_{\mathcal{O}}^{even} \times [-\infty, \infty].$$

The image $\Psi(U)$ is an open subset of M; hence, it is a manifold.

Now let's consider again the $3:1$ map $r_| : \mathfrak{T}_{\mathcal{O}}^{even,\downarrow} \to \mathfrak{T}_{\mathcal{O}}^{\downarrow}$. This induces the map

$$\Phi \circ (r_| \times \mathrm{Id}) : \mathfrak{T}_{\mathcal{O}}^{even,\downarrow} \times [-\infty, \infty] \to F,$$

where Φ is the map from Theorem 5.5. The image of this map is an open subset V of F that contains $\mathfrak{T}_i \setminus \{(+i, +i)\}$. Moreover, V is homeomorphic with the quotient $\Psi(U)$ by the action of the group $\mathbb{Z}_3$. This shows that all the points of $\mathfrak{T}_i \setminus \{(+i, +i), (+i, -i)\}$ are manifold points in F, and that the point $(+i, -i)$ is an orbifold point with group $\mathbb{Z}_3$. □

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We now describe a neighborhood of the singular point $(+i, +i)$.

Proposition 6.2. *The point $(+i, +i)$ has a neighborhood in $\mathfrak{T}$ that is homeomorphic to the cone $C(M)$ over a closed 3–manifold M, where M is a Dehn filling of the complement of the trefoil knot.*

Note that all the possible Dehn fillings of the trefoil knot are described in [Mos71].

Proof. Using the notation of Section 5.1, Let $\beta'_{\mathcal{O}} = \beta_{\mathcal{O}} - 1$, and consider the disc $D_{\beta'_{\mathcal{O}}}$, an open disc in $\mathbb{CP}^1$ contained in $D_{\beta_{\mathcal{O}}}$. We define the closed subset $\mathfrak{T}^*_{\mathcal{O}}$ of $\mathfrak{T}_{\mathcal{O}}$ as the set of tetrahedra in $\mathfrak{T}_{\mathcal{O}}$ such that all vertices are in $\mathbb{CP}^1 \setminus D_{\beta'_c}$. This is a closed neighborhood of $\mathfrak{T}^{\uparrow}_{\mathcal{O}}$, and it is homeomorphic to $\mathfrak{T}^{\uparrow}_{\mathcal{O}}$.

Now consider the set

$$U := \Phi(\mathfrak{T}^*_{\mathcal{O}} \times [0, +\infty]) \subset \mathfrak{T},$$

where Φ is the map from Theorem 5.5. The set U is a closed neighborhood of $(+i, +i)$ in F, and it is easy to see that U is a cone with center in $(+i, +i)$ over the boundary ∂U. We only need to prove that ∂U is homeomorphic to a Dehn filling of the trefoil knot complement.

The boundary ∂U is the union of two pieces, $\Phi(\mathfrak{T}^*_{\mathcal{O}} \times \{0\})$ and $\Phi(\partial\mathfrak{T}^*_{\mathcal{O}} \times [0, +\infty])$. The first piece, $\Phi(\mathfrak{T}^*_{\mathcal{O}} \times \{0\})$, is homeomorphic to $\mathfrak{T}^*_{\mathcal{O}}$ (i.e. homeomorphic to $\mathfrak{T}^{\uparrow}_{\mathcal{O}}$), and by Proposition 5.2, this is homeomorphic to the trefoil knot complement. The second piece is homeomorphic to a solid torus: indeed, $\partial\mathfrak{T}^*_{\mathcal{O}}$ is a torus, $\Phi(\partial\mathfrak{T}_{\mathcal{O}} \times [0, +\infty))$ is homeomorphic to a torus times $[0, +\infty)$, and $\Phi(\partial\mathfrak{T}_{\mathcal{O}} \times \{+\infty\})$ is a circle that completes the solid torus.

From this, we can see that ∂U is a Dehn filling of the trefoil knot complement. □

Corollary 6.3. *The fiber bundle $q\colon \Omega \to \mathbb{H}^2$ is not smooth.*

Proof. The map q is $SL(2, \mathbb{R})$–equivariant. If it were smooth, it would have some regular values, and by $SL(2, \mathbb{R})$–equivariance, all the values would be regular. Hence, it would be a submersion, and this would imply that the fiber would be a smooth manifold, which is impossible because it has four singular points. □

6.2. Cohomology of the fiber F

In this section, we will study the cohomology of $\mathfrak{T}_{\overline{\ell}} \cong F$, and this will determine the cohomology for $\mathfrak{F}$. In particular, we will reprove a result of Dumas–Sanders [DS20]. The proof will include calculations that we will need in later proofs:

Proposition 6.4 (Theorem C in [DS20]). *F is a Poincaré duality space; it is simply connected, and its homology is given by*

- $H^0(F; \mathbb{Z}) \cong \mathbb{Z}$;
- $H^1(F; \mathbb{Z}) = H^3(F; \mathbb{Z}) = 0$;
- $H^2(F; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$;
- $H^4(F; \mathbb{Z}) \cong \mathbb{Z}$;
- $H^i(F; \mathbb{Z}) = 0$ *for all* $i > 4$.

Moreover, for each $i = 0, \ldots, 4$, there is a natural isomorphism $H^i(F; \mathbb{Z}) \cong H_{4-i}(F; \mathbb{Z})$.

To prove Proposition 6.4, we work again with $\mathfrak{T}_{\overline{\ell}}$. We will write $\mathfrak{T}_{\overline{\ell}}$ as a union of two open sets

$$A := \{x \in \mathfrak{T}_{\overline{\ell}} \mid \eta(b_x) \in (-1, +\infty]\}, \qquad B := \{x \in \mathfrak{T}_{\overline{\ell}} \mid \eta(b_x) \in [-\infty, +1)\}.$$

Let

$$Y := A \cap B \cong \mathfrak{T}_{\mathcal{O}} \times (-1, 1).$$

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The first thing we have to prove is the following:

Proposition 6.5. *The open set A deformation retracts to*

$$\{x \in A \mid \eta(b_x) = +\infty\} = \mathfrak{T}_i \cong \mathbb{CP}^1.$$

Similarly, B deformation retracts to

$$\{x \in B \mid \eta(b_x) = -\infty\} = \mathfrak{T}_i \cong \mathbb{CP}^1.$$

Proof. Denote by $\kappa = (\kappa_1, \kappa_2)$ the inverse of the map

$$\Phi|_{\mathfrak{T}_\mathcal{O} \times (-\infty,+\infty)}\colon \mathfrak{T}_\mathcal{O} \times (-\infty,+\infty) \to \mathfrak{T}_\ell,$$

which is a homeomorphism by Theorem 5.5. So for $x \in A \setminus \mathfrak{T}_i$, $\kappa_1(x) \in \mathfrak{T}_\mathcal{O}$, $\kappa_2(x) \in (-1,+\infty)$, and $\Phi(\kappa_1(x), \kappa_2(x)) = x$. Here, $\kappa_2(x) = \eta(b_x)$. We write the retraction as

$$H : A \times [0,+\infty] \to A$$
$$H(x,t) := \begin{cases} x & \text{if } x \in \mathfrak{T}_i \\ \Phi(\kappa_1(x), \kappa_2(x) + t) & \text{if } x \in A \setminus \mathfrak{T}_i. \end{cases}$$

It is easy to check that H is a retraction by deformation; that is, H is continuous, $H(\cdot,0)$ is the identity on A, $H(x,+\infty) \in \mathfrak{T}_i$ and for all $x \in \mathfrak{T}_i$, $H(x,t) = x$. □

We will also use the following version of Poincaré Duality to calculate the homology of the intersection Y. Note that we use the convention that homology and cohomology groups of negative dimension are zero, so the duality statement includes the fact that all the nontrivial homology and cohomology of M lies in the dimension range from 0 to n.

Theorem 6.6 (Poincaré Duality, see Hatcher [Hat02, page 231]). *Let M be a closed orientable n–manifold. Then*

1. *$H_k(M;\mathbb{Z})$ and $H^{n-k}(M;\mathbb{Z})$ are isomorphic.*
2. *Modulo their torsion subgroups, $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ are isomorphic.*
3. *The torsion subgroups of $H_k(M;\mathbb{Z})$ and $H_{n-k-1}(M;\mathbb{Z})$ are isomorphic for $k = 0,\dots,4$.*

Proof of Proposition 6.4. The fact that F is a Poincaré duality space follows from the fact that F is homotopically equivalent to the smooth manifold $\mathfrak{F}$, since both are homotopically equivalent to Ω, as proven in Corollary 4.10 and Theorem 3.1. Since F and $\mathfrak{T}_{\overline{\ell}}$ are homeomorphic, this also tells us that $\mathfrak{T}_{\overline{\ell}}$ is a Poincaré duality space.

For the second part of the result, we will study the homology and cohomology of $\mathfrak{T}_{\overline{\ell}}$, which will suffice to conclude. Through all the proof, we will use the decomposition above for $\mathfrak{T}_{\overline{\ell}} = A \cup B$ and $Y = A \cap B$. Using Proposition 6.5 and the definitions, we can see the following homotopy equivalences:

- $A, B \simeq \mathbb{S}^2$, and
- $Y \simeq \mathrm{SO}(3)/A_4$.

The simple connectivity of $\mathfrak{T}_{\overline{\ell}}$ follows from Seifert–Van Kampen theorem and the decomposition $\mathfrak{T}_{\overline{\ell}} = A \cup B$ described above. Proposition 6.5 shows that the groups $\pi_1(A)$ and $\pi_1(B)$ are trivial, and hence that $\pi_1(\mathfrak{T}_{\overline{\ell}})$ is trivial as well. Since $\mathfrak{T}_{\overline{\ell}}$ is connected, this proves that $\mathfrak{T}_{\overline{\ell}}$ is simply connected.

In order to compute the cohomology of $\mathfrak{T}_{\overline{\ell}}$, we will use the Mayer–Vietoris sequence. We know the cohomology of A and B:

- $H^0(A;\mathbb{Z}) \cong H^0(B;\mathbb{Z}) \cong \mathbb{Z}$;
- $H^2(A;\mathbb{Z}) \cong H^2(B;\mathbb{Z}) \cong \mathbb{Z}$;
- $H^i(A;\mathbb{Z}) = H^i(B;\mathbb{Z}) = 0$ for $i = 1$ and for all $i > 2$.

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In order to compute the cohomology of Y, we remember that it deformation retracts to $\mathrm{SO}(3)/A_4$, which is a Seifert-fibered manifold described in the second line of Table 10.6 in Martelli [Mar22] with $q = -2$. Hence, the cohomology for Y is

- $H^0(Y;\mathbb{Z}) \cong \mathbb{Z}$;
- $H^1(Y;\mathbb{Z}) = 0$;
- $H^2(Y;\mathbb{Z}) \cong \mathbb{Z}_3$;
- $H^3(Y;\mathbb{Z}) \cong \mathbb{Z}$;
- $H^i(Y;\mathbb{Z}) = 0$ for all $i > 3$.

Since Y is connected, we have $H_0(Y;\mathbb{Z}) \cong H^3(Y;\mathbb{Z}) \cong \mathbb{Z}$, and since it is a manifold, and hence a Poincaré duality space, we have that $H_3(Y;\mathbb{Z}) \cong H^0(Y;\mathbb{Z}) \cong \mathbb{Z}$. From Martelli [Mar22], we can see that $H_1(Y;\mathbb{Z}) \cong H^2(Y;\mathbb{Z}) \cong \mathbb{Z}_3$. Finally, again using Poincaré duality (see Theorem 6.6 with $n = 3$), we can see that $H_2(Y;\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$ is free (because its torsion subgroup is isomorphic to the one of $H_0(Y;\mathbb{Z})$), and that $H_2(Y;\mathbb{Z}) = H^1(Y;\mathbb{Z}) = 0$ (because $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}_3$ and, modulo their torsion subgroups, $H_1(M;\mathbb{Z})$ and $H_2(M;\mathbb{Z})$ are isomorphic).

Now we are ready to compute the cohomology of $\mathfrak{T}_{\overline{\ell}}$. We have

- $H^0(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z}$;
- $H^1(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = H^3(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = 0$;
- $H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$;
- $H^4(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z}$;
- $H^i(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = 0$ for all $i > 4$.

Since $\mathfrak{T}_{\overline{\ell}}$ is connected, we have $H_0(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong H^4(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z}$. Using Theorem 6.6, we can see that $H_4(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong H_0(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z}$, that $H_1(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = H_3(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = H^1(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = H^3(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = 0$, and that $H_2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z})$ is free abelian. We now use the following exact sequence coming from the Mayer–Vietoris sequence to see that $H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$$0 = H^1(Y;\mathbb{Z}) \to H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \to H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) \to H^2(Y;\mathbb{Z}) \to H^3(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) = 0.$$

As we said, using the fact that F is homeomorphic to $\mathfrak{T}_{\overline{\ell}}$ via the map g^{-1}, the result follows. □

6.3. The homeomorphism type of $\mathfrak{F}$

In this section, we will prove the following result:

Proposition 6.7. *$\mathfrak{F}$ is homeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, and F and $\mathfrak{T}_{\overline{\ell}}$ are homotopically equivalent to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.*

For the proof, we need deep classification theorems of simply connected smooth 4–manifolds due to Whitehead, Milnor, Milnor–Hausemoller, Freedman, Serre and Donaldson, which use their intersection form. The *intersection form* for a closed oriented 4–manifold N is the map

$$Q_N\colon\ H^2(N;\mathbb{Z}) \times H^2(N;\mathbb{Z}) \to H^4(N;\mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_N(\alpha,\beta) := (\alpha \smile \beta)[N]$, where $\alpha,\beta \in H^2(N,\mathbb{Z})$ and $\smile$ denotes the cohomological cup product of α and β and $[N] \in H_4(N;\mathbb{Z})$ is the fundamental class. See Scorpan [Sco05, Chap. 3] for a more detailed discussion.

This definition of the intersection form only uses the cup product, and this is well defined for all topological spaces, including our singular space $\mathfrak{T}_{\overline{\ell}}$. In our proof, we want to compute the intersection form of the smooth 4–manifold $\mathfrak{F}$, and we will do this by computing the cup product of the homotopically equivalent space $\mathfrak{T}_{\overline{\ell}}$.

Recall the following definitions:

- The intersection form Q_N is called *unimodular* if the matrix representing Q_N is invertible over $\mathbb{Z}$.
- The *rank* of Q_N is defined as $\operatorname{rank}(Q_N) := \dim_{\mathbb{Z}} H^2(N;\mathbb{Z})$.
- The *signature* of Q_N as

$$\operatorname{sign}(Q_N) := \dim_{\mathbb{Z}} H^2_+(N;\mathbb{Z}) - \dim_{\mathbb{Z}} H^2_-(N;\mathbb{Z}),$$

  where $H^2_+(N;\mathbb{Z})$ (resp. $H^2_-(N;\mathbb{Z})$) is defined as the maximal positive-definite (resp. negative-definite) subspace for Q_N.
- The *definiteness* of Q_N can be positive definite, negative definite or indefinite. We say that Q_N is *positive-definite* if for all nonzero α, we have $Q_N(\alpha,\alpha) > 0$, and *negative-definite* if for all nonzero α, we have $Q_N(\alpha,\alpha) < 0$. If there exists classes α, and β such that $Q_N(\alpha,\alpha) > 0$ and $Q_N(\beta,\beta) < 0$, then Q_N is called *indefinite*.
- The *parity* of Q_N can be even or odd. We say that Q_N is *even* if for all classes α, we have $Q_N(\alpha,\alpha)$ is even. Otherwise, we say that Q_N is *odd*.

As proven in Scorpan [Sco05, Sec. 3.2], the intersection form Q_N of a 4–manifold is always unimodular. In addition, the intersection form satisfies the following properties: given two 4–manifolds N_1 and N_2, we have

- $Q_{\overline{N}} = -Q_N$, where $\overline{N}$ is N with opposite orientation.
- $Q_{N_1\#N_2} = Q_{N_1} \oplus Q_{N_2}$, where $N_1\#N_2$ is the connected sum of N_1 and N_2.

Example 6.8. In Scorpan [Sco05, Sec. 3.2], one can see the details of the calculations of the intersection forms for $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $\mathbb{S}^2\times\mathbb{S}^2$ which are simply-connected 4–manifolds with their intersection forms of rank 2 and indefinite signature. The parity is odd for $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, and even for $\mathbb{S}^2\times\mathbb{S}^2$. In fact, their intersection forms are given by

- $Q_{\mathbb{CP}^2\#\overline{\mathbb{CP}}^2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,
- $Q_{\mathbb{S}^2\times\mathbb{S}^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We will use the following classification theorems of smooth 4–manifolds, due to Serre, Freedman and Donaldson:

Theorem 6.9 (Serre, Freedman, Donaldson). *Two smooth simply-connected 4–manifolds are homeomorphic if and only if their intersection forms have the same rank, signature, and parity.*

Theorem 6.10 (Freedman's Classification Theorem [Fre82]). *For any integral symmetric unimodular form Q, there is a closed simply-connected topological 4–manifold that has Q as its intersection form.*

- *If Q is even, there is exactly one such manifold.*
- *If Q is odd, there are exactly two such manifolds, at least one of which does not admit any smooth structures.*

Proof of Theorem 6.7. First, since F, $\mathfrak{T}_{\overline{\ell}}$ and $\mathfrak{F}$ are homotopically equivalent, their intersection forms are isomorphic. Since our description of $\mathfrak{T}_{\overline{\ell}}$ is more concrete, we will discuss $Q_{\mathfrak{T}_{\overline{\ell}}}$ and use that discussion to find the homeomorphism type of $\mathfrak{F}$.

First, we know that the rank of $Q_{\mathfrak{T}_{\overline{\ell}}}$ is 2 because $H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

Second, since there is a self-homeomorphism r of $\mathfrak{T}_{\overline{\ell}}$ reversing the orientation, and since the sign satisfies the following property: $\operatorname{sign}(Q_N) = -\operatorname{sign}(Q_{\overline{N}})$, where $\overline{N}$ is N with opposite orientation, we can see that the signature of the intersection form is $\operatorname{sign}(Q_{\mathfrak{T}_{\overline{\ell}}}) = 0$, and hence, $Q_{\mathfrak{T}_{\overline{\ell}}}$ is indefinite.

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Let us show that $Q_{\mathfrak{T}_{\overline{\ell}}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in an appropriate basis.

From the Mayer–Vietoris sequence in cohomology, we have

$$\begin{array}{ccccccccc} H^1(Y;\mathbb{Z}) & \to & H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) & \to & H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) & \to & H^2(Y;\mathbb{Z}) & \to & H^3(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \\ 0 & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z}_3 & \to & 0 \end{array}$$

Let r be the orientation reversing involution of $\mathfrak{T}_{\overline{\ell}}$. We have that $r(A) = B$. The two maps $H^2(A;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ and $H^2(B;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ are either both zero or both nonzero. By the Mayer-Vietoris sequence, the map $H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is onto; hence, the two maps are both nonzero. We choose the generators of $H^2(A;\mathbb{Z})$ and $H^2(B;\mathbb{Z})$ such that both generators map to 1 in $H^2(Y;\mathbb{Z}) = Z_3$. The map $\zeta\colon\ H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ can then be written as

$$H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z}) \ni (n,m) \longrightarrow \zeta(n,m) = n + m \pmod 3 \in H^2(Y;\mathbb{Z}).$$

Hence, the Mayer–Vietoris sequence and the injective map $\mu\colon H^2(\mathfrak{T}_{\overline{\ell}}) \to H^2(A;\mathbb{Z}) \oplus H^2(B;\mathbb{Z})$ identifies $H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z})$ with the subgroup

$$H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z}) \cong \mathrm{Image}(\mu) = \mathrm{Ker}(\zeta) = \{\ (n,m) \in \mathbb{Z}\oplus\mathbb{Z}\ \mid\ n+m \equiv 0 \pmod 3\ \}.$$

A basis of $H^2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z})$ is given by the elements $v = (2,1)$ and $w = (1,2)$. We now express $Q_{\mathfrak{T}_{\overline{\ell}}}$ as a matrix in this basis:

$$Q_{\mathfrak{T}_{\overline{\ell}}} = \begin{pmatrix} x & z \\ z & y. \end{pmatrix}$$

In order to compute $Q_{\mathfrak{T}_{\overline{\ell}}}$, we consider the elements $(3,0) = 2v - w$ and $(0,3) = 2w - v$. These two elements are $Q_{\mathfrak{T}_{\overline{\ell}}}$–orthogonal because A and B retract to disjoint 2–cycles in $H_2(\mathfrak{T}_{\overline{\ell}};\mathbb{Z})$, and we have that $(3,0)$ maps to 0 in $H^2(B;\mathbb{Z})$ and $(0,3)$ maps to 0 in $H^2(A;\mathbb{Z})$. We have

$$Q_{\mathfrak{T}_{\overline{\ell}}}(2v - w, 2w - v) = -2x - 2y + 5z = 0$$

$$Q_{\mathfrak{T}_{\overline{\ell}}}(2v - w, 2v - w) = 4x + y - 4z = q$$

$$Q_{\mathfrak{T}_{\overline{\ell}}}(2w - v, 2w - v) = x + 4y - 4z = -q,$$

where $q \in \mathbb{Z}$ is the norm of $(3,0)$. The norm of $(0,3)$ is then $-q$ because $r(3,0) = (0,\pm 3)$, and r reverses the orientation.

The determinant of the 3×3 matrix of the coefficients is $27 \neq 0$; hence, the system of equations has at most one solution. An explicit solution is given by $x = \frac{q}{3}$, $y = -\frac{q}{3}$, $z = 0$. Since Q is unimodular, this implies $q = \pm 3$ and we conclude that

$$Q_{\mathfrak{T}_{\overline{\ell}}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\mathfrak{T}_{\overline{\ell}}$ and $\mathfrak{F}$ are homotopy equivalent, we have proven that $\mathfrak{F}$ has the same intersection matrix as $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Since $\mathfrak{F}$ is smooth, Freedman's Classification Theorem tells us that $\mathfrak{F}$ is homeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ (hence that $\mathfrak{T}_{\overline{\ell}}$ and F are homotopy equivalent to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$). □

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