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SUBEQUALIZERS

BY
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Dedicated to the memory of my friend and collaborator Leo Moser

The purpose of this exposition is threefold. Firstly, we wish to show that many concepts and arguments carry over from pre-ordered sets to categories. Secondly, we wish to make some propaganda for the notion of "subequalizer" of two functors, which appears to be more fundamental than Lawvere's so-called "comma-category", in the same sense in which equalizers are more fundamental than pull-backs. Thirdly, we wish to give simple proofs of the adjoint functor theorem and related theorems, which appear to be more economic than those in the literature. The author wishes to take this opportunity to refine some arguments that he has published earlier. He is indebted to Michael Barr, whose presentation of similar proofs in his course has provided the stimulation for preparing this note for publication, to John Isbell for his critical reading of the manuscript and to William Schelter for suggesting a shortcut in one of the proofs.

Some words are called for to apologize for our terminology. Writers of books on categories have by no means agreed on what to call the generalized inverse and direct limits. While "limit" and "colimit" are gaining ground, they are here called "infimum" and "supremum", to bring out the analogy between pre-ordered sets and categories. I also feel a little unhappy about the term "sub-equalizer". This concept makes sense in other hypercategories, but it does not appear to be a special case of a "hyperlimit". If it should turn out to be useful, someone will surely come up with a better name.

1. **Subequalizers of monotone functions.** A *pre-ordered* set $\mathscr{A}$ is a set together with a binary relation $\leq$ on this set which is reflexive and transitive. We do not assume that it is antisymmetric, instead we call two elements A and B of $\mathscr{A}$ *isomorphic* when $A \leq B$ and $B \leq A$ and write "$A \cong B$".

Suppose $F, G: \mathscr{A} \to \mathscr{B}$ are *monotone*, that is, pre-order preserving functions between pre-ordered sets. The *equalizer* $(\mathscr{E}, D)$ of the pair (F, G) consists of the set

$$\mathscr{E} = \{A \in \mathscr{A} \mid F(A) = G(A)\},$$

equipped with the pre-order induced by $\mathscr{A}$, together with the inclusion function D: $\mathscr{E} \to \mathscr{A}$. It has the following universal property: $FD = GD$ and, if D': $\mathscr{E}' \to \mathscr{A}$ is such that $FD' = GD'$, then there exists a unique monotone function E: $\mathscr{E}' \to \mathscr{E}$ such that $D' = DE$.

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We now introduce the *subequalizer* $(\mathscr{E}, D)$ of the pair (F, G). By this we mean the set

$$\mathscr{E} = \{A \in \mathscr{A} \mid F(A) \leq G(A)\},$$

together with the inclusion function $D\colon \mathscr{E} \to \mathscr{A}$. It has the following universal property: $FD \leq GD$ and, if $D'\colon \mathscr{E}' \to \mathscr{A}$ is such that $FD' \leq GD'$, then there exists a unique monotone function $E\colon \mathscr{E}' \to \mathscr{E}$ such that $D' = DE$.

We shall refer to $\mathscr{E}$ as the *subequalizing set* of (F, G). One is often interested in a least element of the set $\mathscr{E}$. Of course, this is unique up to isomorphism.

EXAMPLE 0. Let $\mathscr{A}$ be a pre-ordered set, $F\colon \mathscr{A} \to \mathscr{A}$ a monotone function. Form the subequalizing set $\mathscr{E} = \{A \in \mathscr{A} \mid F(A) \leq A\}$ of the pair $(F, 1_{\mathscr{A}})$. An element A of $\mathscr{A}$ is called a *fixpoint* of F if $F(A) \cong A$. Suppose $\mathscr{E}$ has a least element A_0, then it is easily seen that A_0 is a fixpoint of F, in fact, a least element in the set of all fixpoints of F.

In the following examples F has the form $F = K(B)$, the constant function with value B in $\mathscr{B}$.

EXAMPLE 1. Let $\mathscr{B} = \mathbf{2}$ be the two-element set $\{0, 1\}$ with ordering $0 \leq 1$, and suppose $G\colon \mathscr{A} \to \mathbf{2}$ is monotone. Form the subequalizing set $\mathscr{E} = \{A \in \mathscr{A} \mid 1 \leq G(A)\} = \{A \in \mathscr{A} \mid G(A) = 1\}$ of the pair $(K(1), G)$. If $\mathscr{E}$ has a least element A_0, it is of the form $\{A \in \mathscr{A} \mid A_0 \leq A\}$.

EXAMPLE 2. Let $G\colon \mathscr{A} \to \mathscr{B}$ be monotone, B any element of $\mathscr{B}$. Form the subequalizing set $\{A \in \mathscr{A} \mid B \leq G(A)\}$ of the pair $(K(B), G)$ and assume this has a least element, call it $G^*(B)$. Then $G^*(B) \leq A$ if and only if $B \leq G(A)$. If this is so for each B in $\mathscr{B}$, one says that the functions $G\colon \mathscr{A} \to \mathscr{B}$ and $G^*\colon \mathscr{B} \to \mathscr{A}$ set up a *Galois correspondence* between the pre-ordered sets $\mathscr{A}$ and $\mathscr{B}$.

EXAMPLE 3. Let $\mathscr{X}$ be any pre-ordered set and take $\mathscr{B} = \mathscr{A}^{\mathscr{X}}$ to be the set of all monotone functions from $\mathscr{X}$ to $\mathscr{A}$, equipped with the elementwise order relation. By $K\colon \mathscr{A} \to \mathscr{A}^{\mathscr{X}}$ we mean the function whose value for $A \in \mathscr{A}$ is the constant function $K(A)\colon \mathscr{X} \to \mathscr{A}$ with value A. Given any function $T \in \mathscr{A}^{\mathscr{X}}$, $K(T)\colon \mathscr{A} \to \mathscr{A}^{\mathscr{X}}$ is the constant function with value T. Form the subequalizing set of the pair $(K(T), K)$. If this has a least element A_0, we call A_0 the *supremum* of T. It is easily seen that this is the usual supremum of the set $\{T(X) \mid X \in \mathscr{X}\}$.

A pre-ordered set is said to be *sup-complete* if every subset has a supremum, *inf-complete* if every subset has an infimum. (Actually, these two concepts are equivalent, as we shall see.) In view of the above examples it is of interest to know when a subequalizing set has a least element. Surely this will be the case if it is inf-complete (take the inf of the whole set) or sup-complete (take the sup of the empty set). The following result gives sufficient conditions for this to happen.

LEMMA 0. *Let $F, G\colon \mathscr{A} \to \mathscr{B}$ be monotone functions between pre-ordered sets, $\mathscr{E} = \{A \in \mathscr{A} \mid F(A) \leq G(A)\}$ their subequalizing set. If $\mathscr{A}$ is inf-complete and G pre-*

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serves infs, then $\mathscr{E}$ is inf-complete. If $\mathscr{A}$ is sup-complete and F preserves sups, then $\mathscr{E}$ is sup-complete.

Proof. Let $\mathscr{X}$ be a subset of $\mathscr{E}$. Assuming that A is inf-complete, $\mathscr{X}$ will have an infimum A_0 in $\mathscr{A}$. We claim that A_0 is in $\mathscr{E}$, that is, that $F(A_0) \leq G(A_0)$. Indeed, let $A \in \mathscr{X}$, then $A_0 \leq A$, hence $F(A_0) \leq F(A) \leq G(A)$. Assuming that G preserves infs, we have $G(A_0) = \inf\{G(A) \mid A \in \mathscr{X}\}$. Therefore $F(A_0) \leq G(A_0)$, as was to be shown. The dual statement is proved similarly.

We may apply this result to the four examples mentioned earlier.

APPLICATION 0. (Birkhoff–Tarski.) If $\mathscr{A}$ is an inf-complete, pre-ordered set, any monotone function $F: \mathscr{A} \to \mathscr{A}$ has a fixpoint. (Note that $1_{\mathscr{A}}$ preserves infs.)

APPLICATION 1. If $\mathscr{A}$ is an inf-complete pre-ordered set, and $G: \mathscr{A} \to \mathbf{2}$ is monotone, then there exists an element $A_0 \in \mathscr{A}$ such that $G(A) = 1 \Leftrightarrow A_0 \leq A$, if and only if G preserves infs. (In one direction this follows from the lemma, the other direction is easily checked.)

APPLICATION 2. If $\mathscr{A}$ and $\mathscr{B}$ are pre-ordered sets, $\mathscr{A}$ inf-complete, and $G: \mathscr{A} \to \mathscr{B}$ monotone, then there exists a monotone function $G^*: \mathscr{B} \to \mathscr{A}$ such that $G^*(B) \leq A \Leftrightarrow B \leq G(A)$, if and only if G preserves infs. (See the comment to Application 1.)

APPLICATION 3. If $\mathscr{A}$ is an inf-complete pre-ordered set, then $\mathscr{A}$ is also sup-complete, in fact, any monotone function $T: \mathscr{X} \to \mathscr{A}$ from a pre-ordered set $\mathscr{X}$ has a supremum. (Note that $K(T)$ preserves infs.)

2. **Subequalizers of functors.** Pre-ordered sets are special kinds of categories, so one might wish to generalize the above concepts and results to categories. We shall not make the usual distinction between sets and classes, but we shall often assume that there is given a nonempty universe. Elements of the universe are called *small*, subsets of the universe are called *large*. A category $\mathscr{A}$ is called *locally small* if the set $[A, B]$ of all maps from the object A to the object B is small.

If $F, G: \mathscr{A} \to \mathscr{B}$ are functors between categories, the *equalizer* $(\mathscr{E}, D)$ of the pair (F, G) is well known. $\mathscr{E}$ is the subcategory of $\mathscr{A}$ whose objects A and maps a satisfy $F(A) = G(A)$ and $F(a) = G(a)$, and $D: \mathscr{E} \to \mathscr{A}$ is the inclusion functor. The equalizer of a pair of functors has the same universal property as that of a pair of monotone functions.

We shall now introduce the *subequalizer* of the pair of functors (F, G). By this we mean a triple $(\mathscr{E}, D, t)$, where $\mathscr{E}$ is a category, $D: \mathscr{E} \to \mathscr{A}$ is a functor, and $t: FD \to GD$ is a natural transformation defined as follows. The objects of $\mathscr{E}$ are pairs (A, b), where A is an object of $\mathscr{A}$ and $b: F(A) \to G(A)$ a map in $\mathscr{B}$. The maps $(A, b) \to (A', b')$ in $\mathscr{E}$ are triples (b, a, b'), where $a: A \to A'$ is a map in $\mathscr{A}$ such that $G(a)b = b'F(a)$.

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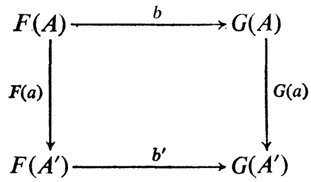


D is given by $D(A, b)=A$ and $D(b, a, b')=a$, and t is given by $t(A, b)=b$.

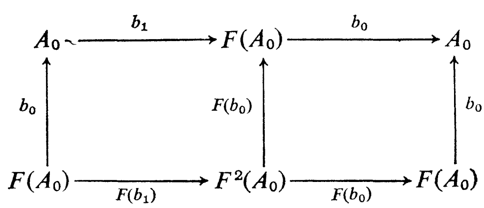
The subequalizer $(\mathscr{E}, D, t)$ has this universal property: D: $\mathscr{E}\to\mathscr{A}$ and t: $FD \to GD$, and if $(\mathscr{E}', D', t')$ is such that D': $\mathscr{E}'\to\mathscr{A}$ and t': $FD'\to GD'$, then there exists a unique functor E: $\mathscr{E}'\to\mathscr{E}$ such that $DE=D'$ and $tE=t'$.

We also call $\mathscr{E}$ the *subequalizing category* of (F, G). One is often interested in an initial object of the category $\mathscr{E}$, that is, an object which admits exactly one map to every object of $\mathscr{E}$. Clearly, an initial object is uniquely determined up to isomorphism.

EXAMPLE 0. Given a category $\mathscr{A}$ and a functor F: $\mathscr{A}\to\mathscr{A}$, form the subequalizing category $\mathscr{E}$ of the pair of functors $(F, 1_{\mathscr{A}})$. Suppose $\mathscr{E}$ has an initial object (A_0, b_0), then for each object (A, b) of $\mathscr{E}$ there exists a unique map $u(b)$: $A_0 \to A$ such that

$$bFu(b) = u(b)b_0.$$

Let us write $b_1=uF(b_0)$: $A_0\to F(A_0)$, then the following squares commute:



Therefore $b_0b_1=u(b_0)=1$, and so also $b_1b_0=F(b_0b_1)=F(1)=1$. Thus b_0: $F(A_0)\to A_0$ is an isomorphism, and we may call A_0 a *fixpoint* of F.

In the following examples, F has the form $F=K(B)$, the constant functor whose value at each object is B and whose value at each map is 1_B.

EXAMPLE 1. Let $\mathscr{B}=Ens$ be the category of small sets, 1 a fixed one-element set, G: $\mathscr{A}\to Ens$ a functor. Form the subequalizing category $\mathscr{E}$ of the pair of functors $(K(1), G)$. Suppose $\mathscr{E}$ has an initial object (A_0, b_0). This means that for each $b\in G(A)$ with A in $\mathscr{A}$ there exists a unique map a: $A_0\to A$ such that $G(a)b_0=b$.

If $\mathscr{A}$ is locally small, that is, if all Hom-sets $[A, A']$ with A, A' objects of $\mathscr{A}$ are elements of the given universe, we may put the above in another way: $G\cong[A_0, -]$, and b_0 is the element of $G(A_0)$ corresponding to the identity map on A_0. One calls the functor G *representable.*

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EXAMPLE 2. Given a functor $G: \mathscr{A} \to \mathscr{B}$ and an object B of $\mathscr{B}$, form the subequalizing category $\mathscr{E}$ of the pair of functors $(K(B), G)$. Suppose the category $\mathscr{E}$ has an initial object $(G^*(B), h(B))$. This means that $h(B): B \to GG^*(B)$ and, whenever $b: B \to G(A)$, there exists a unique map $a: G^*(B) \to A$ such that $G(a)h(B) = b$. The reader will recognize this as a solution of the universal mapping problem for G at B. If this is so for each B in $\mathscr{B}$, it is easily seen that G^* can be made into a functor and that h is a natural transformation. One calls G^* the *left adjoint* of G.

If both $\mathscr{A}$ and $\mathscr{B}$ are locally small, the relationship between G and G^* may also be expressed as a natural isomorphism between the bifunctors $[G^*(-), -]$ and $[-, G(-)]$ from $\mathscr{A}^{\text{opp}} \times \mathscr{B}$ to *Ens*.

EXAMPLE 3. Let $\mathscr{B} = \mathscr{A}^{\mathscr{X}}$ be the category whose objects are functors from $\mathscr{X}$ to $\mathscr{A}$ and whose maps are natural transformations between such functors. By $K: \mathscr{A} \to \mathscr{A}^{\mathscr{X}}$ we mean the functor whose value at the object A of $\mathscr{A}$ is the constant functor $K(A): \mathscr{X} \to \mathscr{A}$ and whose value at the map $a: A \to A'$ is the natural transformation $K(a): K(A') \to K(A')$ defined by $K(a)(X) = a$. Given any functor $T: \mathscr{X} \to \mathscr{A}$, $K(T): \mathscr{A} \to \mathscr{A}^{\mathscr{X}}$ is the constant functor whose value at any object is T and at any map is the natural transformation 1_T. Form the subequalizing category $\mathscr{E}$ of the pair of functors $(K(T), K)$. The objects of the category $\mathscr{E}$ have been called *upper bounds* of T. If $\mathscr{E}$ has an initial object (A_0, u_0) we call this the *supremum* of T, although it is more commonly called the *colimit*. (The dual notions are *infimum* or *limit*.)

If T has a supremum for every functor $T: \mathscr{X} \to \mathscr{A}$ where $\mathscr{X}$ is any small category, that is, an element of the universe, we say that $\mathscr{A}$ is *sup-complete*, the usual term being *cocomplete*. (The dual notions are *inf-complete* or just *complete*.) The reason for restricting $\mathscr{X}$ to be small is that otherwise $\mathscr{A}$ would have to be a pre-ordered set (see [2]).

In view of the above examples, it is of interest to know when a subequalizing category has an initial object. Surely, every sup-complete category $\mathscr{A}$ has an initial object, the sup of the inclusion functor from the empty category into $\mathscr{A}$. However, we cannot conclude that an inf-complete category has an initial object by taking the inf of the identity functor, since the domain of this functor need not be small.

We shall call a full subcategory $\mathscr{C}$ of $\mathscr{A}$ *pre-initial* if every object A of $\mathscr{A}$ admits at least one map $C \to A$ with C in $\mathscr{C}$.

LEMMA 1. *If $\mathscr{A}$ is inf-complete, then $\mathscr{A}$ has an initial object if and only if it has a small pre-initial subcategory.*

Proof. The necessity of the condition is obvious. The following argument combines the proofs of Propositions 1.1 and 1.2 in [5]. Let (A_0, u) be the infimum of the inclusion functor $\mathscr{C} \to \mathscr{A}$. Thus we have a map $u(C): A_0 \to C$ for each C in $\mathscr{C}$. Take an A in $\mathscr{A}$, then by assumption there exists C in $\mathscr{C}$ and $a: C \to A$, hence $au(C): A_0 \to A$. We shall see that this is actually the only map from A_0 to A.

Indeed, suppose $a_1, a_2: A_0 \to A$, and let $k: K \to A_0$ be their equalizer. It will follow that $a_1 = a_2$ if we show that k has a right inverse. Now there exists C_0 in $\mathscr{C}$ and $c: C_0 \to K$. Put $a_0 = kc$.

The result will follow, if we can show that $a_0 u(C_0) = 1$. Now

$$u(C)a_0u(C_0) = u(C),$$

by naturality of u and because $u(C)a_0$ is a map in $\mathscr{C}$. Since (A_0, u) is the infimum of the inclusion functor $\mathscr{C} \to \mathscr{A}$, there exists a unique $x: A_0 \to A_0$ such that $u(C)x = u(C)$ for all C in $\mathscr{C}$. Therefore

$$a_0u(C_0) = x = 1,$$

and our argument is complete.

A functor $H: \mathscr{C} \to \mathscr{A}$ is said to *create* infs provided, whenever $L: \mathscr{X} \to \mathscr{C}$ is a functor such that HL has an inf, then inf $HL = (H(C), Ht)$ where inf $L = (C, t)$. As an immediate consequence of this property we observe that if $\mathscr{A}$ is inf-complete then so is $\mathscr{C}$ and H preserves infs.

LEMMA 2. *Let $(\mathscr{E}, D, t)$ be the subequalizer of the functors $F, G: \mathscr{A} \to \mathscr{B}$. If G preserves infs then D creates them, hence if $\mathscr{A}$ is inf-complete then so is $\mathscr{E}$ and D preserves infs. Dually, if G preserves sups then D creates them, hence if $\mathscr{A}$ is sup-complete then so is $\mathscr{E}$ and D preserves sups.*

Proof. Suppose $L: \mathscr{X} \to \mathscr{E}$, then for each X in $\mathscr{X}$ we have $L(X) = (A_X, b_X)$, where $b_X: F(A_X) \to G(A_X)$. Now $DL(X) = A_X$; suppose inf $DL = (A_0, u)$, where $u(X): A_0 \to A_X$. Suppose also that G preserves infs, then inf $GDL = (G(A_0), Gu)$. Let $b_0: F(A_0) \to G(A_0)$ be the unique map such that the following square commutes for each X in $\mathscr{X}$:

$$\begin{array}{ccc} F(A_0) & \overset{b_0}{\dashrightarrow} & G(A_0) \\ {\scriptstyle Fu(X)}\downarrow & & \downarrow{\scriptstyle Gu(X)} \\ F(A_X) & \underset{b_X}{\longrightarrow} & G(A_X) \end{array}$$

It is easily verified that (A_0, b_0) is the object part of inf L.

From Lemmas 1 and 2 we immediately obtain the following.

PROPOSITION 1. *Let $\mathscr{A}$ be inf-complete and assume that the subequalizing category $\mathscr{E}$ of the pair of functors $F, G: \mathscr{A} \to \mathscr{B}$ has a small pre-initial subcategory. Assume that G preserves infs. Then $\mathscr{E}$ has an initial object.*

The reader is invited to write out in full the condition concerning $\mathscr{E}$.

We shall apply this result to the four examples discussed earlier.

COROLLARY 0. *If $\mathscr{A}$ is inf-complete, any functor $F: \mathscr{A} \to \mathscr{A}$ has a fixpoint provided the subequalizing category of the pair of functors $(F, 1_{\mathscr{A}})$ has a small pre-initial subcategory.*

Proof. $1_{\mathscr{A}}$ preserves infs.

The above condition is not necessary, but a necessary and sufficient condition may be obtained by the same method (see [5]): A functor $F: \mathscr{A} \to \mathscr{A}$ has a fixpoint if and only if there exists a functor $F': \mathscr{A} \to \mathscr{A}$ with small image such that $FF' \cong F'F$.

COROLLARY 1. *Let $\mathscr{A}$ be inf-complete. Then any functor $G: \mathscr{A} \to Ens$ is representable if and only if G preserves infs and the subequalizing category of the pair of functors $(K(1), G)$ has a small pre-initial subcategory.*

The necessity of the conditions is easily checked.

COROLLARY 2. *Let $\mathscr{A}$ be inf-complete. Then any functor $G: \mathscr{A} \to \mathscr{B}$ has a left adjoint if and only if G preserves infs and the subequalizing category of the pair of functors $(K(B), G)$ has a small pre-initial subcategory for each object B of $\mathscr{B}$.*

This is Freyd's "general adjoint functor theorem", minus some unnecessary conditions. The condition on the subequalizing category is called by him the "solution set condition". Corollary 2 can also be deduced from Corollary 1, provided $\mathscr{B}$ is locally small, by looking at the functor $[B, G(-)]: \mathscr{A} \to Ens$. Corollaries 1 and 2 in this generality were first published by Benabou [1].

COROLLARY 3. *Let $\mathscr{A}$ be inf-complete. Then any functor $T: \mathscr{X} \to \mathscr{A}$ has a supremum if and only if the category of upper bounds of T has a small pre-initial subcategory.*

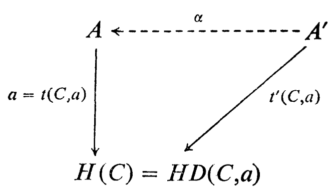
Indeed, $K: \mathscr{A} \to \mathscr{A}^{\mathscr{X}}$ preserves infs. See [5, Corollary to Proposition 6.2].

Example 0 affords an amusing illustration of the well-known fact that an inf-complete category need not be sup-complete.

Let $\mathscr{A}$ be the category of small categories and F the covariant power-set functor. Then Cantor's theorem asserts that F has no fixpoint, hence the subequalizing category $\mathscr{E}$ of the pair $(F, 1_{\mathscr{A}})$ has no initial object. Therefore $\mathscr{E}$ is not sup-complete. On the other hand, Lemma 2 implies that $\mathscr{E}$ is inf-complete.

3. **Subequalizers and adequacy.** Suppose $H: \mathscr{C} \to \mathscr{A}$ is a functor and A is an object of $\mathscr{A}$. Form the subequalizer $(\mathscr{E}, D, t)$ of the pair of functors $(K(A), H)$. In particular, this implies that $t: K(A)D \to HD$ is a lower bound of HD, since $K(A)D$ is itself a constant functor from $\mathscr{E}$ to $\mathscr{A}$ and might equally well be written $K(A)$. (The first "K" denotes the constancy functor $\mathscr{C} \to \mathscr{C}^{\mathscr{A}}$, the second "$K$" denotes the constancy functor $\mathscr{E} \to \mathscr{E}^{\mathscr{A}}$.)

It is a reasonable question to ask: when is $\inf HD = (A, t)$? This is so if and only if for each natural transformation $t': K(A') \to HD$ there exists a unique $\alpha: A' \to A$ such that $t'(C, a) = a\alpha$ for all $a: A \to H(C)$



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Without loss in generality, we may assume that $\mathscr{A}$ is locally small, that is, that the given universe contains all Hom-sets of $\mathscr{A}$ as elements, by picking the universe after $\mathscr{A}$ has been presented. Then we may write

$$t'(C, a) = t^*(C)(a),$$

where $t^*\colon [A, H(-)] \to [A', H(-)]$ is a natural transformation between two functors from $\mathscr{C}$ to *Ens*, the category of all sets in the universe. In fact, it is easily seen that $t' \mapsto t^*$ is a one-to-one correspondence between natural transformations $K(A') \to HD$ and natural transformations $[A, H(-)] \to [A', H(-)]$.

Therefore $\inf HD = (A, t)$ if and only if for each $t^*\colon [A, H(-)] \to [A', H(-)]$ there exists a unique map $\alpha\colon A' \to A$ such that $t^*(C)(a) = a\alpha$ for all $a\colon A \to H(C)$, that is to say, $t^* = [\alpha, H(-)]$. In other words, this means that the functor $A \mapsto [A, H(-)]$ of $\mathscr{A}$ into $Ens^{\mathscr{C}^{\text{opp}}}$ is faithful and full.

We have thus shown the following result (see, for example, [5, Proposition 5.1]).

PROPOSITION 2. *Let $(\mathscr{E}, D, t)$ be the subequalizer of the pair of functors $K(A)$, $H\colon \mathscr{C} \to \mathscr{A}$, and assume that $\mathscr{A}$ is locally small. Then $\inf HD = (A, t)$ if and only if the functor $A \mapsto [A, H(-)]$ of $\mathscr{A}$ into $Ens^{\mathscr{C}^{\text{opp}}}$ is faithful and full.*

Under these conditions H will be called *co-adequate*. (Isbell uses "right adequate" and Ulmer uses "codense".) Weaker than this is the following notion. The functor H is called *cogenerating* if the associated functor $A \mapsto [A, H(-)]$ is faithful. Another way of putting this is this: if α and α' are distinct maps $A' \to A$, then there exists $a\colon A \to H(C)$ with C in $\mathscr{C}$ such that $a\alpha \neq a\alpha'$.

We shall call the object A_0 of $\mathscr{A}$ a *pre-initial* object if the full subcategory $\{A_0\}$ is a pre-initial subcategory of $\mathscr{A}$.

LEMMA 3. *Let $\mathscr{A}$ be inf-complete, $\mathscr{C}$ small. If $H\colon \mathscr{C} \to \mathscr{A}$ is co-adequate then the infimum A_0 of H is a pre-initial object of $\mathscr{A}$. If $H\colon \mathscr{C} \to \mathscr{A}$ is cogenerating, the subobjects of A_0 form a pre-initial subcategory of $\mathscr{A}$.*

Proof. Put $\inf H = (A_0, t_0)$. Let A be a given object of $\mathscr{A}$, and form the subequalizer $(\mathscr{E}, D, t)$ of $K(A)$, $H\colon \mathscr{C} \to \mathscr{A}$.

In the first case, we have $\inf HD = (A, t)$ by Proposition 2. Now $t_0D\colon K(A_0) \to HD$, hence there exists a unique $\alpha\colon A_0 \to A$ such that $t_0D(C, a) = t(C, a)\alpha$ for all $a\colon A \to H(C)$, that is, $t_0(C) = a\alpha$. Clearly, A_0 is a pre-initial object.

In the second case, we put $\inf HD = (A_1, t_1)$, say, and we let $m\colon A \to A_1$ be the unique map such that $t_1(C, a)m = t(C, a) = a$, for all $a\colon A \to H(C)$. Then it is easily seen that m is a monomorphism. Now form the pullback:

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha'\ } & A \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A_0 & \xrightarrow[\ \alpha\ ]{} & A_1 \end{array}$$

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Then m' is a monomorphism also, and so we have a map $\alpha': P \to A$ from a subobject of A_0 to A.

Lemma 3 allows us to infer the existence of a pre-initial object of $\mathscr{A}$, hence the existence of an initial object by Lemma 1. What we are really interested in is an initial object not of $\mathscr{A}$ but of the subequalizing category $\mathscr{E}$ of a pair of functors $F, G: \mathscr{A} \to \mathscr{B}$, at least in the case when $F = K(B)$.

Suppose $H: \mathscr{C} \to \mathscr{A}$ and $F, G: \mathscr{A} \to \mathscr{B}$. Comparing the subequalizer $(\mathscr{E}, D, t)$ of (F, G) with the subequalizer $(\mathscr{E}', D', t')$ of (FH, GH), and using the universal property of the former, we obtain a unique functor $H^*: \mathscr{E}' \to \mathscr{E}$ such that $DH^* = HD'$ and $tH^* = t'$. It is easily seen that $H^*(C, b) = (H(C), b)$ for any $b: FH(C) \to GH(C)$. We shall call $H^*: \mathscr{E}' \to \mathscr{E}$ the functor *induced* by $H: \mathscr{C} \to \mathscr{A}$.

LEMMA 4. *Given $H: \mathscr{C} \to \mathscr{A}$, B in $\mathscr{B}$, $G: \mathscr{A} \to \mathscr{B}$, let $H^*: \mathscr{E}' \to \mathscr{E}$ be the induced functor from the subequalizing category $\mathscr{E}'$ of $K(B), GH: \mathscr{C} \to \mathscr{B}$ to the subequalizing category $\mathscr{E}$ of $K(B), G: \mathscr{A} \to \mathscr{B}$. If H is cogenerating then so is H^*. If H is co-adequate then so is H^*, provided G preserves infs.*

Proof. In the first case, assume that $a_1 \neq a_2: (A, b) \to (A', b')$. This means that $a_1 \neq a_2: A \to A'$ and $G(a_i)b = b'$ for $i = 1, 2$. Since H cogenerates $\mathscr{A}$, there exists C in $\mathscr{C}$ and $a': A' \to H(C)$ such that $a'a_1 \neq a'a_2$. Now $a': (A', b') \to (H(C), G(a')b') = H^*(C, G(a')b')$, hence H^* cogenerates $\mathscr{E}$.

Let us now look at the second case. We are given that $H: \mathscr{C} \to \mathscr{A}$ is a co-adequate, that is to say, whenever $t(C): [A, H(C)] \to [A', H(C)]$ is natural in C, there exists a unique $\alpha: A' \to A$ such that, for all $a: A \to H(C)$, $t(C)(a) = a\alpha$.

We want to show that $H^*: \mathscr{E}' \to \mathscr{E}$ is co-adequate. Thus assume that

$$T(C, \beta): [(A, b), (H(C), \beta)] \to [(A', b'), (H(C), \beta)]$$

is natural in the object (C, β) of $\mathscr{E}'$. We seek a unique $\alpha: (A', b') \to (A, b)$ such that, for all $a: (A, b) \to (H(C), \beta)$, $T(C, \beta)(a) = a\alpha$. For convenience we write $a' = T(C, \beta)(a)$, then $a: A \to H(C)$, $a': A' \to H(C)$, and

$$G(a)b = \beta = G(a')b'.$$

Now write $t(C) = T(C, \beta)$ and check that this is natural in C. Therefore there exists a unique $\alpha: A' \to A$ such that, for all $a: A \to H(C)$, $a' = t(C)(a) = a\alpha$. It remains to show that $\alpha: (A', b') \to (A, b)$, that is, that $G(\alpha)b' = b$.

Now for any $a: A \to H(C)$ we have a: $(A, b) \to (H(C), \beta)$, provided $\beta = G(a)b$. Then

$$G(a)G(\alpha)b' = G(a\alpha)b' = G(a')b' = \beta = G(a)b.$$

We claim that this implies $G(\alpha)b' = b$.

Indeed, let $(\mathscr{E}_A, D_A, t_A)$ be the subequalizer of the pair $K(A), H: \mathscr{C} \to \mathscr{A}$. Then $\inf HD_A = (A, t_A)$, and $t_A(C, a) = a$. Since G preserves infs, also $\inf GHD_A = (G(A), Gt_A)$, and $Gt_A(C, a) = G(a)$. By the universal property of infs, b is uniquely determined by $G(a)b$. The result now follows.

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By saying that the pre-ordered set of subobjects of A is complete we mean that any collection of subobjects, not just a small collection, has an intersection. For small collections this is a consequence of the inf-completeness of $\mathscr{A}$.

PROPOSITION 3. *Assume $\mathscr{C}$ small, $\mathscr{A}$ inf-complete, $\mathscr{B}$ locally small, $H: \mathscr{C} \to \mathscr{A}$, B in $\mathscr{B}$, $G: \mathscr{A} \to \mathscr{B}$ inf-preserving, and let $(\mathscr{E}, D, t)$ be the subequalizer of $K(B)$, $G: \mathscr{A} \to \mathscr{B}$. Then $\mathscr{E}$ has an initial object if either H is co-adequate or H is cogenerating and the subobjects of any object of $\mathscr{A}$ form a complete pre-ordered set.*

Proof. In view of Proposition 1, we need only show that $\mathscr{E}$ has a pre-initial object. By Lemma 2, $\mathscr{E}$ is inf-complete. Moreover $\mathscr{E}'$ is easily seen to be small, since $\mathscr{C}$ is small and $\mathscr{B}$ is locally small.

In the first case, we assume that $\mathscr{H}$ is co-adequate. By Lemma 4, so is $H^*: \mathscr{E}' \to \mathscr{E}$. By Lemma 3, $\mathscr{E}$ then has a pre-initial object.

In the second case, we assume that H is cogenerating. By Lemma 4, so is H^*. By Lemma 3, the subobjects of the infimum (A_0, b_0) of H^* form a pre-initial subcategory of $\mathscr{E}$. Now any subobject of (A_0, b_0) is given by a monomorphism $m: (A, b) \to (A_0, b_0)$ in $\mathscr{E}$. By Lemma 2, $D: \mathscr{E} \to \mathscr{A}$ preserves infs, hence monos (a map being mono if and only if its kernel pair consists of two equal maps), hence $m: A \to A_0$ yields a subobject of A_0 in $\mathscr{A}$.

The intersection (A_1, b_1) of all subobjects of (A_0, b_0) is the infimum of the functor $m \mapsto (A, b)$. (By Lemma 2, this will exist provided the functor $m \mapsto A$ has an infimum, as indeed it has, since the subobjects of A_0 form a complete pre-ordered set.) Then (A_1, b_1) is a pre-initial object of $\mathscr{E}$.

COROLLARY. *Assume $\mathscr{C}$ small, $\mathscr{A}$ inf-complete, $H: \mathscr{C} \to \mathscr{A}$, and suppose that either H is co-adequate or H is cogenerating and the subobjects of any object of $\mathscr{A}$ form a complete pre-ordered set. Then the following conclusions hold*:

(1) *Any inf-preserving functor $G: \mathscr{A} \to$ Ens is representable.*

(2) *Any inf-preserving functor $G: \mathscr{A} \to \mathscr{B}$, where $\mathscr{B}$ is locally small, has a left adjoint.*

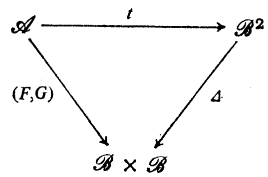
(3) *If $\mathscr{A}$ is locally small then $\mathscr{A}$ is sup-complete.*

This corollary is obtained from Proposition 3 by looking at Examples 1 to 3. However, one can also deduce (2) from (1) and (3) from (2). See [4, §3.12] and [5, §7.1]. Isbell also has a third version of this result in which the notion of "cogenerating" is strengthened.

4. **Further discussion of subequalizers.** (4.1) *Subequalizers as pullbacks.* A natural transformation $t: F \to G: \mathscr{A} \to \mathscr{B}$ is primarily an object function $|A| \to |\mathscr{B}^2|$. One may however define $t(a)$ for any map $a: A \to A'$ by

$$t(a) = G(a)t(A) = t(A')F(a),$$

thus making t into a functor $\mathscr{A} \to \mathscr{B}^2$. Specifically, it is that functor for which the triangle

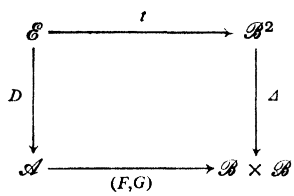


commutes. Here (F, G) and $\Delta=(\text{dom}, \text{cod})$ are defined on objects A of $\mathscr{A}$ and objects $b\colon B\to B'$ of $\mathscr{B}^2$ by

$$(F, G)(A) = (F(A), G(A)), \Delta(b) = (B, B'),$$

and similarly on maps.

Let us now look at the subequalizer $(\mathscr{E}, D, t)$ of $F, G\colon \mathscr{A}\to\mathscr{B}$. Then the natural transformation $t\colon FD\to GD$ is a functor such that the square



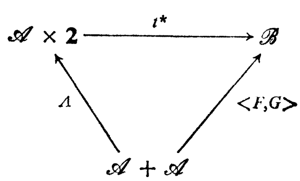
commutes. Moreover, the universal property of subequalizers asserts that this square is a pullback.

(4.2) *Subco-equalizers*. In view of the adjointness $[\mathscr{A}, \mathscr{B}^2]\simeq[\mathscr{A}\times\mathbf{2}, \mathscr{B}]$, it is not surprising that a natural transformation $t\colon F\to G$ may also be regarded as a functor $t^*\colon \mathscr{A}\times\mathbf{2}\to\mathscr{B}$. (Here $\mathbf{2}$ is the category with two objects 0 and 1 and one map $0\to 1$.) Indeed, let $\langle F, G\rangle\colon \mathscr{A}+\mathscr{A}\to\mathscr{B}$ and $\Lambda\colon \mathscr{A}+\mathscr{A}\to\mathscr{A}\times\mathbf{2}$ be defined on objects $(A, 0)$ and $(A, 1)$ of $\mathscr{A}+\mathscr{A}$ by

$$\langle F, G\rangle\,(A, 0) = F(A), \langle F, G\rangle\,(A, 1) = G(A).$$

$$\Lambda(A, 0) = (A, 0), \Lambda(A, 1) = (A, 1),$$

and similarly on maps. Let $t^*\colon \mathscr{A}\times\mathbf{2}\to\mathscr{B}$ be the functor for which the triangle



commutes and which assigns the value $t(a)$ to $(a, 0\to 1)$. Then

$$t^*(A, 0) = F(A), \qquad t^*(A, 1) = G(A), \qquad t^*(A, 0\to 1) = t(A).$$

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Let us now form the pushout:

$$\begin{array}{ccc} \mathscr{A}+\mathscr{A} & \xrightarrow{\langle F,G\rangle} & \mathscr{B} \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle D} \\ \mathscr{A}\times \mathbf{2} & \xrightarrow[t^*]{} & \mathscr{E} \end{array}$$

Then t^* corresponds to a natural transformation $t\colon DF\to DG$, and if $D'\colon \mathscr{B}\to \mathscr{E}'$ and $t'\colon D'F\to D'G$, then there exists a unique functor $E\colon \mathscr{E}\to\mathscr{E}'$ such that $ED=D'$ and $Et=t'$. We call $(\mathscr{E}, D, t)$ the *subco-equalizer* of the pair $F, G\colon \mathscr{A}\to\mathscr{B}$.

Pushout categories are not as easy to describe as pullback categories. To construct the category $\mathscr{E}$ we begin with the category $\mathscr{B}$ and adjoin a family of new maps $t(A)\colon F(A)\to G(A)$, one for each object A of $\mathscr{A}$, together with all maps obtained by composition from the old maps and the new ones, subject to the family of conditions

$$G(a)t(A) = t(A')F(a),$$

one for each map $a\colon A\to A'$ in $\mathscr{A}$.

(4.3) *Comma categories.* As an application of subequalizers one may obtain Lawvere's *comma category* of a pair of functors $F_0\colon \mathscr{A}_0\to\mathscr{B}$, $F_1\colon\mathscr{A}_1\to\mathscr{B}$. This is the subequalizing category of the pair of functors F_0P_0, $F_1P_1\colon \mathscr{A}_0\times\mathscr{A}_1\to\mathscr{B}$, where $P_i\colon\mathscr{A}_0\times\mathscr{A}_1\to\mathscr{A}_i$ are the projection functors $(i=0,1)$. A direct construction of this comma category involves three pullbacks (see Lawvere [7]). It should be pointed out that all our examples, except Example 0, can also be discussed in terms of comma categories. I am not aware of a construction of subequalizing categories in terms of comma categories. Perhaps the latter should be called "subpullbacks", as they are related to subequalizers as pullbacks are to equalizers.

(4.4) *Direct sums of sets.* There may be some expository value in utilizing subequalizers or comma categories in the construction of a *direct sum* or disjoint union of sets. Let $\mathscr{A}$ be a discrete category, that is, a category with no maps other than identity maps. Then a functor $G\colon\mathscr{A}\to Ens$ is just a family of small sets. The direct sum of this family is usually constructed thus:

$$\sum_{A\in\mathscr{A}} G(A) = \{(A,b)\mid A\in\mathscr{A}\ \&\ b\in G(A)\}.$$

If we identify the element b of $G(A)$ with the mapping $1\to G(A)$ with value b, then this is seen to be the set of objects $|\mathscr{E}|$ of the subequalizing category $\mathscr{E}$ of the pair of functors $(K(1), G)$.

It also follows that

$$\prod_{A\in\mathscr{A}} G(A) = \{E\colon\mathscr{A}\to\mathscr{E}\mid DE = 1_{\mathscr{A}}\}.$$

More generally, let $\mathscr{A}$ be any category, no longer assumed to be discrete, then

$$\inf G \cong [1, \inf G] \cong [K(1), G] \cong \{E\colon \mathscr{A} \to \mathscr{E} \mid DE = 1_{\mathscr{A}}\}.$$

(4.5) *Application to embedding problems.* Hedrlin and the author have constructed a full embedding

$$Ens^{\mathscr{A}} \to I\mathscr{R}$$

for any small category $\mathscr{A}$, where I is the set of maps of $\mathscr{A}$ and $I\mathscr{R}$ is the category whose objects are pairs (X, R_I), X being a small set and $R_I = \{R_i \subseteq X \times X \mid i \in I\}$ being a family of binary relations on X, and whose maps are those functions which preserve these relations.

Their argument begins by associating which each functor $G\colon \mathscr{A} \to Ens$ the subequalizing category of the pair $(K(1), G)$. In fact, it can be shown that there are two full embeddings

$$Ens^{\mathscr{A}} \to C(\mathscr{A}) \to I\mathscr{R},$$

where $C(\mathscr{A})$ is the category whose objects are pairs $(\mathscr{E}, D)$, $\mathscr{E}$ being any small category and $D\colon \mathscr{E} \to \mathscr{A}$ being any faithful functor, and whose maps $(\mathscr{E}, D) \to (\mathscr{E}', D')$ are those functors $F\colon \mathscr{E} \to \mathscr{E}'$ for which $D'F = D$.

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