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AN OPTIMIZATION PROBLEM RELATED TO THE ZETA-FUNCTION

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ABSTRACT. S. Golomb noticed that Riemann's zeta function ζ induces a probability distribution $\{n^{-s}/\zeta(s),\ n \in \mathbb{N}\}$ on the positive integers, for any $s > 1$, and studied some of its properties connected to divisibility. The object of this paper is to show that the probability distribution mentioned above is the unique solution of an entropy-maximization problem.

Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{P}$ be the set of primes (not including 1). If $Q = \{q(n), n \in \mathbb{N}\}$, $q(n) \geqq 0$,

$$\sum_n q(n) = 1 \tag{1}$$

is an arbitrary probability distribution on $\mathbb{N}$, the global amount of uncertainty contained by it is measured by Shannon's entropy

$$H = H(Q) = -\sum_n q(n) \ln q(n)$$

Let us solve the following non-linear optimization problem:

(A) *Maximize $H(Q)$ subject to the constraints* (1) *and*

$$\sum_n \ln n \cdot q(n) = K, \qquad (\textit{where } n \in \mathbb{N}, \textit{ and } K > 0). \tag{2}$$

THEOREM. *The unique solution of problem* (A) *is*

$$q(n) = n^{-s}/\zeta(s), \qquad (n \in \mathbb{N}), \tag{3}$$

where

$$\zeta(s) = \sum_n n^{-s}$$

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and s is the unique real number $(s > 1)$ such that

$$K = \sum_p \frac{\ln p}{p^s - 1} \qquad (\textit{where } p \in \mathbb{P}).$$

PROOF. According to Taylor's formula, for any $t > 0$ there is a τ between 1 and t, therefore positive, such that

$$t \ln t = (t - 1) + \frac{1}{2\tau}(t - 1)^2. \tag{4}$$

Introducing Lagrange's multipliers α and β, corresponding to the constraints (1) and (2) (we can suppose $\alpha > 0$ and $\beta > 1$, and we shall explain later why), and applying (4), we have

$$\begin{aligned} (5) \qquad & -H + \alpha \cdot 1 + \beta \cdot K + \sum_n \exp(-\alpha - \beta \ln n) - 1 \\ &= \sum_n \exp(-\alpha - \beta \ln n)[q(n) \exp(\alpha + \beta \ln n)] \ln [q(n) \\ &\quad \exp(\alpha + \beta \ln n)] + \sum_n \exp(-\alpha - \beta \ln n) - 1 \\ &= \frac{1}{2} \sum_n \exp(-\alpha - \beta \ln n) \frac{1}{\tau_n} [q(n) \exp(\alpha + \beta \ln n) - 1]^2 \geqq 0, \end{aligned}$$

where $\tau_n > 0$ for any n, with the equality if and only if

$$q(n) = \exp(-\alpha - \beta \ln n), \tag{6}$$

for any n. For determining β let us introduce (6) into the first constraint (1). We get

$$\exp(-\alpha) = 1/\zeta(\beta), \quad \text{or} \quad \alpha = \ln \zeta(\beta),$$

where we denote

$$\zeta(\beta) = \sum_n n^{-\beta},$$

which is just Riemann's zeta function. The expression (6) becomes

$$q(n) = \frac{1}{\zeta(\beta)} n^{-\beta}, \quad (n \in \mathbb{N}). \tag{7}$$

Introducing (7) into (2) we get

$$\sum_n \ln n \cdot q(n) = \frac{1}{\zeta(\beta)} \sum_n \ln n \cdot n^{-\beta} = -\frac{\zeta'(\beta)}{\zeta(\beta)}. \tag{8}$$

As $\zeta'(\beta)/\zeta(\beta)$ is strictly increasing for $\beta > 1$, let s be the unique value of β for which

$$-\frac{\zeta'(s)}{\zeta(s)} = K.$$

Using von Mangoldt's function on $\mathbb{N}$,

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^m, \ (p \in \mathbb{P},\ m \in \mathbb{N}), \\ 0, & \text{elsewhere,} \end{cases}$$

we can write

$$\text{(9)} \qquad \sum_p \frac{\ln p}{p^s - 1} = \sum_p \ln p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots \right) = \sum_n \Lambda(n) \cdot n^{-s}.$$

But it is well-known (see Hardy and Wright [5], p. 253) that if $s > 1$

$$\text{(10)} \qquad \sum_n \Lambda(n) \cdot n^{-s} = -\frac{\zeta'(s)}{\zeta(s)}$$

Therefore

$$K = \sum_p \frac{\ln p}{p^s - 1}.$$

Returning to (7) we obtain the expression (3). Because (6) must satisfy (1) we have

$$\sum_n \exp(-\alpha - \beta \ln n) = 1,$$

and the inequality (5) gives

$$\text{(11)} \qquad H(Q) \leqq \alpha + \beta \sum_p \frac{\ln p}{p^s - 1},$$

which shows that any $\alpha < 0$ or $\beta < 1$ can only decrease the bound in (11). Introducing

$$\beta = s \quad \text{and} \quad \alpha = \ln \zeta(s)$$

in (11) we get the inequality

$$\text{(12)} \qquad H(Q) \leqq \ln \zeta(s) - \frac{\zeta'(s)}{\zeta(s)},$$

with the equality if and only if the probability distribution Q has the form (3).

REMARKS. (1) Starting from the expression of the zeta-function, S. Golomb [2] noticed that $\{n^{-s}/\zeta(s),\ n \in \mathbb{N}\}$ is a probability distribution on $\mathbb{N}$, for any $s > 1$, and studied some interesting properties of this distribution in number theory. He also computed directly its entropy which is just the upper bound in the above inequality (12); this upper bound is attained only for the probability distribution (3).

According to the above theorem we see that Riemann's zeta-function for $s > 1$ proves to be the normed factor of the unique probability distribution which maximizes the entropy subject to the constraint (2) imposed on the mean value of the logarithm function on $\mathbb{N}$.

(2) Shannon's entropy gives the amount of uncertainty contained by a probability distribution. In the finite discrete case the largest amount of uncertainty is contained by the uniform distribution when no constraint is imposed. According to the above theorem, in the countable case, the zeta-distribution (3) contains the largest amount of uncertainty subject to the constraint imposed on the mean value of the logarithm function. In constraint (2) the right-hand member depends only on the primes. Constraint (2) shows in fact that the logarithm must have the same integral on $\mathbb{N}$ with respect to the probability measure (3) as on $\mathbb{P}$ with respect to the measure defined by $m(p) = 1/(p^s - 1)$.

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