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Small semigroup related structures with infinite properties

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This thesis investigates the boundaries between the finite and infinite properties of finite semigroups and related structures.

A semigroup (or the variety it generates) whose identities are finitely axiomatisable is said to be *finitely based* and the problem of determining when an algebra is finitely based is called the finite basis problem. The first chapter of original content concerns the finite basis problem for finite semigroups obtained by taking Rees quotients of free monoids. We find the smallest possible not finitely based example (reducing the previous bound from 23 elements [**8**] to 9) and show the variety generated by any of these semigroups (or indeed any finite nilpotent semigroup or monoid) has an infinite chain of supervarieties, each generated by a finite semigroup and whose identities alternate between being finitely based and not finitely based. The first examples of pairs of finite aperiodic finitely based semigroups whose direct product is not finitely based are constructed (answering a question of M. Sapir) as well as examples of pairs of finite (aperiodic) semigroups that are not finitely based whose direct product is finitely based (after a similar example was found by O. Sapir; see [**12**]). We also investigate the asymptotic proportion of these semigroups that have a finite basis of identities and find that in a natural sense, "almost all" such semigroups have no such basis. Some of these results are to appear in [**7**] or have been submitted in [**6**].

Probably the most studied subclass of not finitely based finite algebras is the class of inherently not finitely based (INFB) algebras — locally finite algebras that have the property that every locally finite variety containing them is also not finitely based. In Chapter 3 we use a remarkable description of the INFB finite semigroups by M. Sapir (see [**9**] and [**10**]) to show that a regular semigroup or a semigroup whose idempotents form a subsemigroup is INFB if and only if the variety it generates contains the six element Brandt semigroup with adjoined identity element, $\mathbf{B}_2^1$. We then show how to construct all minimal INFB divisors in the class of finite semigroups (there are infinitely many), from which it follows that the smallest INFB finite semigroup whose variety does

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not contain $\mathbf{B}_2^1$ has 56 elements. These results have been submitted for publication in [**5**] and, along with results of the previous chapter, have been announced in [**13**].

In Chapter 4 we find the first example of a finitely based semigroup whose variety has uncountably many subvarieties (the previously known examples were all INFB) and describe which members of some large classes have this property. We also show that a variety with uncountably many subvarieties can be the join of two varieties with comparatively few subvarieties, each containing only finitely based varieties and generated by finite semigroups. These and other results from the chapter are to appear in [**2**]. As a final section we investigate a connection between these results and the construction of varieties which have several almost contradictory properties: they have a recursive basis of identities, n-variable equational theory (for any fixed number n), and word problem; but also have non recursive equational theory, uniform word problem, and membership problem. One of the examples presented answers part of a question of Wells in [**14**]. These results have been submitted in [**4**].

In the final chapter we turn away from problems associated with identities and investigate another rich source of problematic behaviour in finite semigroup theory — embedding problems. The first and second of the problems considered concern the fundamental relations of Green $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$. For example it is shown that there is no algorithm which determines when presented with a finite semigroup S with subset A whether or not S is embeddable in a semigroup (or even a finite semigroup) in which the subset A lies within an $\mathcal{H}$-class (solving a problem of M. Sapir in [**11**]) and some small examples are found which exhibit unusual related properties (following an existence proof in [**11**]). These results have appeared in [**1**]. The last of the embedding problems concerns the potential embeddability of finite semigroup amalgams and ring amalgams. While every finite group amalgam is embeddable in a finite group, it is shown here that the classes of finite semigroup (or ring) amalgams that are embeddable in a semigroup (or ring) or a finite semigroup (finite ring) respectively are not recursive. These results have been submitted in [**3**].

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