Another generalisation of Stewart's Theorem

PANAGIOTIS T. KRASOPOULOS

Introduction

The aim of this Article is to present a natural generalisation of a well-known result from plane geometry, called Stewart's Theorem. We will present our result in the Euclidean space $\mathbb{R}^n$, but we will prove it only for $n = 3$. Since its proof for $n > 3$ is involved, our intention is to give the details in another Article. A search in the relative literature returns many articles which generalise Stewart's Theorem in different directions. We first refer the reader to [1], where a collection of generalisations of Stewart's theorem is presented. In [2], two results are given, which describe a relation between k points in the Euclidean space $\mathbb{R}^n$ and their weighted average. If we apply these theorems to the plane, we straightforwardly get Stewart's Theorem. Another result is given in [3], which is also reproduced in [1]. This result considers $n + 1$ points in $\mathbb{R}^n$ which belong to a hyperplane U of dimension $n - 1$ and another point $A \notin U$, and presents an interesting relation that combines distances between each of these points and A, hypervolumes of simplices and powers of points with respect to hyperspheres. Another relative result which involves convex quadrilaterals on the plane is proved in [4]. Lastly, in [5], a generalisation of [4] is proved for $2k$ points in $\mathbb{R}^n$, and a relation is found between distances of these points and the distance of two corresponding averages (of these points). These articles support the fact that there can be many interesting generalisations of Stewart's Theorem in different directions.

Let us now first recall Stewart's Theorem from plane geometry:

Theorem 1 (Stewart's Theorem): Let a triangle ABC and let a point $G = u_1 B + u_2 C$ on BC, where $u_1, u_2 \geqslant 0$ and $u_1 + u_2 = 1$. Then

$$[AG]^2 = u_1 [AB]^2 + u_2 [AC]^2 - u_1 u_2 [BC]^2. \tag{1}$$

Where $[AB]$ denotes the length of the line segment AB. We should note here that if $G = \frac{1}{2}(B + C)$ is the midpoint of BC then Stewart's Theorem becomes Apollonius' Theorem, a result known from antiquity. In the next Section we will present and prove by using only elementary tools, our result in the Euclidean space $\mathbb{R}^3$.

Generalisation in $\mathbb{R}^3$

In order to present a generalisation in $\mathbb{R}^3$ we first need to consider the generalisation of a triangle in the Euclidean space $\mathbb{R}^3$. This can be a tetrahedron, and so we consider a non-degenerate tetrahedron $ABCD$ with faces ABC, ABD, ACD and BCD. Note that now A, B, C and D are points in the Euclidean space $\mathbb{R}^3$. We also consider a point G on the face BCD such that $G = u_1 B + u_2 C + u_3 D$ with $u_1, u_2, u_3 \geqslant 0$ and $u_1 + u_2 + u_3 = 1$.

https://doi.org/10.1017/mag.2024.116 Published online by Cambridge University Press

Let us also denote by $[ABC]$ the area of the triangle ABC. Finally, we use the notation of a vector as $B - A = \overrightarrow{AB}$. Therefore, our result is stated as follows:

Theorem 2: Let a tetrahedron $ABCD$ and let a point G on the face BCD such that $G = u_1 B + u_2 C + u_3 D$, where $u_1, u_2, u_3 \geqslant 0$ and $u_1 + u_2 + u_3 = 1$. Then

$$u_1 [ABG]^2 + u_2 [ACG]^2 + u_3 [ADG]^2 \\ = u_1 u_2 [ABC]^2 + u_2 u_3 [ACD]^2 + u_3 u_1 [ADB]^2 - u_1 u_2 u_3 [BCD]^2 . \tag{2}$$

Proof: We find it convenient to use the cross product and its properties. It is known that the cross product gives the area of the parallelogram which is spanned by two non-collinear vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The area of a triangle ABC is the half of the area of the corresponding parallelogram and so is given by $[ABC] = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$. Let us first define the following:

$$\vec{b} = \overrightarrow{AC} \times \overrightarrow{AD}, \qquad \vec{c} = \overrightarrow{AD} \times \overrightarrow{AB}, \qquad \vec{d} = \overrightarrow{AB} \times \overrightarrow{AC}.$$

From the above we get

$$[ACD] = \frac{1}{2}\left|\vec{b}\right|, \qquad [ADB] = \frac{1}{2}\left|\vec{c}\right|, \qquad [ABC] = \frac{1}{2}\left|\vec{d}\right|. \tag{3}$$

Our aim now is to write all the areas which appear in (2) as functions of the vectors $\vec{b}$, $\vec{c}$ and $\vec{d}$. We first need to express $\overrightarrow{AG}$ as follows

$$\overrightarrow{AG} = G - A = u_1 B + u_2 C + u_3 D - (u_1 + u_2 + u_3) A = u_1 \overrightarrow{AB} + u_2 \overrightarrow{AC} + u_3 \overrightarrow{AD}.$$

Now we can calculate,

$$[ABG] = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AG}\right|.$$

By substituting $\overrightarrow{AG}$ in the above equation and using the fact that $\overrightarrow{AB} \times \overrightarrow{AG} = \vec{O}$ where $\vec{O}$ is the zero vector and also the fact that the cross product distributes over addition, we get

$$[ABG] = \frac{1}{2}\left|u_2 \vec{d} - u_3 \vec{c}\right|,$$

where we have also used the anti-commutative property of the cross product, i.e. $\overrightarrow{AB} \times \overrightarrow{AD} = -\overrightarrow{AD} \times \overrightarrow{AB} = -\vec{c}$. Similarly, we calculate the areas of $[ACG]$ and $[ADG]$. Thus, we have

$$[ABG] = \frac{1}{2}\left|u_2 \vec{d} - u_3 \vec{c}\right|, \qquad [ACG] = \frac{1}{2}\left|u_3 \vec{b} - u_1 \vec{d}\right|, \qquad [ADG] = \frac{1}{2}\left|u_1 \vec{c} - u_2 \vec{b}\right|. \tag{4}$$

Finally we want to express $[BCD]$ as a function of $\vec{b}$, $\vec{c}$ and $\vec{d}$. Thus

$$[BCD] = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BD}\right|.$$

https://doi.org/10.1017/mag.2024.116 Published online by Cambridge University Press

But we have $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$ and $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$. Hence

$$\overrightarrow{BC} \times \overrightarrow{BD} = \left(\overrightarrow{BA} + \overrightarrow{AC}\right) \times \left(\overrightarrow{BA} + \overrightarrow{AD}\right)$$
$$= \overrightarrow{BA} \times \overrightarrow{AD} + \overrightarrow{AC} \times \overrightarrow{BA} + \overrightarrow{AC} \times \overrightarrow{AD} = \vec{b} + \vec{c} + \vec{d}.$$

Where we have used the fact that $\overrightarrow{BA} = -\overrightarrow{AB}$ and again the anti-commutative property of the cross product. Thus, we have shown that

$$[BCD] = \frac{1}{2}\left|\vec{b} + \vec{c} + \vec{d}\right|. \tag{5}$$

As the next step, we need to square all these areas. For this we need the dot product and its properties. Recall that the dot product is commutative and like the cross product distributes over addition. We will also use freely the fact that $\vec{s}^2 = \vec{s}\vec{s} = \left|\vec{s}\right|^2$. Therefore squaring (3), (4) and (5) we get

$$\begin{aligned}
&[ACD]^2 = \frac{1}{4}\vec{b}^2, \qquad [ADB]^2 = \frac{1}{4}\vec{c}^2, \qquad [ABC]^2 = \frac{1}{4}\vec{d}^2,\\
&[ABG]^2 = \frac{1}{4}\left(u_2^2\vec{d}^2 + u_3^2\vec{c}^2 - 2u_2u_3\vec{c}\vec{d}\right),\\
&[ACG]^2 = \frac{1}{4}\left(u_1^2\vec{d}^2 + u_3^2\vec{b}^2 - 2u_1u_3\vec{b}\vec{d}\right),\\
&[ADG]^2 = \frac{1}{4}\left(u_1^2\vec{c}^2 + u_2^2\vec{b}^2 - 2u_1u_2\vec{b}\vec{c}\right),\\
&[BCD]^2 = \frac{1}{4}\left(\vec{b} + \vec{c} + \vec{d}\right)^2.
\end{aligned} \tag{6}$$

Now using (6) we will prove (2). First, we rewrite (2) as

$$u_1u_2u_3[BCD]^2$$
$$= u_1u_2[ABC]^2 + u_2u_3[ACD]^2 + u_3u_1[ADB]^2 - u_1[ABG]^2 - u_2[ACG]^2 - u_3[ADG]^2.$$

Using (6), the right-hand side of the above equation can be written as

$$\begin{aligned}
&u_1u_2[ABC]^2 + u_2u_3[ACD]^2 + u_3u_1[ADB]^2 - u_1[ABG]^2 - u_2[ACG]^2 - u_3[ADG]^2\\
&= \frac{1}{4}u_1u_2\vec{d}^2 + \frac{1}{4}u_2u_3\vec{b}^2 + \frac{1}{4}u_3u_1\vec{c}^2 - \frac{1}{4}u_1u_2^2\vec{d}^2 - \frac{1}{4}u_1u_3^2\vec{c}^2 - \frac{1}{4}u_2u_1^2\vec{d}^2\\
&\quad - \frac{1}{4}u_2u_3^2\vec{b}^2 - \frac{1}{4}u_3u_1^2\vec{c}^2 - \frac{1}{4}u_3u_2^2\vec{b}^2 + \frac{1}{4}2u_1u_2u_3\left(\vec{c}\vec{d} + \vec{b}\vec{d} + \vec{b}\vec{c}\right)\\
&= \frac{1}{4}u_1u_2(1 - u_1 - u_2)\vec{d}^2 + \frac{1}{4}u_2u_3(1 - u_2 - u_3)\vec{b}^2\\
&\quad + \frac{1}{4}u_3u_1(1 - u_3 - u_1)\vec{c}^2 + \frac{1}{4}2u_1u_2u_3\left(\vec{c}\vec{d} + \vec{b}\vec{d} + \vec{b}\vec{c}\right)\\
&= \frac{1}{4}u_1u_2u_3\left(\vec{b}^2 + \vec{c}^2 + \vec{d}^2 + 2\left(\vec{c}\vec{d} + \vec{b}\vec{d} + \vec{b}\vec{c}\right)\right)\\
&= \frac{1}{4}u_1u_2u_3\left(\vec{b} + \vec{c} + \vec{d}\right)^2 = u_1u_2u_3[BCD]^2.
\end{aligned}$$

Thus we have shown (2) and the proof is complete.

https://doi.org/10.1017/mag.2024.116 Published online by Cambridge University Press

We have now proved the generalisation of Stewart's Theorem in $\mathbb{R}^3$. In the proof, we used only elementary tools like the cross product and the dot product. In the next Section, we will present our generalisation in the Euclidean space $\mathbb{R}^n$.

Generalisation in $\mathbb{R}^n$

We have seen that Stewart's Theorem is a result in $\mathbb{R}^2$ which involves lengths of line segments, and Theorem 2 is a result in $\mathbb{R}^3$ which involves areas of triangles. Respectively, we expect that a result in $\mathbb{R}^4$ will involve volumes of tetrahedra and in $\mathbb{R}^n$ will involve hypervolumes of simplices. Therefore in order to present a generalisation in $\mathbb{R}^n$ we need to consider simplices in $\mathbb{R}^n$. A simplex $AA_1 \dots A_n$ in $\mathbb{R}^n$ is defined by $n + 1$ points in general position, $A, A_1, \dots, A_n \in \mathbb{R}^n$. Let us also note that a simplex $AA_1 \dots A_n$ in $\mathbb{R}^n$ has as its facets (like the sides of a triangle or the faces of a tetrahedron) objects in $\mathbb{R}^n$, which are simplices of dimension $n - 1$. Thus, any combination of n points of its $n + 1$ vertices $A, A_1, \dots, A_n$ is a facet of this simplex. In order to generalise the concept of length, area and volume in more dimensions, we say that a facet, e.g. $AA_1 \dots A_{n-1}$, of this simplex has as its $(n - 1)$ *content* (hypervolume) the quantity $[AA_1 \dots A_{n-1}]$. In this respect, a triangle ABC is a simplex in $\mathbb{R}^2$ and its side AB has 1 *content* (length) equal to $[AB]$. A tetrahedron $ABCD$ is a simplex in $\mathbb{R}^3$ and a face e.g. BCD of it, has 2 *content* (area) equal to $[BCD]$. A simplex $ABCDE$ in $\mathbb{R}^4$ has as its facets tetrahedra and a facet of it, e.g. the tetrahedron $ABCD$, has 3 *content* (volume) equal to $[ABCD]$. As we saw, the same concept holds for simplices in $\mathbb{R}^n$. So a simplex $AA_1 \dots A_n$ in $\mathbb{R}^n$, and a facet of it e.g. $AA_1 \dots A_{n-1}$, has $(n - 1)$ *content* (hypervolume) equal to $[AA_1 \dots A_{n-1}]$. Having all this in mind, we are now ready to present our theorem in $\mathbb{R}^n$:

Theorem 3: Let a simplex $AA_1 \dots A_n$ in $\mathbb{R}^n$ and a point $G = \sum_{i=1}^n u_i A_i$ which belongs to the facet $A_1 \dots A_n$, where $u_1, \dots, u_n \geqslant 0$ and $u_1 + \dots + u_n = 1$. Then

$$\sum_{\{x_1, \dots, x_{n-2}\} \in \mathbf{X}} u_{x_1} \dots u_{x_{n-2}} [AA_{x_1} \dots A_{x_{n-2}} G]^2$$

$$= \sum_{\{y_1, \dots, y_{n-1}\} \in \mathbf{Y}} u_{y_1} \dots u_{y_{n-1}} [AA_{y_1} \dots A_{y_{n-1}}]^2 - \left(\prod_{i=1}^n u_i\right) [A_1 \dots A_n]^2. \qquad (7)$$

Where the set $\mathbf{X}$ contains the sets of all the combinations of $n - 2$ elements from $\{1, \dots, n\}$. Its cardinality is $|\mathbf{X}| = \binom{n}{n-2} = \frac{n(n-1)}{2}$. Correspondingly, the set $\mathbf{Y}$ contains the sets of all the combinations of $n - 1$ elements from $\{1, \dots, n\}$. Its cardinality is $|\mathbf{Y}| = \binom{n}{n-1} = n$.

As we have said, we will not prove Theorem 3 here. Our intention is to present its proof in another article. But for now, we will see how Theorem 3 becomes Stewart's Theorem for $n = 2$ and Theorem 2 for $n = 3$.

https://doi.org/10.1017/mag.2024.116 Published online by Cambridge University Press

Let $n = 2$, then the simplex $AA_1 \ldots A_n$ becomes the triangle AA_1A_2 and we have a point $G = u_1A_1 + u_2A_2$ with $u_1 + u_2 = 1$ and $u_1, u_2 \geqslant 0$. In this case we have that $\mathbf{X} = \varnothing$, and the left-hand side of (7) trivially becomes $[AG]^2$. We also have that $\mathbf{Y} = \{\{1\}, \{2\}\}$, and so the right-hand side of (7) becomes $u_1[AA_1]^2 + u_2[AA_2]^2 - u_1u_2[A_1A_2]^2$. Therefore

$$[AG]^2 = u_1[AA_1]^2 + u_2[AA_2]^2 - u_1u_2[A_1A_2]^2,$$

and we observe that this equation is identical to (1), if we just set $A_1 = B$ and $A_2 = C$.

For $n = 3$ we have a tetrahedron $AA_1A_2A_3$ and a point $G = u_1A_1 + u_2A_2 + u_3A_3$ with $u_1 + u_2 + u_3 = 1$ and $u_1, u_2, u_3 \geqslant 0$. In this case we have $\mathbf{X} = \{\{1\}, \{2\}, \{3\}\}$ and the left-hand side of (7) becomes

$$u_1[AA_1G]^2 + u_2[AA_2G]^2 + u_3[AA_3G]^2.$$

Moreover, $\mathbf{Y} = \{\{1,2\}, \{2,3\}, \{3,1\}\}$ and the right-hand side of (7) becomes

$$u_1u_2[AA_1A_2]^2 + u_2u_3[AA_2A_3]^2 + u_3u_1[AA_3A_1]^2 - u_1u_2u_3[A_1A_2A_3]^2.$$

Hence

$$u_1[AA_1G]^2 + u_2[AA_2G]^2 + u_3[AA_3G]^2$$

$$= u_1u_2[AA_1A_2]^2 + u_2u_3[AA_2A_3]^2 + u_3u_1[AA_3A_1]^2 - u_1u_2u_3[A_1A_2A_3]^2,$$

and this equation is identical to (2) if we set $A_1 = B$, $A_2 = C$ and $A_3 = D$. Consequently, Theorem 3 for $n = 2$ is Stewart's Theorem and for $n = 3$ it is Theorem 2, as was expected.

Acknowledgement

I would like to thank Professor Mowaffaq Hajja for stimulating discussions about this work and simplex geometry generally.

References

1. F. Bellot Rosado, *Quelques généralisations du théoréme de Stewart*, 40e Congrés de la SBPMef Namur (août 2014).
2. T. M. Apostol, M.A. Mnatsakanian, Sums of squares of distances in m-space, *Amer. Math. Monthly* **110** (June-July 2003) pp. 516-526.
3. O. Bottema, Eine Erweiterung der Stewartschen Formel, *Elem. Math.* **34** (1979) pp. 138-140.
4. Ali R. Amir-Moez, J. D. Hamilton, A generalized parallelogram law, *Math. Mag.* **49** (March 1976) pp. 88-89.

5. A. J. Douglas, A generalization of Apollonius theorem, *Math. Gaz.* **65** (March 1981) pp. 19-22.

10.1017/mag.2024.116

Published by
Cambridge University Press
on behalf of The Mathematical Association

PANAGIOTIS T. KRASOPOULOS
Department of Informatics, KEAO
Electronic National Social Security Fund
12 Patision Street,
10677 Athens, Greece
e-mail: *pan_kras@yahoo.gr*

Nemo (continued from page 477)

3. But one day, after standing for a while at the window, looking down on the street where he had first seen the beloved form of Ericson, a certain old mood began to revive in him. He had been working at quadratic equations all the morning; he had been foiled in the attempt to find the true algebraic statement of a very tough question involving various ratios.

4. And how I did cram! I had two years' new work to do in a third of a year. For five weeks I crammed, until simultaneous quadratic equations and chemical formulas fairly oozed from my ears. And then the master of the academy took me aside. He was very sorry, but he was compelled to give me back my tuition fee and to ask me to leave the school.

5. I asked this great creature in what other branches of education she instructed her pupils? 'The modern languages,' says she modestly: 'French, German, Spanish, and Italian, Latin and the rudiments of Greek if desired. English of course; the practice of Elocution, Geography, and Astronomy, and the Use of the Globes, Algebra (but only as far as quadratic equations); for a poor ignorant female, you know, Mr. Snob, cannot be expected to know everything.

6. The equation on the page of his scribbler began to spread out a widening tail, eyed and starred like a peacock's; and, when the eyes and stars of its indices had been eliminated, began slowly to fold itself together again.