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On Dirichlet Spaces With a Class of Superharmonic Weights

Guanlong Bao, Nihat Gökhan Göğüş, and Stamatis Pouliasis

Abstract. In this paper, we investigate Dirichlet spaces $\mathcal{D}_\mu$ with superharmonic weights induced by positive Borel measures μ on the open unit disk. We establish the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$ spaces and we characterize the cases where equality occurs. We define a class of weighted Hardy spaces H^2_μ via the balayage of the measure μ. We show that $\mathcal{D}_\mu$ is equal to H^2_μ if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. As an application, we obtain the reproducing kernel of $\mathcal{D}_\mu$ when μ is an infinite sum of point-mass measures. We consider the boundary behavior and inner-outer factorization of functions in $\mathcal{D}_\mu$. We also characterize the boundedness and compactness of composition operators on $\mathcal{D}_\mu$.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. Let ν be a positive Borel measure on the unit circle $\mathbb{T}$. Motivated by the study of cyclic analytic two-isometries, S. Richter [28] introduced Dirichlet spaces $\mathcal{D}(\nu)$ with harmonic weights. Namely, the space $\mathcal{D}(\nu)$ consists of functions $f \in H(\mathbb{D})$ with $\int_{\mathbb{D}} |f'(z)|^2 P_\nu(z)\, dA(z) < \infty$, where dA denotes the area measure on $\mathbb{D}$ and

$$P_\nu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\nu(\zeta)$$

is a positive harmonic function on $\mathbb{D}$. The theory of $\mathcal{D}(\nu)$ spaces attracted much attention and has been very well developed in recent years. We refer to the recent monograph [15] for a general exposition on $\mathcal{D}(\nu)$ spaces.

A. Aleman [3] introduced Dirichlet spaces with superharmonic weights. By the Riesz decomposition theorem [6, p. 105–106], for every positive superharmonic function ω on $\mathbb{D}$, there are positive Borel measures μ (the Riesz measure of ω) on $\mathbb{D}$ and ν on $\mathbb{T}$ such that ω is equal to the sum of the Green potential of μ and the Poisson integral of ν. More specifically,

$$\omega(z) = \int_{\mathbb{D}} \log\Big|\frac{1-\overline{w}z}{z-w}\Big|\, d\mu(w) + \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\nu(\zeta) := U_\mu(z) + P_\nu(z).$$

A function $f \in H(\mathbb{D})$ belongs to the Dirichlet space $\mathcal{D}_\omega$ induced by the positive superharmonic function ω if $\int_{\mathbb{D}} |f'(z)|^2 \omega(z)\, dA(z) < +\infty$. (See A. Aleman [3] for the

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general theory of $\mathcal{D}_\omega$ spaces.) Recently, D_ω was identified [14] as de Branges–Rovnyak spaces with equal norms, for certain weights ω. For more results on $\mathcal{D}_\omega$ spaces, see [12, 33, 34].

It is well known [3, 14] that $\mathcal{D}_\omega$ spaces are always subsets of the Hardy space H^2. Recall that the Hardy space H^2 is the class of analytic functions $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ on $\mathbb{D}$ such that $\|f\|_{H^2}^2 = \sum_{n=0}^{+\infty} |a_n|^2 < +\infty$. It is well known that the norm in H^2 can be expressed via an area integral in the following way:

$$\|f\|_{H^2}^2 = |f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|}\, dA(z).$$

The theory of $\mathcal{D}_\omega$ depends only on the corresponding weighted functions U_μ and P_ν. As previously mentioned, Dirichlet spaces $\mathcal{D}(\nu)$ with harmonic weights P_ν have been studied extensively. The aim of this paper is to focus on Dirichlet spaces with superharmonic weights U_μ induced by positive Borel measures μ on $\mathbb{D}$. Namely, we investigate the space $\mathcal{D}_\mu$ consisting of functions $f \in H(\mathbb{D})$ with

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z) < +\infty.$$

A norm on $\mathcal{D}_\mu$ can be defined by $\|f\|_{\mathcal{D}_\mu}^2 = \|f\|_{H^2}^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z)$. Equipped with this norm, $\mathcal{D}_\mu$ is a Hilbert space. It is well known (see [6, p. 98]) that $U_\mu \not\equiv +\infty$ if and only if

$$\int_{\mathbb{D}} (1-|z|)\, d\mu(z) < +\infty. \tag{1.1}$$

Thus, throughout this paper, we always assume that μ satisfies condition (1.1). It is worth mentioning that $\mathcal{D}_\mu$ spaces include radial Dirichlet spaces $\mathcal{D}_\omega$, where $\omega(z) = K(|z|)$ and K is a decreasing concave positive function on $[0,1)$ with $\lim_{x\to 1} K(x) = 0$ (see §5). Of course, there exists a $\mathcal{D}_\mu$ that is not equal to any radial Dirichlet space (see Corollary 5.6).

The paper is organized as follows. In Section 2 we establish the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$ spaces. It means that the norm of every function f in $\mathcal{D}_\mu$ is dominated by the product of the area of the image of f and the total mass $\mu(\mathbb{D})$ of μ. We also describe the cases where equality holds in the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$. Note that $\mathcal{D}_\mu$ spaces are always subsets of H^2. Then it is natural to ask if some $\mathcal{D}_\mu$ can be identified as weighted Hardy spaces with equivalent norms. For this purpose, in Section 3 we define the weighted Hardy space H_μ^2 induced by the balayage of μ and we show that $\mathcal{D}_\mu = H_\mu^2$ if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. As an application, we obtain the reproducing kernels of $\mathcal{D}_\mu$ spaces when μ is an infinite sum of point-mass measures. In Section 4 we consider the boundary behavior and inner-outer factorization of functions in $\mathcal{D}_\mu$ spaces. In the last section, we characterize the boundedness and the compactness of composition operators on $\mathcal{D}_\mu$ spaces. We use two equivalent conditions to describe the boundedness of composition operators on $\mathcal{D}_\mu$ spaces. In general, one of the two corresponding conditions for $\mathcal{D}(\nu)$ spaces with harmonic weights cannot be used to describe the boundedness of composition operators on $\mathcal{D}(\nu)$ [31, p. 447].

In this paper, we will write $a \lesssim b$ if there exists a constant C such that $a \le Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b \lesssim a$.

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2 The Alexander–Taylor–Ullman Inequality for $\mathcal{D}_\mu$ Spaces

In this section, we establish the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$ spaces and we characterize the cases where equality holds.

H. Alexander, B. A. Taylor, and J. L. Ullman [5] showed that the norm of a function f in H^2 is dominated by the area of the image of f. Namely, if $f \in H^2$ with $f(0) = 0$, then

$$\|f\|_{H^2}^2 \le \frac{A(f(\mathbb{D}))}{\pi}. \tag{2.1}$$

Later, H. Alexander and R. Osserman [4] proved that the equality in (2.1) holds if and only if f is of the form $f = CI$, where C is a complex constant and I is an inner function with $I(0) = 0$. Here we recall that a bounded analytic function g on $\mathbb{D}$ is called inner if $|g(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$.

The Alexander–Taylor–Ullman inequality (2.1) for H^2 attracted much attention and many different proofs were given[9,21,30,36]. Note that $\mathcal{D}_\mu$ spaces are always subsets of H^2. The purpose of this section is to consider the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$ spaces.

We will denote by G_Ω the Green function of a Greenian domain $\Omega \subseteq \mathbb{C}$, that is, a domain having a Green function [6, p. 89]. Let f be a non-constant analytic function on a Greenian domain Ω such that $f(\Omega)$ is Greenian. We will denote by $m(a)$ the multiplicity of the zero of $f(z) - f(a)$ at $a \in \Omega$ and by $\mathrm{v}(y) = \sum_{f(a)=y} m(a)$ the valency of f at $y \in f(\Omega)$. The following inequality is known as the Lindelöf Principle

$$G_{f(\Omega)}(y_0, f(z)) \ge \sum_{f(a)=y_0} m(a) G_\Omega(a, z),$$

where $z \in \Omega$ and $y_0 \in f(\Omega)$. It is well known [23, Theorem 2.5] that if $f\colon \mathbb{D} \to \mathbb{D}$ is an inner function, then the equality holds in the Lindelöf Principle for every $y_0 \in \mathbb{D}$ except on a set of zero logarithmic capacity. See [9] for a complete characterization of the equality cases in the Lindelöf Principle.

Denote by δ_a the unit point-mass measure at $a \in \mathbb{D}$. We obtain the Alexander–Taylor–Ullman inequality for $\mathcal{D}_\mu$ spaces as follows.

Theorem 2.1 *Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $f \in \mathcal{D}_\mu$. Then*

$$\frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z) \le \frac{\mu(\mathbb{D}) A(f(\mathbb{D}))}{\pi}. \tag{2.2}$$

Also, if $f(0) = 0$, then

$$\|f\|_{D_\mu}^2 \le (1 + \mu(\mathbb{D})) \frac{A(f(\mathbb{D}))}{\pi} \tag{2.3}$$

and the equality holds in (2.3) if and only if the measure μ is of the form

$$\mu = a_0 \delta_0 + \sum_{n=1}^{+\infty} a_n \delta_{z_n}, \quad a_n > 0, z_n \in \mathbb{D},$$

and f is of the form $f = c\phi$, where $c \in \mathbb{C}$ and ϕ is an inner function with $\phi(0) = \phi(z_n) = 0$ for every $n \in \mathbb{N}$.

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Proof From the change of variables formula [2, p. 98] and Lindelöf's principle, we have that for every $w \in \mathbb{D}$,

$$\int_{\mathbb{D}} G_{\mathbb{D}}(z,w)|f'(z)|^2\,dA(z) = \int_{f(\mathbb{D})} \sum_{f(a)=x} G_{\mathbb{D}}(a,w)\,dA(x)$$
$$\le \int_{f(\mathbb{D})} G_{f(\mathbb{D})}(x, f(w))\,dA(x).$$

Also, it is well known ([9, p. 104] or [21, p. 752]) that for every $w \in \mathbb{D}$,

$$\int_{f(\mathbb{D})} G_{f(\mathbb{D})}(x, f(w))\,dA(x) \le \frac{1}{2}A(f(\mathbb{D})).$$

Therefore,

$$\frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\,dA(z) = \frac{2}{\pi}\int_{\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 G_{\mathbb{D}}(z,w)\,dA(z)d\mu(w)$$
$$\le \frac{2}{\pi}\int_{\mathbb{D}} \frac{1}{2}A(f(\mathbb{D}))\,d\mu(w)$$
$$= \frac{\mu(\mathbb{D})A(f(\mathbb{D}))}{\pi},$$

and (2.2) is proved. The inequality (2.3) then follows from the inequalities (2.1) and (2.2).

Suppose that $f(0) = 0$. Then the equality holds in (2.3) if and only if equalities hold in (2.1) and (2.2). The equality in (2.1) holds if and only if $f = c\phi$, where $c \in \mathbb{C}$ and ϕ is an inner function with $\phi(0) = 0$ [4]. The least harmonic majorant of the subharmonic function $|\phi|^2$ on $\mathbb{D}$ is

$$h_\phi(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}|\phi(\zeta)|^2|\,d\zeta| = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\,|d\zeta| = 1, \quad z \in \mathbb{D}.$$

From the Riesz decomposition theorem we obtain that

$$\frac{1}{2\pi}\int_{\mathbb{D}} G_{\mathbb{D}}(z,w)|\phi'(z)|^2\,dA(z) = 1 - |\phi(w)|^2, \quad w \in \mathbb{D}.$$

Therefore,

$$\frac{2}{\pi}\int_{\mathbb{D}} |c\phi'(z)|^2 U_\mu(z)\,dA(z) = |c|^2\frac{2}{\pi}\int_{\mathbb{D}}\int_{\mathbb{D}} |\phi'(z)|^2 G_{\mathbb{D}}(z,w)\,dA(z)d\mu(w)$$
$$= |c|^2\int_{\mathbb{D}} 1 - |\phi(w)|^2\,d\mu(w).$$

Also, since $\mathbb{D} \setminus \phi(\mathbb{D})$ has zero logarithmic capacity [23, Theorem 2.5], we have

$$A(c\phi(\mathbb{D})) = A(c\mathbb{D}) = |c|^2\pi.$$

Therefore, the equality in (2.2) holds for $f = c\phi$ if and only if $\int_{\mathbb{D}} |\phi(w)|^2\,d\mu(w) = 0$, which holds if and only if $\phi = 0$ μ-almost everywhere. Since the zeros of ϕ are isolated, the above equality holds if and only if μ is of the form

$$\mu = a_0\delta_0 + \sum_{n=1}^{+\infty} a_n\delta_{z_n}, \quad a_n > 0,\ z_n \in \mathbb{D},$$

and the inner function ϕ satisfies $\phi(0) = \phi(z_n) = 0$, for every $n \in \mathbb{N}$. ∎

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From Theorem 2.1 it follows that if f is an analytic function on $\mathbb{D}$ with $A(f(\mathbb{D})) < \infty$, then $f \in \mathcal{D}_\mu$ for every finite positive Borel measure μ. We point out that this is not true for some $\mathcal{D}_\mu$ where μ is infinite. For example, let $\mathcal{D}_\omega$ be the weighted Dirichlet space corresponding to the superharmonic function $\omega(z) = (1-|z|^2)^p$, $p \in (0,1)$ and note that $\omega(z) = U_{\mu_\omega}$, where $d\mu_\omega(z) = -\Delta\omega(z)dA(z)$ (see [2, p. 99]) and Δ denotes the Laplace operator. Let $f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{D})$. Then, by [38, p. 23], $f \in \mathcal{D}_\omega$ if and only if $\sum_{n=0}^\infty (n+1)^{1-p}|a_n|^2 < \infty$. Therefore, for $g(z) = \sum_{n=0}^\infty n^{-2} z^{2^n}$, we have that $g \notin \mathcal{D}_\omega$, while $A(g(\mathbb{D})) < \infty$, since g is a bounded analytic function on $\mathbb{D}$. This happens because $\mu_\omega(\mathbb{D}) = -\int_{\mathbb{D}} \Delta\omega(z)\, dA(z) = \infty$.

3 $\mathcal{D}_\mu$ Spaces and a Class of Weighted Hardy Spaces

Since $\mathcal{D}_\mu$ spaces are always subsets of the Hardy space H^2, it is natural to ask if some Dirichlet spaces $\mathcal{D}_\mu$ are equal to certain weighted Hardy spaces. In this section, we give a positive answer to this question. We define a class of weighted Hardy spaces H^2_μ via the balayage of finite positive Borel measures μ on $\mathbb{D}$. We show that $\mathcal{D}_\mu = H^2_\mu$ if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. Applying this relation, we give the reproducing kernel of $\mathcal{D}_\mu$ when μ is an infinite sum of point-mass measures. A measure μ_0 is also constructed such that $\mathcal{D}_{\mu_0} = H^2_{\mu_0}$ and $\mathcal{D}_{\mu_0} \neq H^2$.

3.1 A Class of Weighted Hardy Spaces

In this subsection, we define weighted Hardy spaces H^2_μ via the balayage of finite positive Borel measures μ on $\mathbb{D}$ and we consider Carleson measures for H^2_μ spaces. Before doing that, we recall the balayage of μ and outer functions for H^2 as follows.

Let μ be a finite positive Borel measure on $\mathbb{D}$. The balayage of μ is the function

$$S_\mu(\zeta) = \frac{1}{2\pi}\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(z), \quad \zeta \in \mathbb{T}.$$

From Fubini's theorem it follows that

$$\int_{\mathbb{T}} S_\mu(\zeta)\, |d\zeta| = \mu(\mathbb{D}) < +\infty. \tag{3.1}$$

Let $r \in (0,1)$ with $\mu(r\mathbb{D}) > 0$, where $r\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$. Then

$$S_\mu(\zeta) \geq \frac{1}{2\pi}\int_{r\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(z) \geq \frac{1-r}{2\pi(1+r)}\mu(r\mathbb{D}) > 0, \tag{3.2}$$

for every $\zeta \in \mathbb{T}$. From Fatou's Lemma, we have $S_\mu(\zeta) \leq \liminf_{\zeta_n \to \zeta} S_\mu(\zeta_n)$ for every sequence $\{\zeta_n\} \subseteq \mathbb{T}$ converging to a point $\zeta \in \mathbb{T}$. Thus, S_μ is a positive lower semi-continuous function on $\mathbb{T}$ such that $S_\mu \in L^1(\mathbb{T})$. In fact, by [18, 26], we see that for every lower semicontinuous function ϕ on $\mathbb{T}$ such that $\phi \in L^1(\mathbb{T})$ and $\phi > c$ for some constant $c > 0$, there exists a finite measure μ on $\mathbb{D}$ such that $\phi = S_\mu$ on $\mathbb{T}$.

An outer function for the Hardy space H^2 is a function of the form

$$O(z) = \eta \exp\Big(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log\psi(\zeta)\, \frac{|d\zeta|}{2\pi}\Big), \quad \eta \in \mathbb{T},$$

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where $\psi > 0$ almost everywhere on $\mathbb{T}$, $\log\psi \in L^1(\mathbb{T})$, and $\psi \in L^2(\mathbb{T})$. See [13] for the theory of outer functions. By (3.1) and (3.2),

$$O_\mu(z) = \exp\Big(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{\sqrt{S_\mu(\zeta)}} \frac{|d\zeta|}{2\pi}\Big), \quad z \in \mathbb{D},$$

is an outer function for H^2 with $|O_\mu(\zeta)| = 1/\sqrt{S_\mu(\zeta)}$, a.e. $\zeta \in \mathbb{T}$.

Now we are ready to define a class of weighted Hardy spaces H^2_μ. Let N^+ denote the well-known subset of the Nevanlinna class in [13]. Namely N^+ is the space of functions $f \in H(\mathbb{D})$ such that

$$\lim_{r\to 1} \int_{\mathbb{T}} \log^+ |f(r\zeta)|\,|d\zeta| = \int_{\mathbb{T}} \log^+ |f(\zeta)|\,|d\zeta|.$$

Note that every $f \in N^+$ has nontangential limit $f(\zeta)$ for almost every $\zeta \in \mathbb{T}$. The weighted Hardy space H^2_μ corresponding to a finite positive Borel measure μ is defined by $H^2_\mu = \{f \in N^+ : \int_{\mathbb{T}} |f(\zeta)|^2 S_\mu(\zeta)\,|d\zeta| < +\infty\}$. Equipped with the norm

$$\|f\|^2_{H^2_\mu} = \int_{\mathbb{T}} |f(\zeta)|^2 S_\mu(\zeta)\,|d\zeta|,$$

H^2_μ is a Hilbert space. It is well known [10, p. 68] that, if

$$O_\mu H^2 = \{O_\mu f : f \in H^2\},$$

then $H^2_\mu = O_\mu H^2$ and $\|O_\mu f\|_{H^2_\mu} = \|f\|_{H^2}$ for every $f \in H^2$. Note that O_μ is a bounded function on $\mathbb{D}$. Clearly, $H^2_\mu \subseteq H^2$. If ν is a (possibly infinite) positive Borel measure on $\mathbb{D}$, it follows from Littlewood's theorem [16, p. 94] that $\lim_{r\to 1} U_\nu(r\zeta) = 0$ for almost every $\zeta \in \mathbb{T}$. For finite positive Borel measures μ such that the above limit is zero everywhere on $\mathbb{T}$, the space H^2_μ coincides with the space introduced by E. A. Poletsky and M. I. Stessin [27] via plurisubharmonic exhaustion functions on hyperconvex domains in $\mathbb{C}^n$ (see also [1, 29, 35]).

Let X be a Hilbert space of analytic functions on $\mathbb{D}$. A positive Borel measure ν on $\mathbb{D}$ is a Carleson measure for X if there exists a positive constant C such that $\|f\|_{L^2(\nu)} \le C\|f\|_X$ for all $f \in X$. Carleson measures for H^2 have been characterized by L. Carleson via a geometric condition (see [13, p. 157] or [16, p. 31]). For every arc $I \subseteq \mathbb{T}$ let $S(I) = \{re^{i\theta} \in \mathbb{D} : e^{i\theta} \in I, 1 - \frac{\ell(I)}{2\pi} < r < 1\}$ be the corresponding Carleson box, where $\ell(I)$ is the length of the arc I. Then ν is a Carleson measure for H^2 if and only if there exists a positive constant C such that $\nu(S(I)) \le C\ell(I)$ for every arc $I \subseteq \mathbb{T}$. Using the representation $H^2_\mu = O_\mu H^2$, we obtain a characterization of Carleson measures for H^2_μ . In the next subsection, we will show that some $\mathcal{D}_\mu$ spaces are equal to H^2_μ. The following theorem also characterizes Carleson measures for some $\mathcal{D}_\mu$ spaces (see Corollary 3.5).

Theorem 3.1 *Let ν be a positive Borel measure on $\mathbb{D}$. Then ν is a Carleson measure for H^2_μ if and only if $|O_\mu|^2 d\nu$ is a Carleson measure for H^2, that is, if and only if there exists $C > 0$ such that $\int_{S(I)} |O_\mu|^2\, d\nu \le C\ell(I)$, for every arc $I \subseteq \mathbb{T}$.*

Proof We have that ν is a Carleson measure for H^2_μ if and only if there exists a constant $C > 0$ such that $\big(\int_{\mathbb{D}} |f|^2\, d\nu\big)^{1/2} \le C\|f\|_{H^2_\mu}$, for every $f \in H^2_\mu$. From the equality

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$H^2_\mu = O_\mu H^2$ and the norm equality $\|O_\mu f\|_{H^2_\mu} = \|f\|_{H^2}$, $f \in H^2$, we obtain that the above condition is equivalent with the condition $\big(\int_{\mathbb{D}} |g|^2 |O_\mu|^2\, d\nu\big)^{1/2} \le C\|g\|_{H^2}$, for every $g \in H^2$, which is true if and only if $|O_\mu|^2 d\nu$ is a Carleson measure for H^2. ∎

From the inequality

$$\|f\|^2_{H^2_\mu} = \int_{\mathbb{T}} |f(\zeta)|^2 S_\mu(\zeta)\, |d\zeta| \ge \big(\inf_{\zeta\in\mathbb{T}} S_\mu(\zeta)\big) \|f\|_{H^2}, \quad f \in H^2_\mu,$$

it follows that every Carleson measure for H^2 is Carleson measure for H^2_μ. The converse is not true. In order to give counterexamples, we will use the following theorem [17]. A sequence $\{z_n\} \subseteq \mathbb{D}$ is called an interpolating sequence if, for every bounded sequence $\{a_n\}$, there exists a bounded holomorphic function f on $\mathbb{D}$ such that $f(z_n) = a_n$, $n \in \mathbb{N}$. Equivalently (see [13, p. 149] or [16, p. 278]), $\{z_n\} \subseteq \mathbb{D}$ is an interpolating sequence if and only if there exists $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} \prod_{n\in\mathbb{N}\setminus\{k\}} \Big| \frac{z_n - z_k}{1 - \overline{z_k} z_n} \Big| \ge \delta.$$

It is well known [16, p. 278] that if $\{z_n\} \subseteq \mathbb{D}$ is an interpolating sequence, then the measure $\mu = \sum_{n=1}^{+\infty} (1 - |z_n|^2)\delta_{z_n}$ is a Carleson measure for H^2. For $C > 0$, $\gamma \ge 1$, and $\xi \in \mathbb{T}$, let $R(C, \gamma, \xi) = \{z \in \mathbb{D} : |1 - \overline{\xi} z|^\gamma < C(1 - |z|^2)\}$.

Theorem 3.2 ([17, Theorems 4 and 5]) *Let $\gamma \ge 2$. Then there exist $C > 0$ and an interpolating sequence $\{z_n\}$ contained in $R(C, \gamma, 1)$ such that*

$$\sum_{n=1}^{+\infty} (1 - |z_n|^2)^\beta < +\infty, \quad \beta \in \Big(1 - \frac{1}{\gamma}, +\infty\Big)$$

and

$$\sum_{n=1}^{+\infty} (1 - |z_n|^2)^{1-\frac{1}{\gamma}} = +\infty.$$

As mentioned before, by [18, 26], there exists a finite positive Borel measure μ_α on $\mathbb{D}$ such that $S_{\mu_\alpha}(\zeta) = \frac{1}{|1-\zeta|^{2\alpha}} \in L^1(\mathbb{T})$, $\alpha \in (0, 1/2)$. Note that $|O_{\mu_\alpha}(z)| = |1 - z|^\alpha$. In the following proposition we provide a family of measures that are not Carleson measures for H^2, while they are Carleson measures for $H^2_{\mu_\alpha}$.

Proposition 3.3 *Suppose that $R(C, \gamma, 1)$ is as in Theorem 3.2. Let $\{z_n\} \subseteq R(C, \gamma, 1)$ be an interpolating sequence satisfying*

$$\sum_{n=1}^{+\infty} (1 - |z_n|^2)^{1-\frac{1}{\gamma}} = +\infty.$$

Consider the measure $\lambda = \sum_{n=1}^{\infty} (1 - |z_n|^2)^\beta \delta_{z_n}$, $\beta > 0$. Then

(i) *λ is not a Carleson measure for H^2 for every $\beta \in (1 - \frac{1}{\gamma}, 1)$.*

(ii) *λ is a Carleson measure for $H^2_{\mu_\alpha}$ for every $\beta \in [1 - \frac{2\alpha}{\gamma}, 1)$.*

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Proof (i) Suppose that, for some $0 < \epsilon < \frac{1}{\gamma}$ and $\beta = 1-\epsilon$, λ is a Carleson measure for H^2. Choose any $\alpha \in (\frac{1-\gamma\epsilon}{2}, \frac{1}{2})$. Since $\{z_n\} \subseteq R(C,\gamma,1)$ and $2\alpha > 1-\gamma\epsilon$, we have

$$|1-z_n|^{2\alpha} \leq |1-z_n|^{1-\gamma\epsilon} \leq C^{\frac{1}{\gamma}-\epsilon}(1-|z_n|^2)^{\frac{1}{\gamma}-\epsilon}.$$

Consider the measure $\nu = \sum_{n=1}^{\infty}(1-|z_n|^2)^{1-\frac{1}{\gamma}}\delta_{z_n}$. Then

$$\begin{aligned}\int_{S(I)} |O_{\mu_\alpha}(z)|^2\,d\nu &= \sum_{z_n\in S(I)} |1-z_n|^{2\alpha}(1-|z_n|^2)^{1-\frac{1}{\gamma}}\\ &\leq C^{\frac{1}{\gamma}-\epsilon}\sum_{z_n\in S(I)}(1-|z_n|^2)^{1-\epsilon} = C^{\frac{1}{\gamma}-\epsilon}\lambda(S(I))\\ &\lesssim C^{\frac{1}{\gamma}-\epsilon}\ell(I),\end{aligned}$$

for every arc $I \subseteq \mathbb{T}$, since λ is assumed to be a Carleson measure for H^2. Therefore, $|O_{\mu_\alpha}(z)|^2 d\nu$ is a Carleson measure for H^2 and from Theorem 3.1 we obtain that ν is a Carleson measure for $H^2_{\mu_\alpha}$. But $\|1\|_{H^2_{\mu_\alpha}} = (\int_{\mathbb{T}} S_{\mu_\alpha}(\zeta)\,|d\zeta|)^{1/2} < +\infty$, while

$$\Big(\int_{\mathbb{D}} |1|^2\,d\nu\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{+\infty}(1-|z_n|^2)^{1-\frac{1}{\gamma}}\Big)^{\frac{1}{2}} = +\infty,$$

which contradicts the fact that ν is a Carleson measure for $H^2_{\mu_\alpha}$. We obtain that λ is not a Carleson measure for H^2 for $\beta = 1-\epsilon$. Therefore λ is not a Carleson measure for H^2 for every $\beta \in (1-\frac{1}{\gamma}, 1-\epsilon)$. Since ϵ can be arbitrary small, the conclusion follows.

(ii) Since $\{z_n\} \subseteq R(C,\gamma,1)$, one gets that $|1-z_n|^{2\alpha} \leq C^{\frac{2\alpha}{\gamma}}(1-|z_n|^2)^{\frac{2\alpha}{\gamma}}$. Since $\{z_n\}$ is an interpolating sequence, the measure $\sum_{n=1}^{+\infty}(1-|z_n|^2)\delta_{z_n}$ is a Carleson measure for H^2. Note that $\frac{2\alpha}{\gamma}+\beta \geq 1$. We deduce that

$$\begin{aligned}\int_{S(I)} |O_{\mu_\alpha}(z)|^2\,d\lambda &= \sum_{z_n\in S(I)} |1-z_n|^{2\alpha}(1-|z_n|^2)^{\beta} \leq C^{\frac{2\alpha}{\gamma}}\sum_{z_n\in S(I)}(1-|z_n|^2)^{\frac{2\alpha}{\gamma}+\beta}\\ &\leq C^{\frac{2\alpha}{\gamma}}\sum_{z_n\in S(I)}(1-|z_n|^2) \lesssim C^{\frac{2\alpha}{\gamma}}\ell(I),\end{aligned}$$

for every arc $I \subseteq \mathbb{T}$. Therefore, $|O_{\mu_\alpha}(z)|^2 d\lambda$ is a Carleson measure for H^2 and from Theorem 3.1 we obtain that λ is a Carleson measure for $H^2_{\mu_\alpha}$ for every $\beta \in [1-\frac{2\alpha}{\gamma}, 1)$. ∎

3.2 $\mathcal{D}_\mu$ and H^2_μ Spaces

In this subsection, we show that the equality $\mathcal{D}_\mu = H^2_\mu$ holds if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. Using the relation, we compute the reproducing kernel of $\mathcal{D}_\mu$ when μ is an infinite sum of point-mass measures on $\mathbb{D}$.

Theorem 3.4 *Let μ be a finite positive Borel measure on $\mathbb{D}$. Then $H^2_\mu = \mathcal{D}_\mu \cap L^2(\mu)$ and $\|f\|^2_{\mathcal{D}_\mu} = \|f\|^2_{H^2} + \|f\|^2_{H^2_\mu} - \|f\|^2_{L^2(\mu)}$, for every $f \in H^2_\mu$. The equality $H^2_\mu = \mathcal{D}_\mu$ holds if and only if μ is a Carleson measure for $\mathcal{D}_\mu$; in this case, the norms $\|\cdot\|_{\mathcal{D}_\mu}$ and $\|\cdot\|_{H^2_\mu}$ are equivalent.*

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Proof Let $f \in H^2$ and note that $\Delta|f(z)|^2 = 4|f'(z)|^2$, $z \in \mathbb{D}$. It is well known [13, p. 28] that the least harmonic majorant of the subharmonic function $|f|^2$ on $\mathbb{D}$ is the function

$$h_f(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}|f(\zeta)|^2\,|d\zeta|, \quad z \in \mathbb{D}.$$

From the Riesz decomposition theorem [6, p. 105–106] we obtain that

$$\begin{aligned} |f(z)|^2 &= h_f(z) - \frac{1}{2\pi}\int_{\mathbb{D}} \log\Big|\frac{1-\overline{w}z}{z-w}\Big|\,\Delta|f(w)|^2\,dA(w) \\ &= h_f(z) - \frac{2}{\pi}\int_{\mathbb{D}} \log\Big|\frac{1-\overline{w}z}{z-w}\Big|\,|f'(w)|^2\,dA(w). \end{aligned} \tag{3.3}$$

From the above equality and Fubini's theorem we obtain that

$$\begin{aligned} \int_{\mathbb{T}} |f(\zeta)|^2 S_\mu(\zeta)|d\zeta| &= \int_{\mathbb{T}} |f(\zeta)|^2 \frac{1}{2\pi}\int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta-z|^2}\,d\mu(z)|d\zeta| = \int_{\mathbb{D}} h_f(z)\,d\mu(z) \\ &= \int_{\mathbb{D}} |f(z)|^2\,d\mu(z) \\ &\qquad + \int_{\mathbb{D}} \frac{2}{\pi}\int_{\mathbb{D}} \log\Big|\frac{1-\overline{w}z}{z-w}\Big|\,|f'(w)|^2\,dA(w)d\mu(z) \\ &= \int_{\mathbb{D}} |f(z)|^2\,d\mu(z) + \frac{2}{\pi}\int_{\mathbb{D}} |f'(w)|^2 U_\mu(w)\,dA(w). \end{aligned}$$

This implies that for every $f \in H^2_\mu$,

$$\begin{aligned} \|f\|^2_{\mathcal{D}_\mu} &= \|f\|^2_{H^2} + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\,dA(z) \\ &= \|f\|^2_{H^2} + \|f\|^2_{H^2_\mu} - \|f\|^2_{L^2(\mu)}, \end{aligned}$$

and $H^2_\mu = \mathcal{D}_\mu \cap L^2(\mu)$. The equality $H^2_\mu = \mathcal{D}_\mu$ holds if and only if $\mathcal{D}_\mu \subseteq L^2(\mu)$. By the closed graph theorem, $\mathcal{D}_\mu \subseteq L^2(\mu)$ holds if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. Thus the equality $H^2_\mu = \mathcal{D}_\mu$ holds if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. In this case, by the closed graph theorem again, we obtain that the norms $\|\cdot\|_{\mathcal{D}_\mu}$ and $\|\cdot\|_{H^2_\mu}$ are equivalent. ∎

We will denote by $\mathbb{M}$ the family of finite positive Borel measures μ on $\mathbb{D}$ such that $\mathcal{D}_\mu = H^2_\mu$. Equivalently, $\mu \in \mathbb{M}$ if and only if μ is a Carleson measure for $\mathcal{D}_\mu$. We note that if μ is a Carleson measure for H^2, then $\mathcal{D}_\mu \subseteq H^2 \subseteq L^2(\mu)$ and from Theorem 3.4 we obtain that $\mu \in \mathbb{M}$. From Theorem 3.1 and Theorem 3.4 we obtain the following corollary.

Corollary 3.5 *Suppose that $\mu \in \mathbb{M}$. Then a positive measure ν on $\mathbb{D}$ is a Carleson measure for $\mathcal{D}_\mu$ if and only if $|O_\mu|^2 d\nu$ is a Carleson measure for H^2.*

Applying Theorem 3.4 and following an argument of S. M. Shimorin [34, p. 281], we compute the reproducing kernel of $\mathcal{D}_\mu$ for certain measures μ as follows.

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Theorem 3.6 *Let* $\mu = \sum_{n=1}^{+\infty} a_n \delta_{z_n}$ *be a finite positive measure on* $\mathbb{D}$*, where* $z_n \in \mathbb{D}$ *and* $a_n > 0$*,* $n \in \mathbb{N}$*. If* $\mu \in \mathbb{M}$*, then the reproducing kernel of* $\mathcal{D}_\mu$ *for* $\lambda \in \mathbb{D}$ *with respect to* $\|\cdot\|_{\mathcal{D}_\mu}$ *is*

$$K(z,\lambda) = K_0(z,\lambda) + \sum_{n=1}^{+\infty} \frac{a_n K_0(z,z_n) K_0(z_n,\lambda)}{1 - a_n K_0(z_n,z_n)}, \quad z \in \mathbb{D},$$

where

$$K_0(z,\lambda) = \frac{\overline{T_\mu(\lambda)}}{1-\bar{\lambda} z} T_\mu(z), \quad z \in \mathbb{D},$$

$$T_\mu(z) = \exp\Big(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{\sqrt{1 + S_\mu(\zeta)}} |d\zeta| \Big), \quad z \in \mathbb{D}.$$

Proof From Theorem 3.4, we see that $\mathcal{D}_\mu = H^2_\mu$ and

$$\begin{aligned} \|f\|^2_{\mathcal{D}_\mu} &= \|f\|^2_{H^2} + \|f\|^2_{H^2_\mu} - \|f\|^2_{L^2(\mu)} \\ &= \int_{\mathbb{T}} |f(\zeta)|^2 (1 + S_\mu(\zeta)) |d\zeta| - \sum_{n=1}^{+\infty} a_n |f(z_n)|^2. \end{aligned}$$

For $f \in \mathcal{D}_\mu$, let $\|f\|_0^2 = \int_{\mathbb{T}} |f(\zeta)|^2 (1 + S_\mu(\zeta)) |d\zeta|$ and $\|f\|_n^2 = \|f\|_0^2 - \sum_{i=1}^n a_i |f(z_i)|^2$. Since $\|f\|_n^2 > \|f\|^2_{\mathcal{D}_\mu} > 0$, $n \in \mathbb{N}$, for every $f \in \mathcal{D}_\mu \setminus \{0\}$, and $\mathcal{D}_\mu = H^2_\mu$, $\|\cdot\|_0$ and $\|\cdot\|_n$, $n \in \mathbb{N}$, define norms that make $\mathcal{D}_\mu$ a Hilbert space. The reproducing kernel of $\mathcal{D}_\mu$ for $\lambda \in \mathbb{D}$ with respect to $\|\cdot\|_0$ is [11, Theorem 3.1]

$$K_0(z,\lambda) = \frac{\overline{T_\mu(\lambda)}}{1-\bar{\lambda} z} T_\mu(z), \quad z \in \mathbb{D},$$

where $T_\mu \in H^2$ is the outer function on $\mathbb{D}$ such that $|T_\mu(\zeta)| = 1/\sqrt{1 + S_\mu(\zeta)}$, $\zeta \in \mathbb{T}$, and $T_\mu(0) > 0$, that is,

$$T_\mu(z) = \exp\Big(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{\sqrt{1 + S_\mu(\zeta)}} |d\zeta| \Big), \quad z \in \mathbb{D}.$$

The reproducing kernel of $\mathcal{D}_\mu$ for $\lambda \in \mathbb{D}$ with respect to $\|\cdot\|_1$ is [34, p. 281]

$$K_1(z,\lambda) = K_0(z,\lambda) + \frac{a_1 K_0(z,z_1) K_0(z_1,\lambda)}{1 - a_1 K_0(z_1,z_1)}.$$

Iterating the above formula and using the definition of the norms $\|\cdot\|_n$, we obtain that the reproducing kernel of $\mathcal{D}_\mu$ for $\lambda \in \mathbb{D}$ with respect to $\|\cdot\|_n$ is

$$K_n(z,\lambda) = K_0(z,\lambda) + \sum_{i=1}^{n} \frac{a_i K_0(z,z_i) K_0(z_i,\lambda)}{1 - a_i K_0(z_i,z_i)}.$$

Therefore, from the relations $\|f\|_{n+1} \le \|f\|_n$, $n = 0,1,\dots$ and

$$\|f\|^2_{\mathcal{D}_\mu} = \int_{\mathbb{T}} |f(\zeta)|^2 (1 + S_\mu(\zeta)) |d\zeta| - \sum_{n=1}^{+\infty} a_n |f(z_n)|^2 = \lim_{n\to+\infty} \|f\|_n^2, \quad f \in \mathcal{D}_\mu,$$

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we obtain that (see [34, Lemma 2.4] and references therein) the reproducing kernel of $\mathcal{D}_\mu$ for $\lambda \in \mathbb{D}$ with respect to $\|\cdot\|_{\mathcal{D}_\mu}$ is

$$K(z,\lambda) = \lim_{n\to+\infty} K_n(z,\lambda) = K_0(z,\lambda) + \sum_{n=1}^{+\infty} \frac{a_n K_0(z,z_n)K_0(z_n,\lambda)}{1-a_nK_0(z_n,z_n)}.$$

The proof is complete. ∎

Let μ be a finite positive Borel measure on $\mathbb{D}$. From [3, Proposition 2.6], $\mathcal{D}_\mu = H^2$ if and only if $\sup_{\zeta\in\mathbb{T}} S_\mu(\zeta) < \infty$. Using this result we construct a measure μ_0 such that $\mathcal{D}_{\mu_0} = H^2_{\mu_0}$ and $\mathcal{D}_{\mu_0} \neq H^2$. In fact, consider $\mu_0 = \sum_{n=1}^{\infty}(1-|z_n|)\delta_{z_n}$, where $z_n = 1-2^{-n}$. By [13, Theorem 9.2], $\{z_n\}$ is an interpolating sequence. Hence, μ_0 is a Carleson measure for H^2. As mentioned before Corollary 3.5, $\mathbb{M}$ contains the family of Carleson measures for H^2. Thus $\mu_0 \in \mathbb{M}$, that is, $\mathcal{D}_{\mu_0} = H^2_{\mu_0}$. A direct computation gives that $S_{\mu_0}(1) = \infty$. Therefore, $\mathcal{D}_{\mu_0} \neq H^2$.

4 Boundary Behavior and Inner-outer Factorization of $\mathcal{D}_\mu$ Spaces

In this section, in light of the classical theory of the Hardy space H^2, we consider boundary behavior and inner-outer factorization of functions in $\mathcal{D}_\mu$ spaces. It is worth mentioning that the measures μ in this and the next section can be infinite.

It is well known [13] that the function f in the Hardy space H^2 has non-tangential limit $f(\zeta)$ for almost every ζ on the unit circle $\mathbb{T}$. One of the most essential properties on H^2 is that H^2 has an inner-outer factorization. Namely, every function f in H^2 with $f \not\equiv 0$ can be written as $f = IO$, where I is inner and $O \in H^2$ is outer. Conversely, such a function IO belongs to H^2. Note that $\mathcal{D}_\mu$ spaces are always subsets of H^2. It is natural to consider boundary behavior and inner-outer factorization of functions in $\mathcal{D}_\mu$ spaces.

The following result gives a characterization of $\mathcal{D}_\mu$ spaces by boundary values.

Theorem 4.1 *Let $f \in H^2$ and let μ be a positive Borel measure on $\mathbb{D}$. Then $f \in \mathcal{D}_\mu$ if and only if*

$$\int_{\mathbb{D}}\int_{\mathbb{T}}\int_{\mathbb{T}} |f(\zeta)-f(\eta)|^2 \frac{1-|z|^2}{|\zeta-z|^2}\frac{1-|z|^2}{|\eta-z|^2}\,|d\zeta|\,|d\eta|\,d\mu(z) < \infty.$$

Proof Let $f \in H^2$. From the equality (3.3), we know that

$$\frac{2}{\pi}\int_{\mathbb{D}} \log\Big|\frac{1-\overline{w}z}{z-w}\Big|\,|f'(w)|^2\,dA(w) = \frac{1}{2\pi}\int_{\mathbb{T}} |f(\zeta)|^2\frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta| - |f(z)|^2$$

for all $z \in \mathbb{D}$. From [22, p. 221], one gets that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |f(\zeta)|^2\frac{1-|z|^2}{|\zeta-z|^2}\,|d\zeta| - |f(z)|^2 = \frac{1}{8\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |f(\zeta)-f(\eta)|^2\frac{1-|z|^2}{|\zeta-z|^2}\frac{1-|z|^2}{|\eta-z|^2}\,|d\zeta|\,|d\eta|.$$

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Combining the above formulas and Fubini's theorem, we see that

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z)\, dA(z) = \frac{1}{16\pi} \int_{\mathbb{D}} \int_{\mathbb{T}} \int_{\mathbb{T}} |f(\zeta) - f(\eta)|^2 \frac{1-|z|^2}{|\zeta - z|^2} \frac{1-|z|^2}{|\eta - z|^2} |d\zeta|\,|d\eta| d\mu(z).$$

The conclusion follows. ∎

Checking the proof of Theorem 4.1, we get the following result immediately. See [2, p. 99] for the same characterization of some radial Dirichlet spaces.

Proposition 4.2 *Let $f \in H^2$ and let μ be a positive Borel measure on $\mathbb{D}$. Then $f \in \mathcal{D}_\mu$ if and only if*

$$\int_{\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |f(\zeta)|^2 \frac{1-|w|^2}{|\zeta - w|^2} |d\zeta| - |f(w)|^2 \Big)\, d\mu(w) < \infty.$$

Applying the above characterization of $\mathcal{D}_\mu$, we obtain the following description of $\mathcal{D}_\mu$ spaces via inner-outer factorization. See [8] for recent results related to inner-outer factorization of functions in a class of Möbius invariant spaces.

Theorem 4.3 *Let μ be a positive Borel measure on $\mathbb{D}$ and let $f \in H^2$ with $f \not\equiv 0$. Then $f \in \mathcal{D}_\mu$ if and only if $f = IO$, where I is an inner function and O is an outer function in $\mathcal{D}_\mu$ for which*

$$\int_{\mathbb{D}} |O(w)|^2 (1 - |I(w)|^2)\, d\mu(w) < \infty. \tag{4.1}$$

Proof Let $f \in \mathcal{D}_\mu$. Since $f \not\equiv 0$, f must be of the form IO, where I is an inner function and O is an outer function for H^2. Note that $|I(z)| \le 1$ for all $z \in \mathbb{D}$ and $|I(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. This together with Proposition 4.2 give that

$$\int_{\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |O(\zeta)|^2 \frac{1-|w|^2}{|\zeta - w|^2} |d\zeta| - |O(w)|^2 \Big)\, d\mu(w) \le \int_{\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |O(\zeta)|^2 \frac{1-|w|^2}{|\zeta - w|^2} |d\zeta| - |I(w)O(w)|^2 \Big)\, d\mu(w) < \infty.$$

By Proposition 4.2 again, one gets $O \in \mathcal{D}_\mu$. Furthermore,

$$\begin{aligned} \int_{\mathbb{D}} |O(w)|^2 (1 - |I(w)|^2)\, d\mu(w) &= \int_{\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |O(\zeta)|^2 \frac{1-|w|^2}{|\zeta - w|^2} |d\zeta| - |I(w)O(w)|^2 \Big)\, d\mu(w) \\ &\quad - \int_{\mathbb{D}} \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |O(\zeta)|^2 \frac{1-|w|^2}{|\zeta - w|^2} |d\zeta| - |O(w)|^2 \Big)\, d\mu(w) \\ &< \infty. \end{aligned}$$

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On the other hand, let $O \in \mathcal{D}_\mu$ and let (4.1) hold. Using Proposition 4.2, we obtain that

$$\begin{aligned}\int_{\mathbb{D}}\Big(\frac{1}{2\pi}\int_{\mathbb{T}}|O(\zeta)|^2\frac{1-|w|^2}{|\zeta-w|^2}\,|d\zeta| - |I(w)O(w)|^2\Big)\,d\mu(w)\\ = \int_{\mathbb{D}}|O(w)|^2(1-|I(w)|^2)\,d\mu(w)\\ + \int_{\mathbb{D}}\Big(\frac{1}{2\pi}\int_{\mathbb{T}}|O(\zeta)|^2\frac{1-|w|^2}{|\zeta-w|^2}\,|d\zeta| - |O(w)|^2\Big)\,d\mu(w) < \infty,\end{aligned}$$

which shows that $IO \in \mathcal{D}_\mu$. We finish the proof. ∎

5 Composition Operators on $\mathcal{D}_\mu$ Spaces

Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map of the unit disk $\mathbb{D}$. The function ϕ induces a composition operator C_ϕ acting on $H(\mathbb{D})$ by the formula $C_\phi f(z) = f(\phi(z))$, $z \in \mathbb{D}$, for every $f \in H(\mathbb{D})$. In this section, we characterize the boundedness and the compactness of composition operators on $\mathcal{D}_\mu$ spaces. In fact, $\mathcal{D}_\mu$ spaces include radial Dirichlet spaces. Let K be a decreasing concave function on $[0,1)$ satisfying $\lim_{r\to 1} K(r) = 0$ and let $\omega(z) := K(|z|)$, $z \in \mathbb{D}$. Then ω is a radial superharmonic function on $\mathbb{D}$ and $\omega = U_{-\Delta\omega}$ (see [2, p. 99]). The corresponding Dirichlet spaces $\mathcal{D}_\omega$ with radial superharmonic weights have been studied by several researchers [7, 19, 20]. In particular, for $K(r) = r^p$, $p \in (0,1)$, we obtain the usual Dirichlet type space $\mathcal{D}_p$. Therefore, our results in this section cover the corresponding results for composition operators acting on Dirichlet spaces with radial superharmonic weights [19, 25]. We will give two equivalent conditions to describe the boundedness of composition operators on $\mathcal{D}_\mu$ spaces. From D. Sarason and J-N. O. Silva [31, p. 447], in general one of the two corresponding conditions for $\mathcal{D}(\nu)$ spaces with harmonic weights cannot be used to describe the boundedness of composition operators on $\mathcal{D}(\nu)$.

5.1 Preliminaries

In this subsection, we give an equivalent norm of $\mathcal{D}_\mu$ spaces, which is convenient for the computation. We also consider the submean value property for certain generalized Nevanlinna counting functions. Some test functions in $\mathcal{D}_\mu$ spaces are also given.

For $z, w \in \mathbb{D}$, denote by $\sigma_z(w) = (z-w)/(1-\bar{z}w)$ the Möbius transformation of the unit disk interchanging z and 0.

Lemma 5.1 *Let μ be a positive Borel measure on $\mathbb{D}$, let*

$$V_\mu(z) = \int_{\mathbb{D}}(1-|\sigma_z(w)|^2)\,d\mu(w), \quad z \in \mathbb{D},$$

and let $f \in H^2$. Then $f \in \mathcal{D}_\mu$ if and only if $\int_{\mathbb{D}}|f'(z)|^2 V_\mu(z)\,dA(z) < +\infty$.

Proof It is well known that

$$\int_{\mathbb{D}}|f'(z)|^2(1-|\sigma_a(z)|^2)\,dA(z) \approx \int_{\mathbb{D}}|f'(z)|^2\Big(\log\frac{1}{|\sigma_a(z)|}\Big)\,dA(z),$$

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where $a \in \mathbb{D}$ and $f \in H^2$ (see [16, p. 231]). The conclusion follows by integrating the above relation with respect to μ and applying Fubini's theorem. ∎

In this section, for $f \in \mathcal{D}_\mu$, we use the following norm of f in $\mathcal{D}_\mu$.

$$\|f\|_{\mathcal{D}_\mu} = \Big(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\, dA(z)\Big)^{1/2}.$$

The Nevanlinna counting function of ϕ is defined by $N_\phi(z) = \sum_{\phi(a)=z} \log \frac{1}{|a|}$, $z \in \mathbb{D}$, and the Nevanlinna counting function of ϕ with respect to a Borel measure μ on $\mathbb{D}$ is defined by $N_{\phi,\mu}(z) = \sum_{\phi(a)=z} V_\mu(a)$, $z \in \mathbb{D}$, where, in the above sums, multiplicities are taken into account. Note that $N_{\phi,\mu}(z) = 0$ if $z \notin \phi(\mathbb{D})$. By the change of variable formula ([2, p. 98] or [31, p. 435]), if $f \in H(\mathbb{D})$, then

$$\begin{aligned}\int_{\mathbb{D}} |(C_\phi f)'(z)|^2 V_\mu(z)\, dA(z) &= \int_{\phi(\mathbb{D})} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z)\\ &= \int_{\mathbb{D}} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z).\end{aligned}$$

The following result gives the submean value property of $N_{\phi,\mu}$.

Lemma 5.2 *Let μ be a positive Borel measure on $\mathbb{D}$ and let ϕ be an analytic self-map of $\mathbb{D}$. Then for every disk $B \subseteq \mathbb{D} \smallsetminus \{\phi(0)\}$ with center at z,*

$$N_{\phi,\mu}(z) \le \frac{1}{A(B)} \int_B N_{\phi,\mu}(w)\, dA(w).$$

Proof It follows from Fatou's lemma that V_μ is lower semicontinuous on $\mathbb{D}$. Note that the function $z \to (1 - |\sigma_w(z)|^2)$ is superharmonic on $\mathbb{D}$ for every $w \in \mathbb{D}$. Then V_μ satisfies the supermean value inequality on $\mathbb{D}$. Hence, V_μ is a positive superharmonic function on $\mathbb{D}$. Since

$$V_\mu(z) = \int_{\mathbb{D}} (1 - |\sigma_w(z)|^2)\, d\mu(w) \le 2 \int_{\mathbb{D}} \log \frac{1}{|\sigma_w(z)|}\, d\mu(w) = 2U_\mu(z), \quad z \in \mathbb{D},$$

and the greatest harmonic minorant of U_μ on $\mathbb{D}$ is the zero function [6, p. 98], the greatest harmonic minorant of V_μ on $\mathbb{D}$ is the zero function. From the Riesz decomposition theorem [6, p. 105] we obtain that there exists a positive measure ν on $\mathbb{D}$ such that $V_\mu(z) = U_\nu(z)$, $z \in \mathbb{D}$. Consequently,

$$\begin{aligned}N_{\phi,\mu}(z) = \sum_{\phi(a)=z} U_\nu(a) &= \int_{\mathbb{D}} \sum_{\phi(a)=z} \log \frac{1}{|\sigma_a(w)|}\, d\nu(w)\\ &= \int_{\mathbb{D}} \sum_{\phi(\sigma_w(a))=z} \log \frac{1}{|a|}\, d\nu(w) = \int_{\mathbb{D}} N_{\phi\circ\sigma_w}(z)\, d\nu(w).\end{aligned}$$

From the submean value inequality of the Nevanlinna counting function on $\mathbb{D} \smallsetminus \{\phi(0)\}$ (see [32, p. 190]) and Fubini's theorem we obtain that, for every disk $B \subseteq \mathbb{D} \smallsetminus \{\phi(0)\}$ with center at z,

$$\begin{aligned}N_{\phi,\mu}(z) = \int_{\mathbb{D}} N_{\phi\circ\sigma_w}(z)\, d\nu(w) &\le \frac{1}{A(B)} \int_B \int_{\mathbb{D}} N_{\phi\circ\sigma_w}(a)\, d\nu(w) dA(a)\\ &= \frac{1}{A(B)} \int_B N_{\phi,\mu}(a)\, dA(a).\end{aligned}$$

∎

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We also need the following useful inequality.

Lemma 5.3 ([24, Lemma 2.5]) *Suppose that $s > -1$, $r, t > 0$, and $r + t - s > 2$. If $t < s + 2 < r$, then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\overline{w}z|^r|1-\overline{w}\zeta|^t}\, dA(w) \lesssim \frac{(1-|z|^2)^{2+s-r}}{|1-\overline{\zeta}z|^t},$$

for all $z, \zeta \in \mathbb{D}$.

Recall that we always assume μ satisfy the condition (1.1). Otherwise, $\mathcal{D}_\mu$ spaces are trivial. We give some test functions in $\mathcal{D}_\mu$ spaces as follows.

Lemma 5.4 *Let μ be a positive Borel measure on $\mathbb{D}$. For every $w \in \mathbb{D}$, let*

$$f_w(z) = \frac{\sigma_w(z)}{\sqrt{V_\mu(w)}} - \frac{\sigma_w(0)}{\sqrt{V_\mu(w)}}, \quad z \in \mathbb{D},$$
$$g_w(z) = \frac{1-|w|}{(1-\overline{w}z)\sqrt{V_\mu(w)}}, \quad z \in \mathbb{D},$$

Then $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{D}_\mu} < +\infty$, and $\sup_{w\in\mathbb{D}} \|g_w\|_{\mathcal{D}_\mu} < +\infty$.

Proof From Lemma 5.3 we obtain that for every $a, w \in \mathbb{D}$,

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5.2 The Boundedness of Composition Operators on $\mathcal{D}_\mu$ Spaces

In this subsection, we characterize the boundedness of composition operators on $\mathcal{D}_\mu$ spaces. As an application, we construct a finite positive measure μ_0 such that C_ϕ is not bounded on $\mathcal{D}_{\mu_0}$ even when ϕ is a rotation.

Let $I(z) = z$ be the identity function and let μ be a positive Borel measure on $\mathbb{D}$. The condition (1.1) together with the superharmonicity of the function V_μ gives that

$$\int_{\mathbb{D}} |I'(z)|^2 V_\mu(z)\, dA(z) = \int_{\mathbb{D}} V_\mu(z)\, dA(z) \le \pi V_\mu(0) = \pi \int_{\mathbb{D}} (1-|z|^2)\, d\mu(z) < +\infty.$$

Thus $I \in \mathcal{D}_\mu$. Consequently, $\phi \in \mathcal{D}_\mu$ is a necessary condition for C_ϕ to be bounded on $\mathcal{D}_\mu$.

We characterize the boundedness of composition operators on $\mathcal{D}_\mu$ spaces as follows.

Theorem 5.5 *Let μ be a positive Borel measure on $\mathbb{D}$ and let $\phi \in \mathcal{D}_\mu$ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.*

(i) *C_ϕ is bounded on $\mathcal{D}_\mu$.*
(ii) *$N_{\phi,\mu}(w) = O(V_\mu(w))$, as $|w| \to 1$.*
(iii) *$\frac{1}{A(\Delta_w)} \int_{\Delta_w} N_{\phi,\mu}(z)\, dA(z) = O(V_\mu(w))$, as $|w| \to 1$, where*

$$\Delta_w = \Big\{ z \in \mathbb{D} : |z-w| < \frac{1}{2}(1-|w|) \Big\}.$$

Proof (i) ⇒ (ii). Suppose that C_ϕ is bounded on $\mathcal{D}_\mu$. Then

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z) &= \int_{\mathbb{D}} |(C_\phi f)'(z)|^2 V_\mu(z)\, dA(z)\\ &\lesssim |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\, dA(z),\end{aligned}$$

for all $f \in \mathcal{D}_\mu$. For every $w \in \mathbb{D}$, let

$$f_w(z) = \frac{\sigma_w(z)}{\sqrt{V_\mu(w)}} - \frac{\sigma_w(0)}{\sqrt{V_\mu(w)}}, \quad z \in \mathbb{D}.$$

Applying the previous inequality to the functions f_w and using Lemma 5.4, we obtain that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} N_{\phi,\mu}(z)\, dA(z) \lesssim V_\mu(w),$$

for all $w \in \mathbb{D}$. Now consider $|w| > (1+|\phi(0)|)/2$. Then $\phi(0) \notin \Delta_w$. It is well known [25, p. 684] that $|1-\overline{w}z| \approx (1-|w|^2)$ for $z \in \Delta_w$ and $A(\Delta_w) \approx (1-|w|^2)^2$. Combining these with Lemma 5.2, we obtain that for $|w| > (1+|\phi(0)|)/2$,

$$\begin{aligned}V_\mu(w) &\gtrsim \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} N_{\phi,\mu}(z)\, dA(z) \gtrsim \int_{\Delta_w} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} N_{\phi,\mu}(z)\, dA(z)\\ &\approx \frac{1}{A(\Delta_w)} \int_{\Delta_w} N_{\phi,\mu}(z)\, dA(z)\\ &\gtrsim N_{\phi,\mu}(w).\end{aligned}$$

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(ii) ⇒ (i). Let condition (ii) hold. Then there exist $C > 0$ and $r \in (0,1)$ such that $N_{\phi,\mu}(z) \le CV_\mu(z)$ for all $z \in \mathbb{D} \setminus r\mathbb{D}$. For $f \in \mathcal{D}_\mu$, one gets that

$$\int_{\mathbb{D}\setminus r\mathbb{D}} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^2 V_\mu(z)\, dA(z) < \infty.$$

Since $\phi \in \mathcal{D}_\mu$, we have $\int_{\mathbb{D}} N_{\phi,\mu}(z)\, dA(z) = \int_{\mathbb{D}} |\phi'(z)|^2 V_\mu(z)\, dA(z) < \infty$. Note that $\mathcal{D}_\mu \subseteq H^2$. The Cauchy Formula and the Hölder inequality yield that

$$|f'(z)| = \frac{1}{2\pi}\Big| \int_{\mathbb{T}} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \Big| \lesssim \frac{1}{(1-|z|)^2} \|f\|_{H^2} \lesssim \frac{1}{(1-|z|)^2} \|f\|_{\mathcal{D}_\mu},$$

for all $z \in \mathbb{D}$. Thus,

$$\int_{r\mathbb{D}} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z) \lesssim (1-r)^{-4} \|f\|^2_{\mathcal{D}_\mu} \int_{\mathbb{D}} N_{\phi,\mu}(z)\, dA(z) < \infty.$$

Hence, $\int_{\mathbb{D}} |(C_\phi f)'(z)|^2 V_\mu(z)\, dA(z) = \int_{\mathbb{D}} |f'(z)|^2 N_{\phi,\mu}(z)\, dA(z) < \infty$, which implies that $C_\phi f \in \mathcal{D}_\mu$ for every $f \in \mathcal{D}_\mu$. From the closed graph theorem, we know that C_ϕ is bounded on $\mathcal{D}_\mu$.

(ii) ⇒ (iii). Let $N_{\phi,\mu}(w) = O(V_\mu(w))$, as $|w| \to 1$. Using the superharmonicity of the function V_μ, we obtain that if $|w| \to 1$, then

$$\frac{1}{A(\Delta_w)} \int_{\Delta_w} N_{\phi,\mu}(z)\, dA(z) \lesssim \frac{1}{A(\Delta_w)} \int_{\Delta_w} V_\mu(z)\, dA(z) \lesssim V_\mu(w).$$

(iii) ⇒ (ii). Lemma 5.2 yields the desired result immediately. ∎

Applying Theorem 5.5, we give the following example. Because any composition operator induced by rotation is bounded on Dirichlet spaces with radial weights, the following result also gives examples of $\mathcal{D}_\mu$ spaces which are not equal to any Dirichlet space with radial weight.

Corollary 5.6 *Let $\Omega = \mathbb{D} \cap \{z \in \mathbb{C} : \Re(z) > 0\}$, and let*

$$d\mu_\epsilon(z) = \chi_\Omega(z)/|1-z|^{1+\epsilon} dA(z)$$

for some $\epsilon \in (0,1)$. Let $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and let $\phi(z) = e^{i\theta} z$ be the rotation related to $e^{i\theta}$. Then C_ϕ is not bounded on $\mathcal{D}_{\mu_\epsilon}$.

Proof A direct computation gives that $\mu_\epsilon(\mathbb{D}) < +\infty$. Let $d = \operatorname{dist}(e^{i\theta}, \Omega) > 0$ and set $D(e^{i\theta}, d/2) = \{z \in \mathbb{C} : |z - e^{i\theta}| < d/2\}$. Then for every $z \in \mathbb{D} \cap D(e^{i\theta}, d/2)$,

$$\begin{aligned} V_{\mu_\epsilon}(z) &= \int_\Omega (1 - |\sigma_w(z)|^2)\, d\mu_\epsilon(w) = (1-|z|^2) \int_\Omega \frac{1-|w|^2}{|1-\overline{w}z|^2}\, d\mu_\epsilon(w) \\ &\le (1-|z|^2) \int_\Omega \frac{1}{|z-w|^2}\, d\mu_\epsilon(w) \\ &\le (1-|z|^2) 4\mu_\epsilon(\Omega)/d^2. \end{aligned}$$

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Also, for $r \in (0,1)$,

$$V_{\mu_\epsilon}(r) = (1-r^2)\int_\Omega \frac{1-|w|^2}{|1-\overline{w}r|^2}\,d\mu_\epsilon(w) \geq (1-r^2)\int_{\Delta_r} \frac{1-|w|^2}{|1-\overline{w}r|^2}\frac{1}{|1-w|^{1+\epsilon}}\,dA(w)$$
$$\approx \int_{\Delta_r} \frac{1}{|1-w|^{1+\epsilon}}\,dA(w) \approx \frac{1}{(1-r^2)^{1+\epsilon}}A(\Delta_r) \approx (1-r^2)^{1-\epsilon}.$$

From the above estimates and the fact that ϕ is univalent, we deduce that

$$\lim_{r\to 1}\frac{N_{\phi,\mu_\epsilon}(re^{i\theta})}{V_{\mu_\epsilon}(re^{i\theta})} = \lim_{r\to 1}\frac{V_{\mu_\epsilon}(r)}{V_{\mu_\epsilon}(re^{i\theta})} \gtrsim \lim_{r\to 1}\frac{(1-r^2)^{1-\epsilon}}{1-r^2} = +\infty.$$

By Theorem 5.5, one gets that C_ϕ is not bounded on $\mathcal{D}_{\mu_\epsilon}$. ∎

5.3 The Compactness of Composition Operators on $\mathcal{D}_\mu$ Spaces

In this subsection, we characterize the compactness of composition operators on $\mathcal{D}_\mu$ spaces.

Theorem 5.7 *Let μ be a positive Borel measure on $\mathbb{D}$ and let $\phi\colon\mathbb{D}\to\mathbb{D}$ be analytic. Then the following conditions are equivalent.*

(i) *C_ϕ is compact on $\mathcal{D}_\mu$.*
(ii) *$N_{\phi,\mu}(w) = o(V_\mu(w))$, as $|w|\to 1$.*
(iii) *$\frac{1}{A(\Delta_w)}\int_{\Delta_w} N_{\phi,\mu}(z)\,dA(z) = o(V_\mu(w))$, as $|w|\to 1$, where*

$$\Delta_w = \left\{z\in\mathbb{D} : |z-w| < \frac{1}{2}(1-|w|)\right\}.$$

Proof Checking the proof of Theorem 5.5, it is enough to prove the equivalence between conditions (i) and (ii).

(i) ⇒ (ii). For each $w\in\mathbb{D}$ consider the functions

$$g_w(z) = \frac{1-|w|}{(1-\overline{w}z)\sqrt{V_\mu(w)}}, \quad z\in\mathbb{D}.$$

By Lemma 5.4, $\sup_{w\in\mathbb{D}}\|g_w\|_{\mathcal{D}_\mu} < +\infty$. Clearly,

$$|g_w(z)|^2 \leq \frac{4(1-|w|^2)}{|1-\overline{w}z|^2\int_{\mathbb{D}}(1-|a|^2)\,d\mu(a)}.$$

Then the functions g_w converge to zero as $|w|\to 1$ uniformly on compact subsets on $\mathbb{D}$. Note that C_ϕ is compact on $\mathcal{D}_\mu$. By [37, Lemma 3.7], one gets that

$$\lim_{|w|\to 1}\|C_\phi g_w\|_{\mathcal{D}_\mu} = 0.$$

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Making the change of variables, gives that

$$\begin{aligned}\|C_\phi g_w\|^2_{\mathcal{D}_\mu} &= |g_w(\phi(0))|^2 + \int_{\mathbb{D}} |(g_w)'(z)|^2 N_{\phi,\mu}(z)\, dA(z)\\ &= |g_w(\phi(0))|^2 + \frac{|w|^2(1-|w|)^2}{V_\mu(w)} \int_{\mathbb{D}} \frac{N_{\phi,\mu}(z)}{|1-\overline{w}z|^4}\, dA(z)\\ &\geq |g_w(\phi(0))|^2 + \frac{|w|^2(1-|w|)^2}{V_\mu(w)} \int_{\Delta_w} \frac{N_{\phi,\mu}(z)}{|1-\overline{w}z|^4}\, dA(z).\end{aligned}$$

Recall that $|1-\overline{w}z| \approx 1-|w|$ for all $z \in \Delta_w$. These, together with Lemma 5.2, yield

$$\|C_\phi g_w\|^2_{\mathcal{D}_\mu} \gtrsim |g_w(\phi(0))|^2 + \frac{|w|^2 N_{\phi,\mu}(w)}{V_\mu(w)}.$$

Thus $\lim_{|w|\to 1} \frac{N_{\phi,\mu}(w)}{V_\mu(w)} = 0$.

(ii) ⇒ (i). From [37, Lemma 3.7] it suffices to prove that for any bounded sequence $\{f_n\}$ in $\mathcal{D}_\mu$ that converges to zero uniformly on compact sets of $\mathbb{D}$,

$$\lim_{n\to\infty} \|C_\phi f_n\|_{\mathcal{D}_\mu} = 0.$$

Note that $\lim_{|w|\to 1} \frac{N_{\phi,\mu}(w)}{V_\mu(w)} = 0$. For small $\epsilon > 0$, there exists a positive constant δ such that if $\delta < |w| < 1$, then $N_{\phi,\mu}(w) < \epsilon V_\mu(w)$. There also exists a positive integer M such that if $n > M$, then $|f_n(\phi(0))| < \epsilon$ and $\sup_{|z|\leq\delta} |(f_n)'(z)| < \epsilon$. Consequently, for $n > M$, we deduce that

$$\begin{aligned}\|C_\phi f_n\|^2_{\mathcal{D}_\mu} &= |f_n(\phi(0))|^2 + \int_{|w|\leq\delta} |(f_n)'(w)|^2 N_{\phi,\mu}(w)\, dA(w)\\ &\quad + \int_{\delta<|w|<1} |(f_n)'(w)|^2 N_{\phi,\mu}(w)\, dA(w)\\ &\lesssim \epsilon^2 + \epsilon^2 \int_{\mathbb{D}} N_{\phi,\mu}(w)\, dA(w) + \epsilon \|f_n\|^2_{\mathcal{D}_\mu} \lesssim \epsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \|C_\phi f_n\|_{\mathcal{D}_\mu} = 0$. ■

D. Sarason and J-N. O. Silva [31, Theorem 8] characterized the boundedness and the compactness of composition operators on Dirichlet spaces with harmonic weights as follows.

Theorem 5.8 *Let ν be a positive Borel measure on the unit circle $\mathbb{T}$ and let $\phi \in \mathcal{D}(\nu)$ be an analytic self-map of $\mathbb{D}$.*

(i) *C_ϕ is bounded on $\mathcal{D}(\nu)$ if and only if*

(5.1) $$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \Big(\sum_{\phi(a)=z} P_\nu(a) \Big)\, dA(z) \lesssim P_\nu(w)$$

for all $w \in \mathbb{D}$.

(ii) *C_ϕ is compact on $\mathcal{D}(\nu)$ if and only if*

(5.2) $$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \Big(\sum_{\phi(a)=z} P_\nu(a) \Big)\, dA(z) = o(P_\nu(w)), \quad \text{as } |w| \to 1.$$

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D. Sarason and J-N. O. Silva's conditions (5.1) and (5.2) correspond to our conditions Theorem 5.5 (iii) and Theorem 5.7 (iii), respectively. But as pointed out [31, p. 447], in general the conditions corresponding to Theorem 5.5 (ii) and Theorem 5.7 (ii) cannot be used to describe the boundedness and the compactness of C_ϕ on $\mathcal{D}(\nu)$, respectively.

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Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China
e-mail: glbaoah@163.com

Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla, Istanbul, 34956 Turkey
e-mail: nggogus@sabanciuniv.edu stamatispouliasis@gmail.com

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