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Generalized tilting theory in functor categories

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Abstract

This paper is devoted to the study of generalized tilting theory of functor categories in different levels. First, we extend Miyashita's proof (Math Z 193:113–146,1986) of the generalized Brenner–Butler theorem to arbitrary functor categories $\mathrm{Mod}(\mathcal{C})$ with $\mathcal{C}$ an annuli variety. Second, a hereditary and complete cotorsion pair generated by a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$ is constructed. Some applications of these two results include the equivalence of Grothendieck groups $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$, the existences of a new abelian model structure on the category of complexes $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$, and a t-structure on the derived category $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

1. Introduction

Tilting theory arises as a universal method for constructing equivalences between categories. Since its advent, it has been an essential tool in the study of many areas of mathematics, including algebraic group theory, commutative and noncommutative algebraic geometry, and algebraic topology.

Tilting theory can trace its history back to the article by Bernstein et al. [8], where they used reflection functors to construct recursively all the indecomposable modules from simple modules over a representation-finite hereditary algebra. The major milestone in the development of tilting theory was the article by Brenner and Butler [9]. They introduced the notion of a tilting module over a finite dimensional algebra and established the so-called Brenner–Butler theorem by a tilting module. In this article, the behavior of the associated quadratic forms was investigated as well. Dropping a more restrictive notion of tilting module defined by Brenner and Butler, Happel and Ringel [15] successfully simplified the definition of original tilting modules. A few years later, Miyashita [26] generalized the concept of tilting modules allowing modules of any finite projective dimension and over any ring, for which a generalization of the Brenner–Butler theorem was still valid. Indeed, the authors, like Brenner and Butler [9], Happel and Ringel [15], and Miyashita [26], considered finitely generated tilting modules, obtaining portions of the Brenner–Butler theorem. Colpi and Trlifaj [10] generalized the notion of tilting module to not necessarily finitely generated modules. Later on, Angeleri-Hügel and Coelho [1] did the same with the concept of Miyashita.

On the other hand, functor categories, introduced by Auslander [2], are used as a potent tool for solving some important problems in representation theory. Martsinkovsky and Russell have studied the injective stabilization of additive functors (see [23–25]). Recently, Martínez-Villa and Ortiz-Morales [20–22] initialed the study of tilting theory in arbitrary functor categories with applications to the functor category $\mathrm{Mod}(\mathcal{A})$ for $\mathcal{A}$ a category of modules over a finite dimensional algebra. The first one [20] in a series of three is to deal with the concept of tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, which is the category of contravariant functors from a skeletally small additive category $\mathcal{C}$, to the category of abelian groups. They showed that the Brenner and Butler theorem holds for $\mathcal{T}$. In the second and third papers [21, 22], replacing a tilting subcategory with a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, they continued the project of extending tilting theory to the same functor category with particular focusing on the equivalence of the derived categories of bounded complexes $\mathrm{D}^b(\mathrm{Mod}(\mathcal{C}))$ and $\mathrm{D}^b(\mathrm{Mod}(\mathcal{T}))$.

In the same spirit as in the above-mentioned results of Martínez-Villa and Ortiz-Morales, in this paper, we aim at extending some well-known results, relating generalized tilting modules in module category, to a generalized tilting subcategory of $\mathrm{Mod}(\mathcal{C})$. We now give a brief outline of the contents of this paper.

In Section 2, we collect preliminary notions and results on functor categories that will be useful throughout the paper and we fix notation. We also give an example of a generalized tilting subcategory of $\mathrm{Mod}(\mathcal{C})$ (see Example 2.4).

In Section 3, we are interested in studying a generalized version of the Brenner–Butler theorem in functor category. More precisely, we show in Theorem 3.4 that for a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$ one gets an equivalence between the categories $\mathrm{KE}_e^{\infty}(\mathcal{T})$ and $\mathrm{KT}_e^{\infty}(\mathcal{T})$. As an application of this main theorem, we state in Theorem 3.8 that if $\mathcal{C}$ is an abelian category with enough injectives and $\mathcal{T}$ is an n-tilting subcategory of $\mathrm{mod}(\mathcal{C})$ with pseudokernels, then the Grothendieck groups $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$ are isomorphic.

In Section 4, we prove in Theorem 4.3 that for a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ is a hereditary and complete cotorsion pair. Furthermore, this induces an abelian model structure on $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$, where the trivial objects are the exact complexes, the cofibrant objects are dg-${}^{\perp_\infty}\Leftarrow\mathcal{T}^{\perp_\infty}\Rightarrow$ complexes, and the class of fibrant objects is given by the complexes whose terms are in $\mathcal{T}^{\perp_\infty}$ (see Corollary 4.6).

In Section 5, we use the model structure on $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$ to describe the t-structure on the derived category $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$, induced by a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$ (see Theorem 5.5).

2. Preliminaries

Throughout this paper, $\mathcal{C}$ will be an arbitrary skeletally small additive category, and $\mathrm{Mod}(\mathcal{C})$ will denote the category of additive contravariant functors from $\mathcal{C}$ to the category of abelian groups. It follows from [28, Theorem 1.2 and Proposition 1.9] or [20, Section 1.2] that $\mathrm{Mod}(\mathcal{C})$ is a Grothendieck category with enough projective objects. In addition, $\mathrm{Mod}(\mathcal{C})$ also has enough injective objects by [19, p.384, Theorem B.3]. If $M, N \in \mathrm{Mod}(\mathcal{C})$, we denote the set $\mathrm{Mod}(\mathcal{C})(M, N)$ of natural transformations $M \to N$ by $\mathrm{Hom}_{\mathcal{C}}(M, N)$. Following [3], a functor F is called *representable* if it is isomorphic to $\mathcal{C}(\ \ , C)$ for some $C \in \mathcal{C}$. A functor F is *finitely generated* if there is an epimorphism $\mathcal{C}(\ \ , C) \to F \to 0$ with $C \in \mathcal{C}$. A functor F is *finitely presented*, if there exists a sequence of natural transformations $\mathcal{C}(\ \ , C_1) \to \mathcal{C}(\ \ , C_0) \to F \to 0$ with $C_0, C_1 \in \mathcal{C}$ such that for any $C \in \mathcal{C}$ the sequence of abelian groups $\mathcal{C}(C, C_1) \to \mathcal{C}(C, C_0) \to F(C) \to 0$ is exact. We denote by $\mathrm{mod}(\mathcal{C})$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of finitely presented functors. An object P in $\mathrm{Mod}(\mathcal{C})$ is projective (finitely generated projective) if and only if P is a summand of $\coprod_{i\in I} \mathcal{C}(\ \ , C_i)$ for a (finite) family $\{C_i\}_{i\in I}$ of objects in $\mathcal{C}$ (see [20, Paragraph 3 of Section 1.2]). We recall from [3, p.188] that an *annuli variety* is a skeletally small additive category in which idempotents split.

Let $\mathcal{A}$ be an abelian category and $F \in \mathrm{mod}(\mathcal{A})$, then there is an exact sequence $\mathcal{A}(\ \ , X) \xrightarrow{(\ \ ,f)} \mathcal{A}(\ \ , Y) \longrightarrow F \to 0$, the value of v at F is defined by the following exact sequence $X \xrightarrow{f} Y \to v(F) \to 0$. This assignment extends to the *defect* functor v: $\mathrm{mod}(\mathcal{A}) \to \mathcal{A}$.

Lemma 2.1. *Let $\mathcal{C}$ be an annuli variety and $\mathcal{T}$ a skeletally small full subcategory of* $\mathrm{Mod}(\mathcal{C})$. *We define the following functor:*

$$\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T}), \quad \phi(M) := \mathrm{Hom}(\ \ , M)_{\mathcal{T}},$$

where the contravariant functor $\mathrm{Hom}(\ \ , M)_{\mathcal{T}} : \mathcal{T} \to \mathbf{Ab}$ *is given by* $\mathrm{Hom}(\ \ , M)_{\mathcal{T}}(T) = \mathrm{Hom}(T, M)$ *for any $T \in \mathcal{T}$. Then the following statements hold.*

(1) *The functor* ϕ *has a left adjoint* $-\otimes\mathcal{T}:\ \mathrm{Mod}(\mathcal{T})\to\mathrm{Mod}(\mathcal{C})$.
(2) $\mathcal{T}(\ \ ,T)\otimes\mathcal{T}=T$ *for any* $T\in\mathcal{T}$.
(3) $-\otimes\mathcal{T}|_{\mathrm{mod}(\mathcal{T})}=v$.

Proof. (1) and (2) come from [21, Theorem 3] and [20, Remark 1], respectively.

(3) It suffices to show that (v,ϕ) is an adjoint pair. Let $M\in\mathrm{mod}(\mathcal{T})$ and $N\in\mathrm{Mod}(\mathcal{C})$, we need to find an isomorphism

$$\theta:\mathrm{Hom}_{\mathrm{Mod}(\mathcal{T})}(M,\phi(N))\to\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(v(M),N)$$

which is natural in both M and N. Suppose that there is an exact sequence $\mathcal{T}(\ \ ,T_1)\longrightarrow\mathcal{T}(\ \ ,T_0)\longrightarrow M\to 0$ with $T_1,T_0\in\mathcal{T}$. By the construction of defect functor, we get an exact sequence $T_1\xrightarrow{f}T_0\to v(M)\to 0$ in $\mathrm{Mod}(\mathcal{C})$. It follows from the Yoneda Lemma that $(\mathcal{T}(\ \ ,T_1),\phi(N))\cong(T_1,N)$ and $(\mathcal{T}(\ \ ,T_0),\phi(N))\cong(T_0,N)$. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (M,\phi(N)) \arrow[r] \arrow[d, dotted, "\theta"] & (\mathcal{T}(\ \ ,T_0),\phi(N)) \arrow[r] \arrow[d, "\cong"] & (\mathcal{T}(\ \ ,T_1),\phi(N)) \arrow[d, "\cong"] \\
0 \arrow[r] & (v(M),N) \arrow[r] & (T_0,N) \arrow[r] & (T_1,N).
\end{tikzcd}$$

So θ is an isomorphism and it is easy to check that θ is natural in both M and N. □

Following [20] and [21], given categories $\mathcal{C}$ and $\mathcal{T}$ as in Lemma 2.1, since $\mathrm{Mod}(\mathcal{C})$ and $\mathrm{Mod}(\mathcal{T})$ have enough projective and injective objects, we can define the nth right derived functors of the functors $\mathrm{Hom}_{\mathcal{C}}(M,\ \ )$ and $\mathrm{Hom}_{\mathcal{C}}(\ \ ,N)$, which will be denoted by $\mathrm{Ext}^n_{\mathcal{C}}(M,\ \ )$ and $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,N)$, respectively.

In the same way, the functor $\phi:\mathrm{Mod}(\mathcal{C})\to\mathrm{Mod}(\mathcal{T})$ has an nth right derived functor, denoted by $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}$, and they are defined as $\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}(M)=\mathrm{Ext}^n_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}$. Analogously, the functor $-\otimes\mathcal{T}:\ \mathrm{Mod}(\mathcal{T})\to\mathrm{Mod}(\mathcal{C})$ has an nth left derived functor, denoted by $\mathrm{Tor}^{\mathcal{T}}_n(\ \ ,\mathcal{T})$.

Let $\mathcal{T}$ be a subcategory of $\mathrm{Mod}(\mathcal{C})$. $\mathrm{Add}(\mathcal{T})$ (resp. $\mathrm{add}(\mathcal{T})$) will denote the class of functors isomorphic to summands of (finite) direct sums of objects in $\mathcal{T}$ and $\mathrm{Gen}_n(\mathcal{T})$ will denote the full subcategory consisting of $M\in\mathrm{Mod}(\mathcal{C})$ for which there exists an exact sequence of the form $T_n\to\cdots\to T_2\to T_1\to M\to 0$ with $T_i\in\mathrm{Add}(\mathcal{T})$. For any $i\geqslant 1$, we write

$$\mathcal{T}^{\perp_i}:=\{M\in\mathrm{Mod}(\mathcal{C})\mid\mathrm{Ext}^i_{\mathcal{C}}(T,M)=0\ \textit{for any}\ T\in\mathcal{T}\},$$

$${}^{\perp_i}\mathcal{T}:=\{M\in\mathrm{Mod}(\mathcal{C})\mid\mathrm{Ext}^i_{\mathcal{C}}(M,T)=0\ \textit{for any}\ T\in\mathcal{T}\},$$

$$\mathcal{T}^{\perp_\infty}:=\{M\in\mathrm{Mod}(\mathcal{C})\mid\mathrm{Ext}^i_{\mathcal{C}}(T,M)=0\ \textit{for any}\ T\in\mathcal{T}\ \textit{and any}\ i\geqslant 1\},$$

$${}^{\perp_\infty}\mathcal{T}:=\{M\in\mathrm{Mod}(\mathcal{C})\mid\mathrm{Ext}^i_{\mathcal{C}}(M,T)=0\ \textit{for any}\ T\in\mathcal{T}\ \textit{and any}\ i\geqslant 1\},$$

$${}^{\top_\infty}\mathcal{T}:=\{N\in\mathrm{Mod}(\mathcal{T})\mid\mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T})=0\ \textit{for any}\ T\in\mathcal{T}\ \textit{and any}\ i\geqslant 1\}.$$

We denote by ${}_{\mathcal{T}}\mathcal{X}$ the full subcategory of $\mathcal{T}^{\perp_\infty}$ consisting of functors M such that there is an exact sequence of the form $\cdots\xrightarrow{f_2}T_1\xrightarrow{f_1}T_0\xrightarrow{f_0}M\to 0$ with $T_i\in\mathrm{Add}(\mathcal{T})$ and $\mathrm{Im}\,f_i\in\mathcal{T}^{\perp_\infty}$. It is easy to see that objects in ${}_{\mathcal{T}}\mathcal{X}$ are in $\mathrm{Gen}_n(\mathcal{T})$ for each n.

Next we recall the concept of cotorsion pairs in abelian categories, due to Holm and Jørgensen [16, Section 6].

Definition 2.2. *Let* $\mathcal{A},\mathcal{B}$ *be two classes in* $\mathrm{Mod}(\mathcal{C})$.

(1) *The pair* $(\mathcal{A},\mathcal{B})$ *is called a* cotorsion pair *if* $\mathcal{A}^{\perp_1}=\mathcal{B}$ *and* ${}^{\perp_1}\mathcal{B}=\mathcal{A}$.
(2) *A cotorsion pair* $(\mathcal{A},\mathcal{B})$ *is* generated *by a class* $\mathcal{X}$ *of objects if* $\mathcal{X}^{\perp_1}=\mathcal{B}$.

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(3) *A cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *has* enough projectives*, that is, for every* $M \in \mathrm{Mod}(\mathcal{C})$ *there exists an exact sequence* $0 \to B \to A \to M \to 0$ *with* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*. Dually, we say that a cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *has* enough injectives*, that is, for every* $M \in \mathrm{Mod}(\mathcal{C})$ *there exists an exact sequence* $0 \to M \to B \to A \to 0$ *with* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*.*
(4) *A cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *is* complete *when it has enough injectives and enough projectives.*
(5) *A cotorsion pair* $(\mathcal{A}, \mathcal{B})$ *is* hereditary *if* $\mathcal{A}^{\perp_\infty} = \mathcal{B}$ *and* ${}^{\perp_\infty}\mathcal{B} = \mathcal{A}$*.*

Now we will introduce the notion of a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$.

Definition 2.3. ([21, Definition 6]). *Let* $\mathcal{C}$ *be an annuli variety. A full subcategory* $\mathcal{T}$ *of* $\mathrm{Mod}(\mathcal{C})$ *is* generalized tilting *if the following statements (1)–(3) hold.*

(1) *There exists a fixed integer n such that every object T in* $\mathcal{T}$ *has a projective resolution*

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0,$$

with each P_i *finitely generated.*
(2) *For each pair of objects T and* T' *in* $\mathcal{T}$ *and any positive integer i, we have* $\mathrm{Ext}^i_{\mathcal{C}}(T, T') = 0$*.*
(3) *For each representable functor* $\mathcal{C}(\ \ , C)$*, there is an exact sequence*

$$0 \to \mathcal{C}(\ \ , C) \to T^0_C \to \cdots \to T^{m_C}_C \to 0,$$

with T^i_C *in* $\mathcal{T}$*.*
(3′) *There is a fixed integer m such that each representable functor* $\mathcal{C}(\ \ , C)$ *has an exact sequence*

$$0 \to \mathcal{C}(\ \ , C) \to T^0_C \to \cdots \to T^m_C \to 0,$$

with T^i_C *in* $\mathcal{T}$*.*

For a general subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, we use $\mathrm{pdim}\,\mathcal{T}$ to denote the the supremum of the set of projective dimensions of all T in $\mathcal{T}$. If $\mathcal{T}$ is generalized tilting with $\mathrm{pdim}\,\mathcal{T} \leqslant n$ satisfying condition (3′), and the integer m in condition (3′) equals n, then we say $\mathcal{T}$ is *n-tilting*. It should be pointed that a tilting subcategory $\mathcal{T}$ defined in [20, Definition 8] is exactly 1-tilting when $\mathcal{T}$ is closed under taking direct summands.

Finally, we end this section by showing that there exists a natural example of a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$.

Example 2.4. *Let* Λ *be an artin R-algebra and* $\mathcal{C} = \mathrm{add}\,\Lambda$*. Assume that* $\mathrm{Mod}\,\Lambda$ *has a classical n-tilting module T. Then we have an n-tilting subcategory* $\mathcal{T}$ *of* $\mathrm{Mod}(\mathrm{add}\,\Lambda)$*.*

Proof. Since T is a classical n-tilting module, it follows from [6] that T satisfies the following conditions:

(1) There exists a projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0$ with each P_i finitely generated,

(2) $\mathrm{Ext}^{i\geqslant 1}_{\Lambda}(T, T) = 0$,

(3) There is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}(T)$.

We set $\mathcal{T} = \{\mathcal{C}(\ \ , T') \mid T' \in \mathrm{add}\,T\}$. It is easy to verify that $\mathcal{T}$ is an n-tilting subcategory of $\mathrm{Mod}(\mathcal{C})$. □

3. Equivalences induced by a generalized tilting subcategory

Our purpose in this section is to study category equivalences induced by a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$. First, we observe the following key result, which is vital in proving the main theorem of this section.

Proposition 3.1. *Assume that $\mathcal{T}$ is generalized tilting with* $\mathrm{pdim}\,\mathcal{T} \leqslant n$ *for some integer n. Then the following statements are equivalent for any M in* $\mathrm{Mod}(\mathcal{C})$.

(1) $M \in \mathcal{T}^{\perp\infty}$.
(2) $M \in {}_{\mathcal{T}}\mathcal{X}$.
(3) $M \in \mathrm{Gen}_n(\mathcal{T})$.

Proof. (1) $\Rightarrow$ (2) For each representable functor $\mathcal{C}(\ \ , C)$, since $\mathcal{T}$ is generalized tilting, there is an exact sequence

$$0 \to \mathcal{C}(\ \ , C) \to T_C^0 \to \cdots \to T_C^{m_C} \to 0,$$

with T_C^i in $\mathcal{T}$. Note that $M \in \mathcal{T}^{\perp\infty}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(T_C^0, \mathrm{Tr}_{T_C^0}(M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), \mathrm{Tr}_{T_C^0}(M)) \arrow[d, "\alpha"] & \\
\mathrm{Hom}_{\mathcal{C}}(T_C^0, M) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), M) \arrow[r] & 0,
\end{tikzcd} \tag{3.1}$$

where $\mathrm{Tr}_{T_C^0}(M) = \sum\{\mathrm{Im}\,\psi \mid \psi \in \mathrm{Hom}_{\mathcal{C}}(T_C^0, M)\}$. Then it follows from Diagram (3.1) that α is epic. Since α is also monic, by Yoneda's lemma, we have

$$\mathrm{Tr}_{T_C^0}(M)(C) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), \mathrm{Tr}_{T_C^0}(M)) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(\ \ , C), M) \cong M(C).$$

Thus $M \in \mathrm{Gen}_1(\mathcal{T})$ and there is an exact sequence $0 \to M_1 \to \coprod_{T\in\mathcal{T}} T^{(X_T)} \to M \to 0$ with $X_T = \mathrm{Hom}_{\mathcal{C}}(T, M)$. Moreover, this exact sequence remains exact after applying the functor ϕ to it. Observe that $\mathrm{Ext}_{\mathcal{C}}^{i\geqslant 1}(T, T'^{(X)}) = 0$ for any $T, T' \in \mathcal{T}$ and any set X by [20, Proposition 4]. So $M_1 \in \mathcal{T}^{\perp\infty}$. Now repeating the process to M_1, we obtain that $M \in {}_{\mathcal{T}}\mathcal{X}$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) The case for $n = 0$ is trivial. Now suppose that $n > 0$, then by assumption there is an exact sequence $0 \to N \to T_n \to \cdots \to T_1 \to M \to 0$ with $T_i \in \mathrm{Add}(\mathcal{T})$. Because $\mathcal{T}$ is self-orthogonal, we have $\mathrm{Ext}_{\mathcal{C}}^i(T, M) \cong \mathrm{Ext}_{\mathcal{C}}^{i+n}(T, N)$ for any T in $\mathcal{T}$ and any $i \geqslant 1$ by dimension shift. But the latter equals 0, since $\mathrm{pdim}\,\mathcal{T} \leqslant n$. Therefore, $M \in \mathcal{T}^{\perp\infty}$. □

The following two results, dual to each other, will be used throughout.

Lemma 3.2. *Assume that $\mathcal{T}$ is generalized tilting and M is an object in $\mathcal{T}^{\perp\infty}$. Then $\phi(M) \otimes \mathcal{T} \cong M$ and $\phi(M) \in {}^{\top\infty}\mathcal{T}$.*

Proof. Since $M \in \mathcal{T}^{\perp\infty}$, $M \in {}_{\mathcal{T}}\mathcal{X}$ by Proposition 3.1, in particular there is an exact sequence

$$\cdots \xrightarrow{f_{n+1}} T_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0 \tag{3.2}$$

with $T_i = \coprod_{T\in\mathcal{T}} T^{(X_i)}$ and each $\mathrm{Im}\,f_i \in \mathcal{T}^{\perp\infty}$. Applying the functor ϕ to (3.2) yields the following exact sequence

$$\cdots \xrightarrow{\phi(f_{n+1})} \phi(T_n) \xrightarrow{\phi(f_n)} \cdots \xrightarrow{\phi(f_2)} \phi(T_1) \xrightarrow{\phi(f_1)} \phi(T_0) \xrightarrow{\phi(f_0)} \phi(M) \to 0.$$

Thanks to [20, Theorem 2], the functor ϕ preserves direct sums. Thus, each $\phi(T_i)$ is projective in $\mathrm{Mod}(\mathcal{T})$. Moreover, by Lemma 2.1, we have the following commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & \phi(T_n)\otimes\mathcal{T} \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \phi(T_0)\otimes\mathcal{T} \arrow[r] \arrow[d, "\cong"] & \phi(M)\otimes\mathcal{T} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & T_n \arrow[r] & \cdots \arrow[r] & T_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}
$$

Consequently, we obtain that $\phi(M)\otimes\mathcal{T}\cong M$ and $\phi(M)\in {}^{\top_\infty}\mathcal{T}$. □

Analogously, dualizing the proof of the above lemma, we have the following

Lemma 3.3. *Assume that $\mathcal{T}$ is generalized tilting and N is an object in ${}^{\top_\infty}\mathcal{T}$. Then $N\cong\phi(N\otimes\mathcal{T})$ and $N\otimes\mathcal{T}\in\mathcal{T}^{\perp_\infty}$.*

Proof. Consider the following projective resolution of N

$$\cdots\to P_i\to\cdots\to P_1\to P_0\to N\to 0 \tag{3.3}$$

with $P_i=\coprod_{j_i\in J_i}\mathcal{T}(\ \ ,T_{j_i})$. Since $(-\otimes\mathcal{T},\phi)$ forms an adjoint pair by Lemma 2.1, the functor $-\otimes\mathcal{T}$ preserves direct sums and $(\coprod_{j_i\in J_i}\mathcal{T}(\ \ ,T_{j_i}))\otimes\mathcal{T}\cong\coprod_{j_i\in J_i}T_{j_i}$. By the assumption of N, applying the functor $-\otimes\mathcal{T}$ to (3.3) yields the following exact sequence

$$\cdots\to P_n\otimes\mathcal{T}\cdots\to P_1\otimes\mathcal{T}\to P_0\otimes\mathcal{T}\to N\otimes\mathcal{T}\to 0.$$

Note that $\mathcal{T}$ is generalized tilting, we have that $P_i\otimes\mathcal{T}\in\mathcal{T}^{\perp_\infty}$ for any $i\geqslant 0$. Moreover, we have the following commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\cong"] & P_0 \arrow[r] \arrow[d, "\cong"] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \phi(P_n\otimes\mathcal{T}) \arrow[r] & \cdots \arrow[r] & \phi(P_1\otimes\mathcal{T}) \arrow[r] & \phi(P_0\otimes\mathcal{T}) \arrow[r] & \phi(N\otimes\mathcal{T}) \arrow[r] & 0.
\end{tikzcd}
$$

So we get that $N\cong\phi(N\otimes\mathcal{T})$ and $N\otimes\mathcal{T}\in\mathcal{T}^{\perp_\infty}$. □

In order to present the main theorem in this section, we need to introduce the following notions.

$$\mathrm{KE}_e^n(\mathcal{T}):=\{M\in\mathrm{Mod}(\mathcal{C})\mid \mathrm{Ext}^i_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}=0\ \textit{if}\ 0\leqslant i\leqslant n\ \textit{and}\ i\neq e\},$$

$$\mathrm{KT}_e^n(\mathcal{T}):=\{N\in\mathrm{Mod}(\mathcal{T})\mid \mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T})=0\ \textit{if}\ 0\leqslant i\leqslant n\ \textit{and}\ i\neq e\},$$

$$\mathrm{KE}_e^\infty(\mathcal{T}):=\{M\in\mathrm{Mod}(\mathcal{C})\mid \mathrm{Ext}^i_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}=0\ \textit{if}\ 0\leqslant i<\infty\ \textit{and}\ i\neq e\},$$

$$\mathrm{KT}_e^\infty(\mathcal{T}):=\{N\in\mathrm{Mod}(\mathcal{T})\mid \mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T})=0\ \textit{if}\ 0\leqslant i<\infty\ \textit{and}\ i\neq e\}.$$

Theorem 3.4. *Assume that $\mathcal{T}$ is generalized tilting and e is a non-negative integer. Then there are two category equivalences*

$$\mathrm{KE}_e^\infty(\mathcal{T}) \underset{\mathrm{Tor}^{\mathcal{T}}_e(\ \ ,\mathcal{T})}{\overset{\mathrm{Ext}^e_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}}{\rightleftarrows}} \mathrm{KT}_e^\infty(\mathcal{T}).$$

Proof. We can apply Lemmas 3.2 and 3.3 to conclude that the equivalence holds for $e=0$. Now assume that $e\geqslant 1$ and $M\in\mathrm{KE}_e^\infty(\mathcal{T})$. Consider an injective resolution of M

$$0\to M\xrightarrow{f_0} I_0\xrightarrow{f_1} I_1\xrightarrow{f_2}\cdots\xrightarrow{f_i} I_i\xrightarrow{f_{i+1}}\cdots. \tag{3.4}$$

Since $\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,\mathrm{Im}\,f_e)_{\mathcal{T}}\cong\mathrm{Ext}^{i+e}_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}=0$ for any $i\geqslant 1$. It follows from Lemma 3.2 that $\phi(\mathrm{Im}\,f_e)\otimes\mathcal{T}\cong\mathrm{Im}\,f_e$ and $\phi(\mathrm{Im}\,f_e)\in{}^{\top_\infty}\mathcal{T}$. Applying the functor ϕ to (3.4), we get an exact sequence

$$0\to\phi(I_0)\to\phi(I_1)\to\cdots\to\phi(I_{e-1})\to\phi(\mathrm{Im}\,f_e)\to X\to 0$$

with $X \cong \mathrm{Ext}^1_{\mathcal{C}}(\ \ , \mathrm{Im}\, f_{e-1})_{\mathcal{T}} \cong \mathrm{Ext}^e_{\mathcal{C}}(\ \ , M)_{\mathcal{T}}$. Because every term except X in the exact sequence belongs to ${}^{\top_\infty}\mathcal{T}$, the ith-homology can be computed by it. Therefore, we obtain that $\mathrm{Tor}^{\mathcal{T}}_i(X, \mathcal{T}) = 0$ for any $0 \leqslant i < \infty$ and $i \neq e$, $\mathrm{Tor}^{\mathcal{T}}_e(X, \mathcal{T}) \cong M$.

Conversely, suppose that $N \in \mathrm{KT}^{\infty}_e(\mathcal{T})$. Consider a projective resolution of N

$$\cdots \xrightarrow{g_{i+1}} P_i \xrightarrow{g_i} \cdots \xrightarrow{g_2} P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} N \to 0 \tag{3.5}$$

with $P_i = \coprod_{j_i \in J_i} \mathcal{T}(\ \ , T_{j_i})$. Since $\mathrm{Tor}^{\mathcal{T}}_i(\mathrm{Im}\, g_e, \mathcal{T}) \cong \mathrm{Tor}^{\mathcal{T}}_{i+e}(N, \mathcal{T}) = 0$ for any $i \geqslant 1$. It follows from Lemma 3.3 that $\mathrm{Im}\, g_e \cong \phi(\mathrm{Im}\, g_e \otimes \mathcal{T})$ and $\mathrm{Im}\, g_e \otimes \mathcal{T} \in \mathcal{T}^{\perp_\infty}$. Applying the functor $- \otimes \mathcal{T}$ to (3.5), we get an exact sequence

$$0 \to Y \to \mathrm{Im}\, g_e \otimes \mathcal{T} \to P_{e-1} \otimes \mathcal{T} \to \cdots \to P_1 \otimes \mathcal{T} \to P_0 \otimes \mathcal{T} \to 0$$

with $Y \cong \mathrm{Tor}^{\mathcal{T}}_1(\mathrm{Im}\, g_{e-1}, \mathcal{T}) \cong \mathrm{Tor}^{\mathcal{T}}_e(N, \mathcal{T})$. Because every term except Y in the exact sequence belongs to $\mathcal{T}^{\perp_\infty}$, the ith-cohomology can be computed by it. Therefore, we obtain that $\mathrm{Ext}^i_{\mathcal{C}}(\ \ , Y)_{\mathcal{T}} = 0$ for any $0 \leqslant i < \infty$ and $i \neq e$, $\mathrm{Ext}^e_{\mathcal{C}}(\ \ , Y)_{\mathcal{T}} \cong N$. $\square$

Given a 1-tilting subcategory $\mathcal{T}$, Martínez-Villa and Ortiz-Morales in [20, Theorem 3] proved that ϕ and $- \otimes \mathcal{T}$ induce an equivalence between $\mathrm{KE}^1_0(\mathcal{T})$ and $\mathrm{KT}^1_0(\mathcal{T})$. We generalize this result to n-tilting subcategory $\mathcal{T}$ as follows.

Corollary 3.5. *Assume that $\mathcal{T}$ is n-tilting. Then for any $0 \leqslant e \leqslant n$, there are two category equivalences*

$$\mathrm{KE}^n_e(\mathcal{T}) \underset{\mathrm{Tor}^{\mathcal{T}}_e(\ \ ,\mathcal{T})}{\overset{\mathrm{Ext}^e_{\mathcal{C}}(\ \ ,-)_{\mathcal{T}}}{\rightleftarrows}} \mathrm{KT}^n_e(\mathcal{T}).$$

Proof. Since $\mathrm{pdim}(\mathcal{T}) \leqslant n$, it is obvious that $\mathrm{KE}^n_e(\mathcal{T}) = \mathrm{KE}^{\infty}_e(\mathcal{T})$. For each representable functor $\mathcal{C}(\ \ , C)$, there is an exact resolution

$$0 \to \mathcal{C}(\ \ , C) \to T^0 \to \cdots \to T^n \to 0$$

with T^i in $\mathcal{T}$. Then we get a projective resolution of $(\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}}$:

$$0 \to (T_n, \ \ )_{\mathcal{T}} \to \cdots \to (T_1, \ \ )_{\mathcal{T}} \to (T_0, \ \ )_{\mathcal{T}} \to (\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}} \to 0.$$

Then [20, Proposition 14] implies that $\mathrm{Tor}^{\mathcal{T}}_i((\mathcal{C}(\ \ , C), \ \ )_{\mathcal{T}}, N) \cong \mathrm{Tor}^{\mathcal{T}}_i(N, \mathcal{T})(C) = 0$ for any $N \in \mathrm{Mod}(\mathcal{T})$ and $i \geqslant n + 1$. So the two categories $\mathrm{KT}^n_e(\mathcal{T})$ and $\mathrm{KT}^{\infty}_e(\mathcal{T})$ coincide. Finally, the conclusion follows by Theorem 3.4. $\square$

According to [21], we say $\mathcal{C}$ has *pseudokernels* if given a map $f : C_1 \to C_0$ in $\mathcal{C}$, there is a map $g : C_2 \to C_1$ in $\mathcal{C}$ such that the sequence of representable functors $\mathcal{C}(\ \ , C_2) \xrightarrow{\mathcal{C}(\ ,g)} \mathcal{C}(\ \ , C_1) \xrightarrow{\mathcal{C}(\ ,f)} \mathcal{C}(\ \ , C_0)$ is exact. Since $\mathrm{Mod}(\mathcal{C})$ is an abelian category, $\mathcal{C}$ has pseudokernels if and only if $\mathrm{Ker}(, f)$ is finitely generated for each $f : C_1 \to C_0$ in $\mathcal{C}$. Next we turn to investigating the invariance of Grothendieck groups under generalized tilting. To this end, we need the following.

Definition 3.6. ([20, Definition 10]) *Let $\mathcal{C}$ be a skeletally small preadditive category with pseudokernels. Let's define by* $|\,\mathrm{mod}(\mathcal{C})|$ *the set of isomorphism classes of objects in* $|\,\mathrm{mod}(\mathcal{C})|$. *Let $\mathcal{A}$ be the free abelian group generated by* $|\,\mathrm{mod}(\mathcal{C})|$ *and $\mathcal{R}$ the subgroup of $\mathcal{A}$ generated by relations* $[M] - [K] - [L]$ *where* $0 \to K \to M \to L \to 0$ *is a short exact sequence in* $\mathrm{mod}(\mathcal{C})$. *Then, the Grothendieck group of $\mathcal{C}$ is* $\mathcal{K}_0(\mathcal{C}) = \mathcal{A}/\mathcal{R}$.

It was proved in [4] that $\mathrm{mod}(\mathcal{C})$ is abelian if and only if $\mathcal{C}$ has pseudokernels. We will use this result to show the following proposition.

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Proposition 3.7. *Let $\mathcal{C}$ be an annuli variety and $\mathcal{T}$ a generalized tilting subcategory of* $\mathrm{mod}(\mathcal{C})$. *Assume $\mathcal{C}$ and $\mathcal{T}$ have pseudokernels. Then the following statements hold.*

(1) $\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}} \in \mathrm{mod}(\mathcal{T})$ *for any* $M \in \mathrm{mod}(\mathcal{C})$ *and any* $i \geqslant 0$.
(2) $\mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T}) \in \mathrm{mod}(\mathcal{C})$ *for any* $N \in \mathrm{mod}(\mathcal{T})$ *and any* $i \geqslant 0$.

Proof. (1) Since $\mathcal{T}$ is generalized tilting, we may assume that $\mathrm{pdim}\,\mathcal{T} \leqslant n$ for some integer n. Let $M \in \mathrm{mod}(\mathcal{C})$, then there is an exact sequence $0 \to K_1 \to \mathcal{C}(\ \ ,C_0) \to M \to 0$ with $K_1 \in \mathrm{mod}(\mathcal{C})$. Applying the functor ϕ to this exact sequence gives rise to the following exact sequence

$$0 \to \phi(K_1) \to \phi(\mathcal{C}(\ \ ,C)) \to \phi(M) \to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,K_1)_{\mathcal{T}} \to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,\mathcal{C}(\ \ ,C_0))_{\mathcal{T}}$$

$$\to \mathrm{Ext}^1_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}} \to \cdots \to \mathrm{Ext}^n_{\mathcal{C}}(\ \ ,K_1)_{\mathcal{T}} \to \mathrm{Ext}^n_{\mathcal{C}}(\ \ ,\mathcal{C}(\ \ ,C_0))_{\mathcal{T}} \to \mathrm{Ext}^n_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}} \to 0.$$

It follows from the proof of [21, Proposition 6] that $\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,K_1)_{\mathcal{T}}$ is in $\mathrm{mod}(\mathcal{T})$ for any $i \geqslant 0$. On the other hand, by [21, Lemma 5], we know that $\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,\mathcal{C}(\ \ ,C_0))_{\mathcal{T}}$ is also in $\mathrm{mod}(\mathcal{T})$ for any $i \geqslant 0$. So $\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}} \in \mathrm{mod}(\mathcal{T})$ for any $i \geqslant 0$ since $\mathrm{mod}(\mathcal{T})$ is abelian.

(2) Let $N \in \mathrm{mod}(\mathcal{T})$ and

$$\cdots \xrightarrow{f_{i+1}} \mathcal{T}(\ \ ,T_i) \xrightarrow{f_i} \cdots \xrightarrow{f_2} \mathcal{T}(\ \ ,T_1) \xrightarrow{f_1} \mathcal{T}(\ \ ,T_0) \xrightarrow{f_0} N \to 0$$

a projective resolution of N. Set $L_i = \mathrm{Im}\, f_i$ and split the resolution in short exact sequences: $0 \to L_{i+1} \to \mathcal{T}(\ \ ,T_i) \to L_i \to 0$. Thus it follows from the long homology sequence that there are exact sequences:

$$0 \to \mathrm{Tor}^{\mathcal{T}}_1(N,\mathcal{T}) \to L_1 \otimes \mathcal{T} \to T_0 \to N \otimes \mathcal{T} \to 0,$$

$$0 \to \mathrm{Tor}^{\mathcal{T}}_1(L_1,\mathcal{T}) \to L_2 \otimes \mathcal{T} \to T_1 \to L_1 \otimes \mathcal{T} \to 0.$$

Since $T_1 \in \mathrm{mod}(\mathcal{C})$, $L_1 \otimes \mathcal{T}$ is finitely generated. Thus, $\mathrm{Im}(L_1 \otimes \mathcal{T} \to T_0)$ is finitely generated and so $N \otimes \mathcal{T}$ is finitely presented. Similarly, as $L_1 \in \mathrm{mod}(\mathcal{C})$, $L_1 \otimes \mathcal{T}$ is finitely presented. Note that $\mathrm{mod}(\mathcal{C})$ is abelian. We get that $\mathrm{Tor}^{\mathcal{T}}_1(N,\mathcal{T})$ is finitely presented. Because $\mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T}) \cong \mathrm{Tor}^{\mathcal{T}}_1(L_{i-1},\mathcal{T})$ for any $i \geqslant 2$, $\mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T}) \in \mathrm{mod}(\mathcal{C})$. □

We now come to the first application in this section, our method in the following has its origin in [26, Theorem 1.19].

Theorem 3.8. *Let $\mathcal{C}$ be an abelian category with enough injectives and $\mathcal{T}$ an n-tilting subcategory of* $\mathrm{mod}(\mathcal{C})$ *with pseudokernels. Then the Grothendieck groups $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$ are isomorphic.*

Proof. We define two group homomorphisms

$$F : K_0(\mathcal{C}) \to K_0(\mathcal{T}), \quad [M] \mapsto \sum_{i \geqslant 0} (-1)^i [\mathrm{Ext}^i_{\mathcal{C}}(\ \ ,M)_{\mathcal{T}}],$$

$$G : K_0(\mathcal{T}) \to K_0(\mathcal{C}), \quad [N] \mapsto \sum_{i \geqslant 0} (-1)^i [\mathrm{Tor}^{\mathcal{T}}_i(N,\mathcal{T})].$$

It is easily seen by Proposition 3.7 that F and G are well defined. For any $M \in \mathrm{mod}(\mathcal{C})$, since $\mathrm{mod}(\mathcal{C})$ has enough injectives by [29, Section 6], we have an injective resolution

$$0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} I_i \xrightarrow{f_{i+1}} \cdots .$$

in $\mathrm{mod}(\mathcal{C})$. Set $K_i = \mathrm{Im}\, f_i$. Then $K_n \in \mathrm{KE}^{\infty}_0(\mathcal{T}) \bigcap \mathrm{mod}(\mathcal{C})$. Thus, $K_0(\mathcal{C})$ is generated by the set of all $[X]$ with $X \in \mathrm{KE}^{\infty}_0(\mathcal{T}) \bigcap \mathrm{mod}(\mathcal{C})$. Dually we have that $K_0(\mathcal{T})$ is generated by the set of all $[Y]$ with

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$Y \in \mathrm{KT}_0^{\infty}(\mathcal{T}) \bigcap \mathrm{mod}(\mathcal{T})$. Since $GF([X]) = [X]$ and $FG([Y]) = [Y]$ by Theorem 3.4, we conclude that $K_0(\mathcal{C})$ and $K_0(\mathcal{T})$ are isomorphic. □

Given $M \in \mathrm{Mod}(\mathcal{C})$, since M has an injective envelope by [19, Theorem B.3], we have a minimal injective resolution $0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} \cdots$. Then $\mathrm{cTr}_{\mathcal{T}}\, M := \mathrm{Coker}\, \phi(f_0)$ is called the *cotranspose* of M with respect to $\mathcal{T}$. The notion is analogous to the cotranspose of a module with respect to a semidualizing bimodule defined in [32, Defintion 3.1]. Using the tool of cotransposes, Tang and Huang in [32, Proposition 3.2] established the so-called dual Auslander sequence. We will apply Theorem 3.4 to conclude that the dual Auslander sequence still holds in functor categories, but the approach used here is different.

Corollary 3.9. *Assume that $\mathcal{T}$ is generalized tilting. Then for any $M \in \mathrm{Mod}(\mathcal{C})$, there is an exact sequence*

$$0 \to \mathrm{Tor}_2^{\mathcal{T}}(\,\mathrm{cTr}_{\mathcal{T}}\, M, \mathcal{T}) \to \phi(M) \otimes \mathcal{T} \to M \to \mathrm{Tor}_1^{\mathcal{T}}(\,\mathrm{cTr}_{\mathcal{T}}\, M, \mathcal{T}) \to 0.$$

Proof. Let $0 \to M \to I_0 \to I_1 \to \cdots$ be a minimal injective resolution of M. Applying the functor ϕ to it yields an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \phi(M) \arrow[r] & \phi(I_0) \arrow[rr] \arrow[dr] & & \phi(I_1) \arrow[r] & \mathrm{cTr}_{\mathcal{T}}\, M \arrow[r] & 0 \\
& & & K \arrow[ur] & & &
\end{tikzcd} \tag{3.6}$$

where $K = \mathrm{Im}(\phi(I_0) \to \phi(I_1))$. By Theorem 3.4, now applying the functor $\otimes\mathcal{T}$ to Diagram (3.6) gives rise to the following diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & & \\
& \mathrm{Tor}_2^{\mathcal{T}}(\mathrm{cTr}_{\mathcal{T}}\, M, \mathcal{T}) \arrow[r, equal] \arrow[d] & \mathrm{Tor}_2^{\mathcal{T}}(\mathrm{cTr}_{\mathcal{T}}\, M, \mathcal{T}) \arrow[d] & & & \\
& \phi(M) \otimes \mathcal{T} \arrow[r, equal] \arrow[d] & \phi(M) \otimes \mathcal{T} \arrow[d] & & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d, equal] & \mathrm{cTr}_{\mathcal{T}}\, M \otimes \mathcal{T} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1^{\mathcal{T}}(\mathrm{cTr}_{\mathcal{T}}\, M, \mathcal{T}) \arrow[r] \arrow[d] & K \otimes \mathcal{T} \arrow[r] \arrow[d] & I_1 \arrow[r] & \mathrm{cTr}_{\mathcal{T}}\, M \otimes \mathcal{T} \arrow[r] & 0 \\
& 0 & 0. & & &
\end{tikzcd}$$

Therefore the left most column in the above diagram is as desired. □

4. Cotorsion pair and model category structure

Our goal in this section is to construct a cotorsion pair induced by a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$, allowing us to provide a model category structure on the category of complexes $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$. For the definition of a model structure, we refer to the book by Hoevy [18].

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The following lemma is straightforward, but we include a proof as we have not been able to find a suitable reference for it.

Lemma 4.1. *Suppose that $\mathcal{T}$ is a subcategory of* $\mathrm{Mod}(\mathcal{C})$. *Then the following statements hold.*

(1) *If $\mathcal{T}$ is closed under cokernels of monomorphisms and contains all injective objects in* $\mathrm{Mod}(\mathcal{C})$, *then ${}^{\perp_\infty}\mathcal{T} = {}^{\perp_1}\mathcal{T}$.*
(2) *If $\mathcal{T}$ is closed under kernels of epimorphisms and contains all projective objects in* $\mathrm{Mod}(\mathcal{C})$, *then $\mathcal{T}^{\perp_\infty} = \mathcal{T}^{\perp_1}$.*

Proof. We show the first statement of the lemma. The second statement follows from a dual argument.

(1) It is enough to show that ${}^{\perp_1}\mathcal{T} \subseteq {}^{\perp_\infty}\mathcal{T}$. Let $X \in {}^{\perp_1}\mathcal{T}$ and $M \in \mathcal{T}$, then we have an exact sequence $0 \to M \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ with I_i injective. Set $K_i = \mathrm{Ker}(I_i \to I_{i+1})$. By assumption, both I_i and K_i are in $\mathcal{T}$. So $\mathrm{Ext}^i_{\mathcal{C}}(X, M) \cong \mathrm{Ext}^1_{\mathcal{C}}(X, K_{i-1}) = 0$ for any $i > 1$. □

Now we consider the relation between $\mathrm{Gen}_n(\mathcal{T})$ and $\mathrm{Gen}_n({}_{\mathcal{T}}\mathcal{X})$.

Lemma 4.2. *Assume that $\mathcal{T}$ is a subcategory of* $\mathrm{Mod}(\mathcal{C})$. *If there is an exact sequence $0 \to L \to K_n \to \cdots \to K_1 \to M \to 0$ in* $\mathrm{Mod}(\mathcal{C})$ *with $K_i \in {}_{\mathcal{T}}\mathcal{X}$, then there is an exact sequence $0 \to U_n \to V_n \to L \to 0$ for some $U_n \in {}_{\mathcal{T}}\mathcal{X}$, and some V_n such that there is an exact sequence $0 \to V_n \to T_n \to \cdots \to T_1 \to M \to 0$ with $T_i \in$* $\mathrm{Add}(\mathcal{T})$.

Proof. The proof is modeled on [33, Lemma 3.5(1)]. We shall prove the statement by induction on n. When $n = 1$, we have an exact sequence $0 \to L \to K_1 \to M \to 0$ with $K_1 \in {}_{\mathcal{T}}\mathcal{X}$. Thus, we have another exact sequence $0 \to U_1 \to T_1 \to K_1 \to 0$ with $T_1 \in \mathrm{Add}(\mathcal{T})$ and $U_1 \in {}_{\mathcal{T}}\mathcal{X}$. Consider the following pull-back diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & U_1 \arrow[r, equal] \arrow[d] & U_1 \arrow[d] & & \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd} \tag{4.1}$$

Then the middle row and left column in Diagram (4.1) are the desired exact sequences. Now assume that the conclusion is true for $n-1$. We will show that the conclusion holds for n. Set $L' = \mathrm{Coker}(L \to K_n)$. Then, by the induction assumption, there is an exact sequence $0 \to U_{n-1} \to V_{n-1} \to L' \to 0$ for some $U_{n-1} \in {}_{\mathcal{T}}\mathcal{X}$, and some V_{n-1} such that there is an exact sequence $0 \to V_{n-1} \to T_{n-1} \to \cdots \to T_1 \to M \to 0$ with $T_i \in \mathrm{Add}(\mathcal{T})$. Then we can construct the following pullback diagram

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$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U_{n-1} \arrow[r, equal] \arrow[d] & U_{n-1} \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & V_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & K_n \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd} \tag{4.2}$$

Since $U_{n-1}, K_n \in {}_{\mathcal{T}}\mathcal{X}$ and ${}_{\mathcal{T}}\mathcal{X}$ is closed under extensions by [11, Lemma 8.2.1], we get that $X \in {}_{\mathcal{T}}\mathcal{X}$. Thus, there is an exact sequence $0 \to U_n \to T_n \to X \to 0$ with $T_n \in \mathrm{Add}(\mathcal{T})$ and $U_n \in {}_{\mathcal{T}}\mathcal{X}$. Consider the following pull-back diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & U_n \arrow[r, equal] \arrow[d] & U_n \arrow[d] & & \\
0 \arrow[r] & V_n \arrow[r] \arrow[d] & T_n \arrow[r] \arrow[d] & V_{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & V_{n-1} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd} \tag{4.3}$$

Now the left column in Diagram (4.3) is the desired exact sequence. □

Our main aim in this section is to show that the following holds.

Theorem 4.3. *Suppose that $\mathcal{T}$ is a generalized tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$ *with* $\mathrm{pdim}(\mathcal{T}) \leqslant n$. *Then the following statements hold.*

(1) $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ *is a hereditary and complete cotorsion pair.*
(2) $\mathrm{pdim}({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})) \leqslant n$.
(3) ${}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}) \cap \mathcal{T}^{\perp_\infty} = \mathrm{Add}(\mathcal{T})$.

Proof. (1) Since $\mathcal{T}^{\perp_\infty}$ is closed under cokernels of monomorphisms and contains all injective objects, it follows from Lemma 4.1(1) that ${}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}) = {}^{\perp_1}(\mathcal{T}^{\perp_\infty})$. On the other hand, since ${}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})$ is closed under kernels of epimorphisms and contains all projective objects, we have $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}))^{\perp_1} = ({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}))^{\perp_\infty}$ Lemma 4.1(2). Thus, $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}))^{\perp_1} = \mathcal{T}^{\perp_\infty}$. Hence, $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ forms a hereditary cotorsion pair. For each $T \in \mathcal{T}$, by assumption, there is a projective resolution $0 \to P_n(T) \to \cdots \to P_1(T) \to P_0(T) \to T \to 0$. Set $K_i(T) = \mathrm{Im}(P_i(T) \to P_{i-1}(T))$ for $1 \leqslant i \leqslant n$ and $\mathcal{X} = \{K_i(T) \mid T \in \mathcal{T}\} \cup \mathcal{T}$. Obviously, $\mathcal{X}$ is a set and $\mathcal{X}^{\perp_1} = \mathcal{T}^{\perp_\infty}$. It implies that $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ is generated by a set $\mathcal{X}$. We conclude by [16] that $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ is a complete cotorsion pair.

(2) Let $M \in \mathrm{Mod}(\mathcal{C})$, there is an exact sequence $0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to K \to 0$ with I_i injective. Then $I_i \in {}_{\mathcal{T}}\mathcal{X}$ by Proposition 3.1. Thus, $K \in \mathrm{Gen}_n(\mathcal{T})$ by Lemma 4.2. It follows from Proposition 3.1 again that $K \in \mathcal{T}^{\perp_\infty}$. Given $X \in {}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})$, we know that $\mathrm{Ext}^{i+n}_{\mathcal{C}}(X, M) \cong \mathrm{Ext}^i_{\mathcal{C}}(X, K) = 0$ for any $i \geqslant 1$. Therefore, the result holds.

(3) $\mathrm{Add}(\mathcal{T}) \subseteq {}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}) \cap \mathcal{T}^{\perp_\infty}$ is trivial. Conversely, let $M \in {}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}) \cap \mathcal{T}^{\perp_\infty}$, then there is an exact sequence $0 \to K \to T \to M \to 0$ with $T \in \mathrm{Add}(\mathcal{T})$ and $K \in \mathcal{T}^{\perp_\infty}$ by Proposition 3.1. So the sequence splits, which implies that $M \in \mathrm{Add}(\mathcal{T})$. □

Let $\mathcal{A}$ be an abelian category. A complex $A = (A_n, d_n^A)$ is a sequence $\cdots \to A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \to \cdots$ with $A_n \in \mathcal{A}$ and $d_n \in \mathrm{Hom}_{\mathcal{A}}(A_n, A_{n-1})$ satisfying $d_n^A d_{n+1}^A = 0$ for any $n \in \mathbb{Z}$. We denote by $\mathrm{C}(\mathcal{A})$ the category of complexes. A morphism $f : X \to Y$ is said to be a *quasi-isomorphism* if the induced morphism $\mathrm{H}(f) : \mathrm{H}(X) \to \mathrm{H}(Y)$ is an isomorphism. Given a complex $A = (A_n, d_n^A)$, the *suspension* of A, denoted ΣA, is the complex given by $(\Sigma A)_n = A_{n-1}$ and $d_n^{\Sigma A} = -d_{n-1}^A$. For complexes X and Y, we define the homomorphism complex $\mathrm{Hom}(X, Y) \in \mathrm{C}(\mathcal{A})$ to be the complex

$$\cdots \to \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X_k, Y_{k+n-1}) \to \cdots,$$

where $(\delta_n f) = d_{k+n}^Y f_k - (-1)^n f_{k-1} d_k^X$. We let $\mathrm{Ext}^1_{\mathrm{C}(\mathcal{A})}(Y, X)$ to be the group of (equivalence classes) of short exact sequences $0 \to X \to Z \to Y \to 0$ in $\mathrm{C}(\mathcal{A})$. Recall that a morphism $f : X \to Y$ of complexes is called *null-homotopic* if there exists $s_n \in \mathrm{Hom}_{\mathcal{A}}(X_{n-1}, Y_n)$ such that $f_n = d_{n+1}^Y s_{n+1} + s_n d_n^X$ for each $n \in \mathbb{Z}$. For morphisms $f, g : X \to Y$ in $\mathrm{C}(\mathcal{A})$, we denote $f \sim g$ if $f - g$ is null-homotopic. We denote by $\mathrm{K}(\mathcal{A})$ the *homotopic category*, that is, the category consisting of complexes such that the morphism set between $X, Y \in \mathrm{C}(\mathcal{A})$ is given by $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X, Y) = \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(X, Y)/\sim$. Furthermore, there is a corresponding *derived category* $\mathrm{D}(\mathcal{A})$, which is also triangulated.

In order to obtain an abelian model structure, we have to introduce the following classes in $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$.

Definition 4.4. ([12]). *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathcal{C}$. Let X be a complex.*

1. *X is called an $\mathcal{A}$* complex *if it is exact and $Z_n(X) \in \mathcal{A}$ for all n.*
2. *X is called a $\mathcal{B}$* complex *if it is exact and $Z_n(X) \in \mathcal{B}$ for all n.*
3. *X is called a* dg-$\mathcal{A}$ complex *if $X_n \in \mathcal{A}$ for each n, and* $\mathrm{Hom}(X, B)$ *is exact whenever B is a $\mathcal{B}$ complex.*
4. *X is called a* dg-$\mathcal{B}$ complex *if $X_n \in \mathcal{B}$ for each n, and* $\mathrm{Hom}(A, X)$ *is exact whenever A is an $\mathcal{A}$ complex.*

We denote the class of $\mathcal{A}$ complexes by $\tilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $dg\tilde{\mathcal{A}}$. Similarly, the class of $\mathcal{B}$ complexes is denoted by $\tilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes is denoted by $dg\tilde{\mathcal{B}}$.

Inspired by [5, Theorem 2.5], we present the following theorem.

Theorem 4.5. *Suppose that $\mathcal{T}$ is a generalized tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$. *Let $\mathcal{A} = {}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})$, $\mathcal{B} = \mathcal{T}^{\perp_\infty}$. Then there is an abelian model structure on* $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$ *given as follows:*

1. *Weak equivalences are quasi-isomorphisms,*
2. *Cofibrations (trivial cofibrations) consist of all the monomorphisms f such that $\mathrm{Ext}^1_{\mathrm{C}(\mathrm{Mod}(\mathcal{C}))}(\mathrm{Coker}\, f, X) = 0$ for any $X \in \tilde{\mathcal{B}}$($\mathrm{Coker}\, f \in \tilde{\mathcal{A}}$),*
3. *Fibrations (trivial fibrations) consist of all the epimorphisms g such that $\mathrm{Ext}^1_{\mathrm{C}(\mathrm{Mod}(\mathcal{C}))}(X, \mathrm{Ker}\, g) = 0$ for any $X \in \tilde{\mathcal{A}}$($\mathrm{Ker}\, g \in \tilde{\mathcal{B}}$).*

The homotopy category of this model category is $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

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Proof. Since Mod($\mathcal{C}$) is a Grothendieck category, it is a bicomplete abelian category by [30, Chapter V] and [20, Section 1.2]. Because $({}^{\perp\infty}(\mathcal{T}^{\perp\infty}), \mathcal{T}^{\perp\infty})$ is a hereditary and complete cotorsion pair by Theorem 4.3, it follows from [34, Theorem 2.4] that there are two induced complete cotorsion pairs $(\tilde{\mathcal{A}}, \text{dg-}\mathcal{B})$ and $(\text{dg-}\mathcal{A}, \tilde{\mathcal{B}})$. So we claim that $X \in \text{dg-}\mathcal{A}$ if and only if $\mathrm{Ext}^1_{\mathrm{C(Mod}(\mathcal{C}))}(X, B) = 0$ for any $B \in \tilde{\mathcal{B}}$ and $Y \in \text{dg-}\mathcal{B}$ if and only if $\mathrm{Ext}^1_{\mathrm{C(Mod}(\mathcal{C}))}(A, Y) = 0$ for any $A \in \tilde{\mathcal{A}}$.

Next, we know from [13, Corollary 3.8] that there is an abelian model structure on C(Mod($\mathcal{C}$)). Furthermore, weak equivalences, cofibrantions, and fibrations are described exactly as in the statements. Observe that exact dg-$\mathcal{A}$ complexes are exactly $\mathcal{A}$ complexes by [34, Theorem 2.5]. In particular, $f : M \to N$ is a trivial cofibration if and only if there is an exact sequence $0 \to M \xrightarrow{f} N \to L \to 0$ such that f is a quasi-isomorphism and Coker $f \in \text{dg-}\mathcal{A}$ if and only if there is an exact sequence $0 \to M \xrightarrow{f} N \to L \to 0$ such that $L \in \tilde{\mathcal{A}}$. The case for trivial fibrations can be shown similarly. The last statement follows from [14, Introduction]. □

According to [17, 18], suppose that an abelian category $\mathcal{A}$ has a model structure, X is *trivial* if $0 \to X$ is a weak equivalence, X is *cofibrant* if $0 \to X$ is a cofibration and X is *fibrant* if $X \to 0$ is a fibration.

Corollary 4.6. *In the notations of Theorem 4.5, then we have the following statements.*

(1) *X is trivial if and only if X is exact.*
(2) *C is a cofibrant if and only if $C \in \text{dg-}\mathcal{A}$.*
(3) *F is a fibrant if and only if $F \in \text{dg-}\mathcal{B}$ if and only if F has all the terms in $\mathcal{B}$.*

Proof. (1), (2) and the first equivalence of (3) follow from Theorem 4.5 easily. We only need to show that $F \in \text{dg-}\mathcal{B}$ if and only if F all the terms in $\mathcal{B}$. If $F \in \text{dg-}\mathcal{B}$, then F has all the terms in $\mathcal{B}$. Conversely, if F has all the terms in $\mathcal{B}$, since $(\tilde{\mathcal{A}}, \text{dg-}\mathcal{B})$ is a complete cotorsion pair by the proof of Theorem 4.5, there is an exact sequence $0 \to F \to B \to A \to 0$ with $B \in \text{dg-}\mathcal{B}$ and $A \in \tilde{\mathcal{A}}$. The fact that the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary implies that $A_n \in \mathcal{B}$ for each n. For any $T \in \mathcal{T}$ and any $i \in \mathbb{Z}$, since T has finite projective dimension, we have that $\mathrm{Ext}^{j\geqslant 1}_{\mathcal{C}}(T, Z_i(A)) = 0$ by dimension shifting. Thus, $A \in \tilde{\mathcal{B}}$. Let $X \in \tilde{\mathcal{A}}$. Since all components of F are in $\mathcal{B}$ and all components of X are in $\mathcal{A}$, we deduce that the sequence $0 \to \mathrm{Hom}(X, F) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, A) \to 0$ is exact. Observe that $A \in \tilde{\mathcal{B}} \subseteq \text{dg-}\mathcal{B}$ and $B \in \text{dg-}\mathcal{B}$. The complexes $\mathrm{Hom}(X, B)$ and $\mathrm{Hom}(X, A)$ are exact. So is $\mathrm{Hom}(X, F)$. Therefore, $F \in \text{dg-}\mathcal{B}$. □

Corollary 4.7. *In the notations of Theorem 4.5, let F be a complex with terms in $\mathcal{B}$ and let C be cofibrant in the model structure induced by $\mathcal{T}$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{K(\,Mod}(\mathcal{C}))}(C, F) \cong \mathrm{Hom}_{\mathrm{D(\,Mod}(\mathcal{C}))}(C, F).$$

In particular, this applies to the complexes C bounded below and with terms in $\mathcal{A}$.

Proof. We know by Corollary 4.6(3) that F is a fibrant. Then it follows from [14] that $\mathrm{Hom}_{\mathrm{K(Mod}(\mathcal{C}))}(C, F) \cong \mathrm{Hom}_{\mathrm{D(Mod}(\mathcal{C}))}(C, F)$. In particular, if C is a bounded below complex with terms in $\mathcal{A}$, then $C \in \text{dg-}\mathcal{A}$ by [12, Lemma 3.4(1)]. It implies that C is a cofibrant by Corollary 4.6(2). So the equivalence also applies to C. □

5. A t-structure induced by a generalized tilting subcategory

In this section, we mainly show that there exists a t-structure on the derived category D(Mod($\mathcal{C}$)), relating a generalized tilting subcategory $\mathcal{T}$ of Mod($\mathcal{C}$). For the sake of completeness, let us recall the definition of a t-structure.

Definition 5.1. ([7]). *Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ with the following properties: write $\mathcal{D}^{\leqslant n} = \Sigma^{-n}\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant n} = \Sigma^{-n}\mathcal{D}^{\geqslant 0}$ for $n \in \mathbb{Z}$.*

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(1) $\mathrm{Hom}_{\mathcal{D}}(X,Y)=0$ *for all* $X\in\mathcal{D}^{\leqslant 0}$ *and* $Y\in\mathcal{D}^{\geqslant 1}$.
(2) $\mathcal{D}^{\leqslant 0}\subseteq\mathcal{D}^{\leqslant 1}$ *and* $\mathcal{D}^{\geqslant 0}\supseteq\mathcal{D}^{\geqslant 1}$.
(3) *For every object* $Z\in\mathcal{D}$ *there is a distinguished triangle* $X\to Z\to Y\to\Sigma X$ *with* $X\in\mathcal{D}^{\leqslant 0}$ *and* $Y\in\mathcal{D}^{\geqslant 1}$.

Notation 5.2. *Suppose that* $\mathcal{T}$ *is a generalized tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$. *For any* $k\in\mathbb{Z}$, *we denote by* $\mathcal{D}_{\mathcal{T}}^{\leqslant k}$ *and* $\mathcal{D}_{\mathcal{T}}^{\geqslant k}$ *the full subcategories of* $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ *given by*

$$\mathcal{D}_{\mathcal{T}}^{\leqslant k}=\{X\in\mathrm{D}(\mathrm{Mod}(\mathcal{C}))\mid\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^iT,X)=0\ \textit{for any}\ i<k\ \textit{and}\ T\in\mathcal{T}\},$$

$$\mathcal{D}_{\mathcal{T}}^{\geqslant k}=\{Y\in\mathrm{D}(\mathrm{Mod}(\mathcal{C}))\mid\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^iT,Y)=0\ \textit{for any}\ i>k\ \textit{and}\ T\in\mathcal{T}\}.$$

Let $\mathcal{A}$ be an abelian category and $\mathcal{F}$ be a class of objects in $\mathcal{A}$. Then a morphism $\varphi:F\to M$ of $\mathcal{A}$ is called an $\mathcal{F}$-*precover* of M if $F\in\mathcal{F}$ and $\mathrm{Hom}_{\mathcal{A}}(F',F)\to\mathrm{Hom}_{\mathcal{A}}(F',M)\to 0$ is exact for all $F'\in\mathcal{F}$. If every object of $\mathcal{A}$ has an $\mathcal{F}$-precover, $\mathcal{F}$ is said to be *precovering* [11]. Given $k\in\mathbb{Z}$, we say that a complex $X\in\mathrm{C}^{[k,\infty]}(\mathcal{F})$ if $X_i=0$ for $i<k$ and $X_i\in\mathcal{F}$ for $i\geqslant k$.

Lemma 5.3. *Suppose that* $\mathcal{A}$ *is an abelian category and* $\mathcal{F}$ *is a class of objects in* $\mathcal{A}$. *If* $\mathcal{F}$ *is precovering, then for every complex* X *in* $\mathrm{D}(\mathcal{A})$ *and every* $k\in\mathbb{Z}$, *there is a chain map* $f:F\to X$ *with* $F\in\mathrm{C}^{[k,\infty]}(\mathcal{F})$ *such that* $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma^iF,f)$ *is an isomorphism for any* $F\in\mathcal{F}$ *and any* $i\geqslant k$.

Proof. Given a complex $X:=\cdots\to X_{n+1}\xrightarrow{d_{n+1}^X}X_n\xrightarrow{d_n^X}X_{n-1}\to\cdots$ with $X_n\in\mathcal{A}$. We will inductively construct a chain map $f:X\to F$

$$\begin{tikzcd}
\cdots \arrow[r] & F_{k+1} \arrow[r, "d_{k+1}^F"] \arrow[d, "f_{k+1}"] & F_k \arrow[r] \arrow[d, "f_k"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_{k+1} \arrow[r, "d_{k+1}^X"] & X_k \arrow[r, "d_k^X"] & X_{k-1} \arrow[r, "d_{k-1}^X"] & X_{k-2} \arrow[r] & \cdots
\end{tikzcd}$$

such that $F_i\in\mathcal{F}$ for $i\geqslant 0$. Consider an $\mathcal{F}$-precover of $\mathrm{Ker}\,d_k^X$, $F_k\xrightarrow{\varphi_k}\mathrm{Ker}\,d_k^X$, and let f_k be the composition of φ_k with the inclusion $\mathrm{Ker}\,d_k^X\to X_k$. By induction construct $f_{i+1}:F_{i+1}\to X_{i+1}$ as follows. Having defined $f_i:F_i\to X_i$ and $d_i^F:F_i\to F_{i-1}$. Let $\lambda_i:K_i\to F_i$ be the kernel of d_i^F and let $g_i=f_i\lambda_i$. Consider the pullback P_{i+1} of the maps g_i and d_{i+1}^X. Let $\varphi_{i+1}:F_{i+1}\to P_{i+1}$ be an $\mathcal{F}$-precover of P_{i+1} and let f_{i+1} be the obvious composition. All the maps used in the inductive step are depicted in the following diagram

$$\begin{tikzcd}
F_{i+1} \arrow[rr, "d_{i+1}^F"] \arrow[d, "\varphi_{i+1}"] \arrow[dd, bend right, "f_{i+1}"'] & & F_i \arrow[dd, "f_i"] \\
P_{i+1} \arrow[r, "\alpha_{i+1}"] \arrow[d, "\beta_{i+1}"] & K_i \arrow[ur, "\lambda_i"] \arrow[dr, "g_i"] & \\
X_{i+1} \arrow[rr, "d_{i+1}^X"] & & X_i,
\end{tikzcd}$$

where $d_{i+1}^F=\lambda_i\alpha_{i+1}\varphi_{i+1}$. It is easy to see that F is a complex with all terms in $\mathcal{F}$ and f is a chain map between F and X. Now we claim that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma^iF',X)=0$ for any $F'\in\mathcal{F}$ and any $i\geqslant k$. If $i\geqslant k$, given a map $h:\Sigma^iF'\to X$ in $\mathrm{C}(\mathcal{A})$, our task is to prove that

(a) h factors through $f:F\to X$, and
(b) If h is null-homotopic, so is any such factorization $t:\Sigma^iF'\to F$.

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Regarding (a), since h is a chain map, $d_i^X h_i = 0$. Using the pullback property of P_i, we construct a map $\theta_i : F' \to P_i$ such that $\beta_i\theta_i = h_i$ and $\alpha_i\theta_i = 0$. We can factor θ_i further through the $\mathcal{F}$-precover $\varphi_i : F_i \to P_i$ to obtain a map $t_i : F' \to F_i$. Thus, $d_i^F t_i = \lambda_{i-1}\alpha_i\varphi_i t_i = \lambda_{i-1}\alpha_i\theta_i = 0$. It means that $t : \Sigma^i F' \to F$ is a chain map. On the other hand, as $f_i t_i = \beta_i\varphi_i t_i = \beta_i\theta_i = h_i$, we conclude that $h = ft$.

Regarding (b), assume that h is null-homotopic and $h = ft$ for some $t : \Sigma^i F' \to F$ in $\mathrm{C}(\mathcal{A})$, we will show that t is also null-homotopic. Since h is null-homotopic, there exists a map $s_{i+1} : F' \to X_{i+1}$ such that $d_{i+1}^X s_{i+1} = h_i$. Note that t is a chain map and $\lambda_i : K_i \to F_i$ is the kernel of d_i^F. We get a map $\gamma_i : F' \to K_i$ with $\lambda_i\gamma_i = t_i$. Then $g_i\gamma_i = f_i\lambda_i\gamma_i = f_i t_i = h_i = d_{i+1}^X s_{i+1}$. By the pullback property of P_{i+1}, there is a map $\delta_{i+1} : F' \to P_{i+1}$ such that $s_{i+1} = \beta_{i+1}\delta_{i+1}$ and $\gamma_i = \alpha_{i+1}\delta_{i+1}$. We further factor δ_{i+1} through the $\mathcal{F}$-precover $\varphi_{i+1} : F_{i+1} \to P_{i+1}$ to obtain a map $\eta_{i+1} : F' \to F_{i+1}$. Then $d_{i+1}^F\eta_{i+1} = \lambda_i\alpha_{i+1}\varphi_{i+1}\eta_{i+1} = \lambda_i\alpha_{i+1}\delta_{i+1} = \lambda_i\gamma_i = t_i$. So t is null-homotopic, and we are done. □

Next, we can give a description of the complexes in $\mathcal{D}_{\mathcal{T}}^{\leqslant k}$.

Proposition 5.4. *Let $\mathcal{T}$ be a generalized tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$, *$k \in \mathbb{Z}$ and $\mathcal{D}_{\mathcal{T}}^{\leqslant k}$ as in Notation 5.2. For a complex in* $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$, *the following statements are equivalent.*

(1) $X \in \mathcal{D}_{\mathcal{T}}^{\leqslant k}$.

(2) *X is isomorphic in* $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ *to a complex B of the form*

$$\cdots \to B_{k+2} \to B_{k+1} \to B_k \to 0 \to \cdots$$

with $B_i \in \mathcal{T}^{\perp_\infty}$ for every $i \geqslant k$.

(3) *X is isomorphic in* $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ *to a complex T as in (2), but with $T_i \in \mathrm{Add}(\mathcal{T})$ for every $i \geqslant k$.*

Proof. (3) ⇒ (2) is obvious.

(2) ⇒ (1) For any $T \in \mathcal{T}$ and $i < k$, we have that $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, X) = \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, B) = \mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, B) = \mathrm{H}_i\,\mathrm{Hom}(T, B) = 0$ by Corollary 4.7. So $X \in \mathcal{D}_{\mathcal{T}}^{\leqslant k}$.

(1) ⇒ (3) Every complex X has a fibrant replacement in the model structure defined by the generalized tilting subcategory $\mathcal{T}$, by Corollary 4.6 we can assume that X has terms in $\mathcal{B}$. It follows from Proposition 3.1 that every term of X has an $\mathrm{Add}(\mathcal{T})$-precover. Then by Lemma 5.3 there is a chain map $f : T \to X$ with $T \in \mathrm{C}^{[k,\infty]}(\mathrm{Add}(\mathcal{T}))$ and $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma^i T', f)$ is an isomorphism for any $T' \in \mathcal{T}$ and any $i \geqslant k$. For $i < k$. By assumption on X we know that $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T', X) = 0$ for any $T' \in \mathcal{T}$. Note that $\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T', T) = \mathrm{H}_i\,\mathrm{Hom}(T', T) = 0$ for any $T' \in \mathcal{T}$. Thus, we say that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma^i T', f)$ is an isomorphism for any $i \in \mathbb{Z}$. Let $\mathrm{Cone}(f)$ be the mapping cone of f. It is easy to see that $\mathrm{Cone}(f)$ is a fibrant by Corollary 4.6(3). From the triangle $T \xrightarrow{f} X \to \mathrm{Cone}(f) \to \Sigma T$ in $\mathrm{K}(\mathrm{Mod}(\mathcal{C}))$, we obtain $\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T', \mathrm{Cone}(f)) = 0$ for any $T' \in \mathcal{T}$ and any $i \in \mathbb{Z}$. Moreover, by Corollary 4.7, $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T', \mathrm{Cone}(f)) = 0$ for any $T' \in \mathcal{T}$ and any $i \in \mathbb{Z}$. Because $\mathcal{T}$ satisfies the condition (3) of Definition 2.3, $\mathrm{Cone}(f) = 0$ in $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$. Consequently, f becomes an isomorphism in $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ and T satisfies (3). □

Motivated by [5, Theorem 3.5], we can assign a t-structure to a generalized tilting subcategory $\mathcal{T}$ of $\mathrm{Mod}(\mathcal{C})$.

Theorem 5.5. *Let $\mathcal{T}$ be a generalized tilting subcategory of* $\mathrm{Mod}(\mathcal{C})$ *and $k \in \mathbb{Z}$. Then $(\mathcal{D}_{\mathcal{T}}^{\leqslant k}, \mathcal{D}_{\mathcal{T}}^{\geqslant k})$ forms a t-structure on the derived category* $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

Proof. We will show that $(\mathcal{D}_{\mathcal{T}}^{\leqslant 0}, \mathcal{D}_{\mathcal{T}}^{\geqslant 0})$ is a t-structure, since it is routine to check that the shifted pair $(\mathcal{D}_{\mathcal{T}}^{\leqslant k}, \mathcal{D}_{\mathcal{T}}^{\geqslant k})$ is also a t-structure. The proof follows the pattern of that of [31, Theorem 4.5], but in a dual manner. By the definition of $\mathcal{D}_{\mathcal{T}}^{\leqslant 0}$ and $\mathcal{D}_{\mathcal{T}}^{\geqslant 0}$, it is easy to verify that $\mathcal{D}_{\mathcal{T}}^{\leqslant 0}$(resp. $\mathcal{D}_{\mathcal{T}}^{\geqslant 0}$) is closed under Σ(resp. Σ^{-1}). Thus, we only have to show the conditions (1) and (3) of Definition 5.1

In order to prove (1) of Definition 5.1, we assume that $X \in \mathcal{D}_{\mathcal{T}}^{\leqslant 0}$ and $Y \in \mathcal{D}_{\mathcal{T}}^{\geqslant 1}$. Let $X' = \Sigma X$ and $Y = \Sigma^{-1} Y'$ for some $Y' \in \mathcal{D}_{\mathcal{T}}^{\geqslant 0}$. Then $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(X, Y) = 0$ is equivalent to $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(X', Y') = 0$.

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In view of Proposition 5.4, X' has the form $\cdots \to X_2 \to X_1 \to 0 \to \cdots$ with $X_i \in \mathrm{Add}(\mathcal{T})$ for every $i \geqslant 1$. For every $n \geqslant 1$, we shall denote by $\tau_{\leqslant n}X'$ the brutally truncated complex $\cdots 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to 0$. As $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T', Y') = 0$ for any $T' \in \mathcal{T}$ and any $i > 0$, by induction on n, we know from the triangle $\tau_{\leqslant n-1}X' \to \tau_{\leqslant n}X' \to \Sigma^n X_n \to \Sigma(\tau_{\leqslant n-1}X')$ that $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\tau_{\leqslant n}X', Y') = 0 = \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma(\tau_{\leqslant n}X'), Y')$ for all $n \geqslant 1$. Since $X' = \varinjlim \tau_{\leqslant n}X'$, by [27, Proposition 11.7], there is a triangle in $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ of the form

$$\coprod_{n\geqslant 1} \tau_{\leqslant n}X' \to \coprod_{n\geqslant 1} \tau_{\leqslant n}X' \to X' \to \coprod_{n\geqslant 1} \Sigma(\tau_{\leqslant n}X').$$

Hence, $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(X', Y') = 0$.

Now we will prove (3) of Definition 5.1. Let $X \in \mathrm{D}(\mathrm{Mod}(\mathcal{C}))$, in view of Corollary 4.6, we may assume that X has all the terms in $\mathcal{T}^{\perp_\infty}$. By Lemma 5.3, there is a chain map $f : F \to X$ with $F \in \mathrm{C}^{[0,\infty]}(\mathrm{Add}(\mathcal{T}))$ and $\mathrm{Hom}_{\mathrm{K}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, f)$ is an isomorphism for any $T \in \mathcal{T}$ and any $i \geqslant 0$. By Corollary 4.7, the same is true for $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, f)$. Furthermore, it is straightforward to check that $F \in \mathcal{D}^{\leqslant 0}_{\mathcal{T}}$. Let $\mathrm{Cone}(f)$ be the mapping cone of f, that is, we have a triangle

$$F \xrightarrow{f} X \to \mathrm{Cone}(f) \to \Sigma F. \tag{5.1}$$

Then necessarily $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, \mathrm{Cone}(f)) = 0$ for any $i > 0$. Indeed, if we apply $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, \quad)$ to (5.1), we get an exact sequence

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, F) \xrightarrow{f'} \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, X) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, \mathrm{Cone}(f))$$
$$\to \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, \Sigma F).$$

Now f' is an isomorphism and $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, \Sigma F) = 0$ since $F \in \mathcal{D}^{\leqslant 0}_{\mathcal{T}}$. Thus, $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(T, \mathrm{Cone}(f)) = 0$. Hence, $\mathrm{Cone}(f) \in \mathcal{D}^{\geqslant 1}_{\mathcal{T}}$. □

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