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ABSTRACT

Laumon introduced the local Fourier transform for ℓ -adic Galois representations of local fields, of equal characteristic p different from ℓ , as a powerful tool for studying the Fourier–Deligne transform of ℓ -adic sheaves over the affine line. In this article, we compute explicitly the local Fourier transform of monomial representations satisfying a certain ramification condition, and deduce Laumon’s formula relating the ε -factor to the determinant of the local Fourier transform under the same condition.

1. Introduction

1.1 In his seminal article [Lau87], Laumon introduced the local Fourier transform for ℓ -adic Galois representations of local fields, of equal characteristic p different from ℓ , providing a powerful tool for studying the Fourier–Deligne transform of ℓ -adic sheaves over the affine line. He used it to prove that the constant of the functional equation of the L -function associated to an ℓ -adic representation of a function field is a product of local constants, also known as ε -factors. As a key step, he gave a cohomological interpretation of the ε -factor in terms of the determinant of the local Fourier transform. In this article, we compute explicitly the local Fourier transform of monomial representations satisfying a certain ramification condition, and deduce Laumon’s formula for ε -factors under the same condition. Our approach, inspired by our ramification theory [AS09], is local and geometric, while Laumon’s approach is global, combining arithmetic and geometric arguments.

1.2 One of the main innovations of [Lau87], leading to the local Fourier transform, is Laumon’s *principle of stationary phase*, which has its origins in the classical theory of asymptotic integrals (cf. [Katz89]). We briefly recall the classical theory [Die68, IV, §4]. Given two functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$, we are interested in studying the asymptotic behavior at ∞ of the integral, depending on a real parameter t ,

$$I(t) = \int f(x)e^{it\varphi(x)} dx.$$

If the derivative of φ does not vanish at any point in $\text{Supp}(f)$, then $I(t)$ is rapidly decreasing at ∞ . It follows that if φ has only finitely many critical points in $\text{Supp}(f)$, then the asymptotic behavior of $I(t)$ at ∞ is a finite sum of contributions, one from each critical point of φ in $\text{Supp}(f)$. If, moreover, all critical points of φ are non-degenerate (i.e. the second derivative of φ does not vanish at these points), then one can give a very explicit description of $I(t)$ as t tends to ∞ .

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1.3 Let k be a perfect field of characteristic p , let $A = \text{Spec}(k[x])$ and $\check{A} = \text{Spec}(k[\check{x}])$ be the affine lines over k (equipped with coordinates x and \check{x}), let ℓ be a prime number different from p , and let $\psi_0: \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial additive character. We denote by P and \check{P} the projective lines over k , completions of A and \check{A} , respectively, and by $\infty \in P$ and $\check{\infty} \in \check{P}$ the points at infinity. For closed points $z \in P$ and $\check{z} \in \check{P}$, we denote by T_z and $\check{T}_{\check{z}}$ the henselizations of P and \check{P} at z and \check{z} , respectively, and by τ_z and $\check{\tau}_{\check{z}}$ their generic points. Let \mathcal{F} be a $\overline{\mathbb{Q}}_\ell$ -sheaf over A . The analogue of the integral $I(t)$ is provided by the Fourier–Deligne transform $\mathfrak{F}_{\psi_0}(\mathcal{F})$, which is a complex of ℓ -adic sheaves on \check{A} (cf. §3.2); in fact, the precise analogue of $I(t)$ is the sheaf $\mathcal{H}^1(\mathfrak{F}_{\psi_0}(\mathcal{F}))$ (where t is replaced by \check{x}). The ‘asymptotic behavior’ of this sheaf at $\check{\infty}$ is encoded in its restriction to $\check{\tau}_\infty$, which corresponds to an ℓ -adic representation of the absolute Galois group of the function field $k(\check{\tau}_\infty)$ of \check{T}_∞ . Let U be a dense open subscheme of A such that $A - U \subset A(k)$. If \mathcal{F} is the extension by 0 of a smooth $\overline{\mathbb{Q}}_\ell$ -sheaf over U , Laumon proved that we have a canonical decomposition

$$\mathcal{H}^1(\mathfrak{F}_{\psi_0}(\mathcal{F}))|_{\check{\tau}_\infty} \simeq \bigoplus_{z \in P-U} \mathfrak{F}_{\psi_0}^{(z, \check{\infty})}(\mathcal{F}|_{\tau_z}), \tag{1.3.1}$$

where the factor $\mathfrak{F}_{\psi_0}^{(z, \check{\infty})}(\mathcal{F}|_{\tau_z})$ is the *local Fourier transform* of $\mathcal{F}|_{\tau_z}$ at $(z, \check{\infty})$. The latter transformation is a functor from $\overline{\mathbb{Q}}_\ell$ -sheaves over τ_z to $\overline{\mathbb{Q}}_\ell$ -sheaves over $\check{\tau}_\infty$, defined by Laumon using vanishing cycles (cf. §3.4).

1.4 We assume in the following that $p > 2$. Let S be the spectrum of a henselian discrete valuation ring, η (respectively, s) be its generic (respectively, closed) point, $v: S \rightarrow A$ and $\check{v}: S \rightarrow \check{P}$ be two morphisms such that $\check{v}(s) = \check{\infty}$. We put $z = v(s)$ and assume for simplicity (only in the introduction) that $z \in A(k)$. We denote by $f: S \rightarrow T_z$ and $\check{f}: S \rightarrow \check{T}_\infty$ the morphisms induced by v and \check{v} respectively (cf. §1.3). Assume that f and \check{f} are finite and étale at η . Let \mathcal{G} be a $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over η . Our main theorem (Theorem 3.7) says that if $(\mathcal{G}, f, \check{f})$ is a *Legendre triple*, a condition that is defined below, then we have a canonical isomorphism of sheaves over $\check{\tau}_\infty$

$$\mathfrak{F}_{\psi_0}^{(z, \check{\infty})}(f_*\mathcal{G}) \simeq \check{f}_* \left(\mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K} \left(-\frac{1}{2} \frac{dc}{db} \right) \otimes \mathcal{Q} \right), \tag{1.4.1}$$

where the rank-one sheaf between brackets on the right-hand side is defined as follows: the pull-backs of the coordinates x and \check{x} by f and \check{f} define two functions on η , denoted respectively by b and c . The sheaf $\mathcal{L}_{\psi_0}(bc)$ is the Artin–Schreier sheaf over η associated to the additive character ψ_0 and the function bc (cf. §3.1). The sheaf $\mathcal{K}(-(1/2)(dc/db))$ is the Kummer sheaf over η associated to the unique non-trivial character $\kappa_0: \mu_2(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and the function $-(1/2)(dc/db)$ (cf. §3.3 and Remark 3.8). Finally, \mathcal{Q} is the $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over $\text{Spec}(k)$ corresponding to the quadratic Gauss sum defined by ψ_0 and κ_0 (see §3.3).

We also prove variants of (1.4.1) for the local Fourier transforms $\mathfrak{F}_{\psi_0}^{(\infty, \check{\infty})}$ and $\mathfrak{F}_{\psi_0}^{(\infty, \check{0})}$ (Theorem 3.9).

1.5 The notion of a Legendre triple relies on ramification theory. Let R be the completion of the local ring of S , K be its fraction field, t be a uniformizer of R , ord be the valuation of K normalized by $\text{ord}(t) = 1$. For any $y \in K$, we put $y' = dy/dt$. Ramification theory of Artin–Schreier–Witt sheaves over $\text{Spec}(K)$ is described in terms of the Kato filtration on the

ring $W_{m+1}(K)$ of Witt vectors of length $m + 1$, and the homomorphism of de Rham–Witt $F^m d: W_{m+1}(K) \rightarrow \Omega_K^1$. We refer to § 2 for a short review of this theory, and to [AS09, Kato89] for more details.

Let $a = (a_0, \dots, a_m)$ be a non-zero element of $W_{m+1}(K)$, α be the element of K defined by the equation $F^m d(a) = \alpha dt$ (see (2.4.2)), b, c be non-zero elements of K (for which we will take the functions provided by § 1.4). We say that (a, b, c) is a *strong Legendre triple* if the following relations are satisfied:

$$p^{m-i} \text{ord}(a_i) \geq -n = \text{ord}(t\alpha) \quad (\forall 0 \leq i \leq m), \tag{1.5.1}$$

$$F^m d(a) + cdb = 0, \tag{1.5.2}$$

$$2\text{ord}(tb'/b) + p\text{ord}(tc'/c) < (p - 2)n. \tag{1.5.3}$$

The inequalities in (1.5.1) mean that a belongs to the level n of the Kato filtration of $W_{m+1}(K)$; the equality in (1.5.1) implies that n is the Swan conductor of the sheaf of rank one over $\text{Spec}(K)$ defined by a . We may consider equation (1.5.2) as an analogue of the Legendre transform, used in the method of the saddle point [Die68, IX, § 1], and the ramification condition (1.5.3) as an analogue of the convexity condition required for this method (or, equivalently, the non-degenerate critical points condition in the principle of stationary phase).

In general, a and b are given, and c will be defined by the equations above. To allow more flexibility, we replace (1.5.2) by the following weaker but sufficient condition:

$$2\text{ord}(\alpha + cb') \geq -n + \text{ord}(tc'/c); \tag{1.5.4}$$

we say then that (a, b, c) is a *Legendre triple* (cf. § 2.13).

1.6 We take again the notation and assumptions of § 1.4. We say that (\mathcal{G}, b, c) , or $(\mathcal{G}, f, \check{f})$, is a *Legendre triple* if we can write \mathcal{G} as a tensor product of two $\overline{\mathbb{Q}}_\ell$ -sheaves of rank one over η , $\mathcal{G} \simeq \mathcal{G}_t \otimes \mathcal{G}_w$, where \mathcal{G}_t is tamely ramified at s and \mathcal{G}_w is trivialized by a cyclic extension of order p^{m+1} of η ($m \geq 0$) and satisfies the following conditions: there exists $a \in W_{m+1}(K)$ such that (a, b, c) is a Legendre triple and the pull-back of \mathcal{G}_w to $\text{Spec}(K)$ is associated to a (cf. Definition 2.16). It follows in particular that \mathcal{G} is wildly ramified.

Suppose given the pair (\mathcal{G}, f) , we would like to compute the local Fourier transform $\mathcal{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G})$. In order to apply (1.4.1), since the morphism \check{f} is completely determined by c , the problem is to find a non-zero function c over η such that (\mathcal{G}, b, c) is a Legendre triple. It is clear that we can first choose a satisfying (1.5.1), and then choose c satisfying (1.5.4); but, in general, c may not satisfy (1.5.3). In fact, there are pairs (\mathcal{G}, b) such that the sheaf $\mathcal{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G})$ is not monomial; therefore, (1.4.1) implies that there is no c such that (\mathcal{G}, b, c) is a Legendre triple for such pairs (\mathcal{G}, b) . On the other hand, there are extreme cases for which (1.5.3) is implied by the two other relations, and hence there exists c such that (\mathcal{G}, b, c) is a Legendre triple. Indeed, (1.5.1) and (1.5.4) imply that we have the following relation

$$\text{deg}(\check{f}) = \text{sw}(f_* \mathcal{G}) + \text{deg}(f), \tag{1.6.1}$$

where $\text{deg}(-)$ is the degree and $\text{sw}(-)$ is the Swan conductor (cf. (5.2.3)). If f is tamely ramified, $\text{sw}(f_* \mathcal{G}) = \text{sw}(\mathcal{G}) \geq 1$ and $\text{sw}(\mathcal{G}) + \text{deg}(f)$ is prime to p , then \check{f} is tamely ramified, and hence (1.5.3) is satisfied. Note that under the assumptions on tameness, we have $\text{ord}(tb'/b) = \text{ord}(tc'/c) = 0$.

A special case of formula (1.4.1) was conjectured by Laumon and Malgrange [Lau87, 2.6.3] and proved by Fu [Fu07]. It corresponds to the extreme case where \mathcal{G}_w is an Artin–Schreier sheaf

(i.e. $m = 0$) of Swan conductor s , and f is a tamely ramified morphism of degree r , such that $1 \leq s < p$, $r + s$ is prime to p , and k is algebraically closed.

1.7 The main idea and the key technical tool for the proof of (1.4.1) come from our theory of ramification [AS09]. We denote by pr_1 and pr_2 the canonical projections of $\eta \times_k \eta$, by $j: \eta \times_k \eta \rightarrow S \times_k S$ the canonical injection, and put $b_1 = \text{pr}_1^*(b)$ and $c_2 = \text{pr}_2^*(c)$. The proof of (1.4.1) is made in two steps. The first and most important step is a calculus of vanishing cycles under the Legendre conditions. The second step is a computation of the dimension of the local Fourier transform, which holds in general without any restriction on the sheaf.

First, we study the complex of vanishing cycles of the sheaf $j_!(\text{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\psi_0}(b_1 c_2))$ relatively to the second projection $S \times_k S \rightarrow S$. By adapting our method in [AS09], we prove that, under the Legendre conditions, this complex can be explicitly described over an open subscheme of a suitable blow-up of $S \times_k S$ along a closed subscheme of the diagonal $S \rightarrow S \times_k S$ (cf. Proposition 4.11). Figuratively speaking, we kill the ramification by blowing-up in the diagonal, which was the leitmotiv of [AS02, AS03, AS09]. From this, we deduce that the sheaf $\check{f}^*(\mathfrak{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G}))$ over η has a direct factor isomorphic to

$$\mathcal{D} = \mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{H} \left(-\frac{1}{2} \frac{dc}{db} \right) \otimes \mathcal{L},$$

and the morphism $\check{f}_*(\mathcal{D}) \rightarrow \mathfrak{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G})$, induced by the trace homomorphism $\check{f}_* \check{f}^* \rightarrow \text{id}$, is injective (cf. Proposition 4.3).

Second, in order to prove that the morphism $\check{f}_*(\mathcal{D}) \rightarrow \mathfrak{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G})$ is an isomorphism, it is enough to show that the rank of $\mathfrak{F}_{\psi_0}^{(z, \infty)}(f_* \mathcal{G})$ is equal to the degree of \check{f} . By (1.6.1), the latter relation is a special case of a general formula proved by Laumon [Lau87, 2.4.3]: namely, for any $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} over τ_z , we have

$$\text{rk}(\mathfrak{F}_{\psi_0}^{(z, \infty)}(\mathcal{F})) = \text{sw}(\mathcal{F}) + \text{rk}(\mathcal{F}). \tag{1.7.1}$$

We give in the appendix (Proposition B.6) another proof of this equation using a formula of Deligne–Kato that computes the dimension of the nearby cycle complex of a sheaf on a smooth curve over a strictly henselian trait. Deligne considered the case where the sheaf has no vertical ramification [Lau81, 5.1.1], and Kato extended the formula to the general case [Kato87a, 6.7]. We give in the appendix (Theorem A.13) a brief review of Kato’s formula for rank-one sheaves, which is enough for our application, by using his refined Swan conductors. The latter fits perfectly in our ramification theory as proved in [AS09], and hence in the general philosophy of this article.

1.8 Formula (1.4.1) has strong relations with the theory of ε -factors. First, it was suggested by explicit formulas for ε -factors of quasi-characters (8.7.3) (cf. also [Hen84]). Second, it implies Laumon’s formula relating ε -factors and local Fourier transforms. More precisely, if k is finite, \mathcal{F} is a $\overline{\mathbb{Q}}_\ell$ -sheaf over τ_0 and $\mathcal{F}_!$ is the extension of \mathcal{F} by 0 to T_0 , then Laumon [Lau87, 3.6.2] proved that we have

$$(-1)^d \det(\text{Rec}_{\check{T}_\infty}(\check{x}^{-1}), \mathfrak{F}^{(0, \infty)}(\mathcal{F})) = \varepsilon(T_0, \mathcal{F}_!, dx), \tag{1.8.1}$$

where d is the dimension of $\mathfrak{F}_{\psi_0}^{(0, \infty)}(\mathcal{F})$, $\varepsilon(T_0, \mathcal{F}_!, dx)$ is the ε -factor of the sheaf $\mathcal{F}_!$ over T_0 , and $\text{Rec}_{\check{T}_\infty}$ is the reciprocity isomorphism of class field theory for the completion of the function field $k(\check{\tau}_\infty)$ of \check{T}_∞ (cf. §9.1).

Under the assumptions of §1.4, if $(\mathcal{G}, f, \check{f})$ is a Legendre triple and $z = 0$, we give in Theorem 9.2 a new proof of (1.8.1) for the sheaf $\mathcal{F} = f_*(\mathcal{G})$. We deduce it from (1.4.1) by using three ingredients. The first is a classical explicit formula for ε -factors involving Gauss sums (Proposition 8.7). The second is a variation on Witt’s explicit reciprocity law due to Fontaine (Proposition 8.11). The third ingredient is new (Proposition 8.8); it is an explicit formula for the Langlands λ -factor which appears in the induction formula for ε -factors. We prove the latter by using Deligne’s formula for the ε -factor of an orthogonal representation in terms of its second Stiefel–Whitney class [Del76, 1.5], and Serre’s formula for the second Stiefel–Whitney class of induced representations in terms of the Hasse–Witt invariant of quadratic forms [Ser84].

1.9 This article is divided into two parts and augmented by two appendices. The first part, with a strong geometric flavor, is devoted to the proof of (1.4.1). Section 2 develops the necessary tools from ramification theory of Artin–Schreier–Witt sheaves. It contains in particular a computation of Witt vectors (Proposition 2.7) that plays a crucial role in the following. In §3, we review the definition of the local Fourier transform and state the main theorems (Theorems 3.7 and 3.9). Section 4 is the heart of the article. It contains the analysis of a complex of vanishing cycles by blowing-up in the diagonal mentioned in §1.7. The proofs of the main theorems are given in §5. The second part, with a more arithmetic flavor, is devoted to the proof of (1.8.1). It starts by a brief review of Stiefel–Whitney classes and a formula of Serre in §6, followed by a short complement on refined logarithmic differentials in §7. In §8, we review the theory of ε -factors and develop the necessary ingredients for the proof of (1.8.1). Finally, the proof of this formula is completed in §9. In Appendix A we review the Deligne–Kato formula for the dimension of the nearby cycle complex of a sheaf of rank one on a smooth curve over a strictly henselian trait. In Appendix B we apply this formula to compute the dimension of the local Fourier transform.

1.10 C. Sabbah proved an explicit formula for the local Fourier transform of a formal germ of meromorphic connection of one complex variable using a blow-up technique [Sab08]. The relation between our approaches is not clear. During the preparation of this article, we learned from M. Strauch that he made some expectations on a local principle of stationary phase, without giving precise formulas. He is motivated by applications to the cohomology of Lubin–Tate spaces. We are grateful to an anonymous referee for his thorough reading of the manuscript and helpful comments. The first author would like to acknowledge the hospitality of the Department of Mathematical Sciences at the University of Tokyo where this work was achieved.

We finish this section with some notation and conventions.

1.11 In this article (except in §§7 and 8 and the appendix), we fix a prime number $p > 2$, a perfect field k of characteristic p and an algebraic closure \bar{k} of k . For q a power of p , we denote by \mathbb{F}_q the unique subfield of \bar{k} with q elements. We also fix a prime number ℓ different from p , an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field \mathbb{Q}_ℓ of ℓ -adic numbers and a non-trivial additive character $\psi_0: \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$. For every integer $m \geq 0$, we fix an injective homomorphism $\psi_m: \mathbb{Z}/p^{m+1}\mathbb{Z} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that for any $a \in \mathbb{F}_p$, we have $\psi_m(p^m a) = \psi_0(a)$, where $p^m a$ denotes the embedding $\mathbb{F}_p \rightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$ induced by the multiplication by p^m on \mathbb{Z} .

1.12 If X is a scheme and $x \in X$, we denote by $\kappa(x)$ the residue field of X at x and by $i_x: \text{Spec}(\kappa(x)) \rightarrow X$ the canonical morphism.

1.13 For a scheme X , a ‘ $\overline{\mathbb{Q}}_\ell$ -sheaf over X ’ stands for a ‘constructible $\overline{\mathbb{Q}}_\ell$ -sheaf over X ’ in the sense of [Del80, 1.1.1]. We denote by $\mathbf{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ the derived category of ℓ -adic sheaves defined in [Del80, 1.1.2 and 1.1.3].

2. Calculus on Witt vectors

2.1 Let R be a complete discrete valuation ring of equal characteristic p , with residue field k , equipped with a uniformizer t , K be the fraction field of R , ord be the valuation of K normalized by $\text{ord}(t) = 1$. We identify R with the ring of power series $k[[t]]$. For any $x \in K$, we denote by $x^{(i)}$ the i th iterated derivative of x relatively to t ($i \geq 1$); we put $x^{(0)} = x$ and $x' = x^{(1)}$.

2.2 The module Ω_R^1 is free of rank one over R , and hence complete and separated. We identify it with a submodule of Ω_K^1 . For $a \in K^\times$, we put $d \log(a) = da/a \in \Omega_K^1$. We denote by $\Omega_R^1(\log)$ the sub- R -module of Ω_K^1 generated by Ω_R^1 and the elements of the form $d \log a$ for $a \in R - \{0\}$. Then $\Omega_R^1(\log)$ is a free R -module of rank one generated by $d \log(t)$. We put $\Omega_k^1(\log) = \Omega_R^1(\log) \otimes_R k = k \cdot d \log(t)$. We define an increasing exhaustive filtration on Ω_K^1 by setting, for $n \in \mathbb{Z}$, $\text{fil}_n \Omega_K^1 = t^{-n} \Omega_R^1(\log)$. We have

$$\text{Gr}_n \Omega_K^1 = \text{fil}_n \Omega_K^1 / \text{fil}_{n-1} \Omega_K^1 \simeq (t^{-n} R / t^{-n+1} R) \cdot d \log(t).$$

2.3 Let m be an integer ≥ 0 , $W_{m+1}(K)$ be the ring of Witt vectors of length $m + 1$. Following [Bry83, Kato89], we define an increasing exhaustive filtration on the group of Witt vectors $W_{m+1}(K)$ by setting, for $n \in \mathbb{Z}$, $\text{fil}_n W_{m+1}(K)$ to be the subgroup of elements (x_0, \dots, x_m) such that

$$p^{m-i} \text{ord}(x_i) \geq -n \quad \text{for all } 0 \leq i \leq m. \tag{2.3.1}$$

We put

$$\text{Gr}_n W_{m+1}(K) = \text{fil}_n W_{m+1}(K) / \text{fil}_{n-1} W_{m+1}(K).$$

Let $V: W_{m+1}(K) \rightarrow W_{m+2}(K)$ be the verschiebung morphism. We have

$$V(\text{fil}_n W_{m+1}(K)) \subset \text{fil}_n W_{m+2}(K).$$

2.4 Let $F: W_{\bullet+1} \Omega_K^1 \rightarrow W_{\bullet} \Omega_K^1$ be the Frobenius morphism of the de Rham–Witt complex of K over k . The homomorphism

$$F^m d: W_{m+1}(K) \rightarrow \Omega_K^1 \tag{2.4.1}$$

is given by the formula

$$F^m d(x_0, \dots, x_m) = \sum_{i=0}^m x_i^{p^{m-i}-1} dx_i. \tag{2.4.2}$$

Therefore, for any integer n , we have

$$F^m d(\text{fil}_n W_{m+1}(K)) \subset \text{fil}_n \Omega_K^1. \tag{2.4.3}$$

We deduce a canonical homomorphism

$$\text{gr}_n(F^m d): \text{Gr}_n W_{m+1}(K) \rightarrow \text{Gr}_n \Omega_K^1. \tag{2.4.4}$$

2.5 The exact sequence

$$0 \longrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z} \longrightarrow W_{m+1} \xrightarrow{F-1} W_{m+1} \longrightarrow 0 \tag{2.5.1}$$

induces a surjective homomorphism

$$\delta_{m+1} : W_{m+1}(K) \rightarrow H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}). \tag{2.5.2}$$

We define an increasing exhaustive filtration on $H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ by setting (for $n \in \mathbb{Z}$)

$$\text{fil}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = \delta_{m+1}(\text{fil}_n W_{m+1}(K)). \tag{2.5.3}$$

We put

$$\text{Gr}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = \text{fil}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) / \text{fil}_{n-1} H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}).$$

By [Kato89, 3.2] and [AS09, 10.7], for any integer $n \geq 1$, there exists a unique homomorphism

$$\psi_{m,n} : \text{Gr}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \rightarrow \text{Gr}_n \Omega_K^1 \tag{2.5.4}$$

making the following diagram commutative.

$$\begin{array}{ccc} \text{Gr}_n W_{m+1}(K) & \xrightarrow{-\text{gr}_n(F^m d)} & \text{Gr}_n \Omega_K^1 \\ \text{gr}_n(\delta_{m+1}) \downarrow & \nearrow \psi_{m,n} & \\ \text{Gr}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) & & \end{array} \tag{2.5.5}$$

For any $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, the Swan conductor of χ , $\text{sw}(\chi)$, is the smallest integer $n \geq 0$ such that $\chi \in \text{fil}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ (cf. [Bry83, corollary of Theorem 1] and [Kato89]). Kato defined the *refined Swan conductor* of χ , $\text{rsw}(\chi)$, as the image of the class of χ by the homomorphism

$$\psi_{m,\text{sw}(\chi)} : \text{Gr}_{\text{sw}(\chi)} H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \rightarrow \text{Gr}_{\text{sw}(\chi)} \Omega_K^1.$$

2.6 Let $k(\theta)$ be the field of rational functions in one variable θ over k , $R_L = k(\theta)[[t]]$ be the ring of power series in the variable t over $k(\theta)$, L be the fraction field of R_L . We consider L as an extension of K by the k -homomorphism $v : K \rightarrow L$ defined by $v(t) = t$. Let r be an integer greater than or equal to one, $u : K \rightarrow L$ be the k -homomorphism defined by $u(t) = t(1 + t^r\theta)$. In [AS09, 13.4], we proved that for any integer n , the group homomorphism

$$u - v : W_{m+1}(K) \rightarrow W_{m+1}(L) \tag{2.6.1}$$

maps $\text{fil}_n W_{m+1}(K)$ to $\text{fil}_{n-r} W_{m+1}(L)$, and we expressed the induced homomorphism on the graded pieces

$$\text{Gr}_n W_{m+1}(K) \rightarrow \text{Gr}_{n-r} W_{m+1}(L).$$

We refine this result as follows.

PROPOSITION 2.7. *Let n be an integer, $a = (a_0, \dots, a_m) \in \text{fil}_n W_{m+1}(K)$, α be the element of K such that $F^m da = \alpha dt$. Then $\text{ord}(t\alpha) \geq -n$ and we have*

$$u(a) - a \equiv V^m \left(\sum_{i=1}^{p-1} \frac{t^i \alpha^{(i-1)}}{i!} (t^r \theta)^i \right) \pmod{\text{fil}_{n-pr} W_{m+1}(L)}. \tag{2.7.1}$$

First, we prove some preliminary results. We define a sequence of polynomials ($n \geq 0$)

$$Q_n \in \mathbb{Z} \left[\frac{1}{p} \right] [X_0, \dots, X_n, Y_0, \dots, Y_n]$$

by the inductive formula

$$\sum_{i=0}^n p^i (X_i(1 + Y_i))^{p^{n-i}} = \sum_{i=0}^n p^i X_i^{p^{n-i}} + \sum_{i=0}^n p^i Q_i^{p^{n-i}}. \tag{2.7.2}$$

Observe that for a commutative ring A and elements $x = (x_0, \dots, x_m)$, $y = (y_0, \dots, y_m)$ and $z = (z_0, \dots, z_m)$ of $W_{m+1}(A)$ such that $z_i = x_i(1 + y_i)$ for all $0 \leq i \leq m$, we have

$$z - x = (Q_0(x, y), Q_1(x, y), \dots, Q_m(x, y)). \tag{2.7.3}$$

We denote by $\lambda(Y)$ the shifted p -truncated logarithm, that is, the polynomial of $\mathbb{Z}_{(p)}[Y]$ defined by

$$\lambda(Y) = \sum_{i=1}^{p-1} (-1)^{i+1} \frac{Y^i}{i}. \tag{2.7.4}$$

LEMMA 2.8.

- (i) The polynomials Q_n belong to the ideal of $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ generated by (Y_0, \dots, Y_n) .
- (ii) If we attach the weight p^i to the variable X_i and the weight 0 to the variable Y_i , the polynomial Q_n is homogeneous of weight p^n .
- (iii) We have the following relation in $\mathbb{Z}_{(p)}[X_0, \dots, X_n, Y_0, \dots, Y_n]$

$$Q_n \equiv \sum_{i=0}^{n-1} X_i^{p^{n-i}} \lambda(Y_i) + X_n Y_n \pmod{(p) + (Y_0, \dots, Y_n)^p}. \tag{2.8.1}$$

Propositions (i) and (ii) are easy. We prove proposition (iii). Since Q_i belongs to the ideal (Y_0, \dots, Y_i) , we have

$$p^n Q_n \equiv \sum_{i=0}^n p^i X_i^{p^{n-i}} ((1 + Y_i)^{p^{n-i}} - 1) \pmod{(Y_0, \dots, Y_n)^p}.$$

So the required relation follows from the following congruence, for $i \leq n - 1$ and $1 \leq j \leq p - 1$,

$$\frac{1}{p^{n-i}} \binom{p^{n-i}}{j} \equiv (-1)^{j+1} \frac{1}{j} \pmod{p}.$$

LEMMA 2.9.

- (i) For any $x \in K$, we have

$$u(x) - x \equiv \sum_{i=1}^{p-1} \frac{t^i x^{(i)}}{i!} (t^r \theta)^i \pmod{xt^{pr} R_L}. \tag{2.9.1}$$

- (ii) For any $x \in K^\times$, we have

$$\text{ord} \left(\frac{u(x)}{x} - 1 \right) \geq r. \tag{2.9.2}$$

- (iii) For any $x \in K^\times$, if we put $y = x'/x$, we have

$$\lambda \left(\frac{u(x)}{x} - 1 \right) \equiv \sum_{i=1}^{p-1} \frac{t^i y^{(i-1)}}{i!} (t^r \theta)^i \pmod{t^{pr} R_L}. \tag{2.9.3}$$

(i) Since $\text{ord}(tx') \geq \text{ord}(x)$ for any $x \in K$, we are reduced by the Leibniz rule to proving (2.9.1) for $x = t$, for $x = t^{-1}$ and for $x \in R^\times$. The first two cases are obvious, and the last one follows from Taylor expansion.

(ii) It follows immediately from part (i) and the fact that $\text{ord}(t^i x^{(i)}) \geq \text{ord}(x)$ for all $i \geq 1$.

(iii) Since both sides of (2.9.3) define group homomorphisms from K^\times to $R_L/t^{pr}R_L$, it is enough to prove (2.9.3) for $x = t$ and for $x \in R^\times$. For $x = t$, both sides are equal to $\lambda(t^r\theta)$. For $x \in R^\times$, we are reduced by truncation to the case where x is a polynomial in t with a non-vanishing constant term. Then after replacing k by an algebraic closure, we are further reduced to the case where $x = 1 - ct$ with $c \in k$. In this case, both sides of (2.9.3) are equal since

$$\lambda\left(\frac{-ct}{1-ct}t^r\theta\right) = \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{i} \left(\frac{-ct}{1-ct}\right)^i (t^r\theta)^i, \tag{2.9.4}$$

and, for $1 \leq i \leq p-1$,

$$\left(\frac{-c}{1-ct}\right)^{(i-1)} = (-1)^{i-1}(i-1)! \left(\frac{-c}{1-ct}\right)^i. \tag{2.9.5}$$

2.10 We can now prove 2.7. We set $b = (b_0, \dots, b_m) \in L^{m+1}$, where $b_i = (u(a_i)/a_i) - 1$ if $a_i \neq 0$, and $b_i = 0$ if $a_i = 0$; so we have $b_i \in t^r R_L$ (see (2.9.2)). It follows from 2.8(ii) and (2.8.1) that we have

$$p^{m-i}\text{ord}(Q_i(a, b)) \geq -n + p^{m-i}r, \tag{2.10.1}$$

$$\text{ord}\left(Q_m(a, b) - \sum_{i=0}^{m-1} a_i^{p^{m-i}} \lambda(b_i) - u(a_m) + a_m\right) \geq -n + pr. \tag{2.10.2}$$

We put $c_i = a'_i/a_i$ if $a_i \neq 0$, and $c_i = 0$ if $a_i = 0$. It follows from (2.9.1) and (2.9.3) that we have

$$\begin{aligned} & \sum_{i=0}^{m-1} a_i^{p^{m-i}} \lambda(b_i) + u(a_m) - a_m \\ & \equiv \sum_{i=0}^{m-1} a_i^{p^{m-i}} \sum_{j=1}^{p-1} \frac{t^j c_i^{(j-1)}}{j!} (t^r\theta)^j + \sum_{j=1}^{p-1} \frac{t^j a_m^{(j)}}{j!} (t^r\theta)^j \pmod{t^{pr-n}R_L} \\ & \equiv \sum_{j=1}^{p-1} t^j \left(\sum_{i=0}^{m-1} a_i^{p^{m-i}} c_i^{(j-1)} + a_m^{(j)} \right) \frac{(t^r\theta)^j}{j!} \pmod{t^{pr-n}R_L}. \end{aligned} \tag{2.10.3}$$

The proposition follows since we have by definition

$$\alpha = \sum_{i=0}^{m-1} a_i^{p^{m-i}} c_i + a'_m. \tag{2.10.4}$$

COROLLARY 2.11. We keep the notation of 2.7; moreover, let b, c be non-zero elements of K , $\nu(b) = \text{ord}(tb'/b)$, $\nu(c) = \text{ord}(tc'/c)$. Assume that $\alpha + cb' = 0$ and $\nu(b) + \nu(c) < (p-2)r$. Then we have

$$u(a) - a + V^m(c(u(b) - b)) \in \text{fil}_{n-\nu(c)-2r}(\mathbb{W}_{m+1}(L)) \tag{2.11.1}$$

and

$$u(a) - a + V^m(c(u(b) - b)) \equiv V^m\left(\frac{1}{2}t\alpha \frac{tc'}{c}(t^r\theta)^2\right) \pmod{\text{fil}_{n-\nu(c)-2r-1}(\mathbb{W}_{m+1}(L))}. \tag{2.11.2}$$

Observe first that we have

$$\alpha' + cb^{(2)} = \alpha \frac{c'}{c}, \tag{2.11.3}$$

and, for any $i \geq 2$,

$$\text{ord}(t^i \alpha^{(i-1)} + t^i b^{(i)} c) \geq \text{ord}\left(t \alpha \frac{tc'}{c}\right). \tag{2.11.4}$$

Indeed, the equation $\alpha + cb' = 0$ implies immediately (2.11.3) and the following equation

$$t^i \alpha^{(i-1)} + ct^i b^{(i)} = -\sum_{j=1}^{i-1} \binom{i-1}{j} t^i b^{(i-j)} c^{(j)}.$$

The relation $\text{ord}(tz') \geq \text{ord}(z)$ for any $z \in K$, implies that each term of the right-hand side has bigger valuation than $t^2 b' c' = -t^2 \alpha c' / c$.

We have $\text{ord}(t\alpha) \geq -n$ (Proposition 2.7), $\text{ord}(bc) \geq -n - \nu(b)$ and $\text{ord}(t^2 \alpha c' / c) \geq -n + \nu(c)$. Hence, we deduce from (2.7.1) and (2.9.1) that we have

$$\begin{aligned} &u(a) - a + V^m(c(u(b) - b)) \\ &\equiv V^m\left(\sum_{i=2}^{p-1} \frac{t^i}{i!} (\alpha^{(i-1)} + cb^{(i)})(t^r \theta)^i\right) \pmod{\text{fil}_{n+\nu(b)-pr} W_{m+1}(L)}. \end{aligned} \tag{2.11.5}$$

The corollary follows from (2.11.5), (2.11.3), (2.11.4) and the assumptions.

We can replace the condition $\alpha + cb' = 0$ of 2.11 by a weaker condition (2.12.2) as follows.

COROLLARY 2.12. *We keep the notation of 2.7; moreover, let b, c be non-zero elements of K , $\nu(b) = \text{ord}(tb'/b)$, $\nu(c) = \text{ord}(tc'/c)$. Assume that the following conditions are satisfied*

$$\text{ord}(t\alpha) = -n, \tag{2.12.1}$$

$$\text{ord}(\alpha + cb') \geq -n + \nu(c) + r, \tag{2.12.2}$$

$$\nu(b) + \nu(c) < (p - 2)r. \tag{2.12.3}$$

Then we have

$$u(a) - a + V^m(c(u(b) - b)) \in \text{fil}_{n-\nu(c)-2r}(W_{m+1}(L)) \tag{2.12.4}$$

and

$$u(a) - a + V^m(c(u(b) - b)) \equiv V^m\left(\frac{1}{2} t \alpha \frac{tc'}{c} (t^r \theta)^2\right) \pmod{\text{fil}_{n-\nu(c)-2r-1}(W_{m+1}(L))}. \tag{2.12.5}$$

Let c_0 be the element of K such that $\alpha + c_0 b' = 0$. Since $\alpha \neq 0$, we have $c_0 \neq 0$ and

$$1 - \frac{c}{c_0} = \frac{\alpha + cb'}{\alpha}.$$

We deduce that $\text{ord}(1 - c/c_0) > \nu(c) + r > 0$; in particular, we have $\text{ord}(c) = \text{ord}(c_0)$. The relation $\text{ord}(tz') \geq \text{ord}(z)$ for any $z \in K$, implies that

$$\text{ord}\left(t\left(\frac{c'}{c} - \frac{c'_0}{c_0}\right)\right) = \text{ord}\left(t \frac{c'c_0 - cc'_0}{c_0^2}\right) > \nu(c) + r, \tag{2.12.6}$$

and, hence, we have $\nu(c) = \nu(c_0)$. By (2.9.1), we have

$$(c - c_0)(u(b) - b) = \sum_{i=1}^{p-1} (c - c_0) t^i \frac{b^{(i)}}{i!} (t^r \theta)^i \pmod{(c - c_0) b t^{pr} R_L}.$$

Taking into account the relations $\text{ord}(tz') \geq \text{ord}(z)$ for any $z \in K$,

$$\text{ord}(b(c - c_0)) > \text{ord}(bc_0) = -n - \nu(b), \quad \text{ord}(b'(c - c_0)) \geq -n + \nu(c) + r$$

which is (2.12.2), and (2.12.3), we deduce that

$$\text{ord}((c - c_0)(u(b) - b)) > -n + \nu(c) + 2r. \tag{2.12.7}$$

The proposition follows from (2.12.3), (2.12.6), (2.12.7) and 2.11.

2.13 Let a be a non-zero element of $W_{m+1}(K)$, let b, c be non-zero elements of K . We denote by α the element of K such that $F^m da = \alpha dt$, $n = -\text{ord}(t\alpha)$, $\nu(b) = \text{ord}(tb'/b)$ and $\nu(c) = \text{ord}(tc'/c)$. We say that (a, b, c) is a *Legendre triple* if the following conditions are satisfied

$$a \in \text{fil}_n W_{m+1}(K), \tag{2.13.1}$$

$$2\text{ord}(\alpha + cb') \geq -n + \nu(c), \tag{2.13.2}$$

$$2\nu(b) + p\nu(c) < (p - 2)n. \tag{2.13.3}$$

Under these conditions, n is finite (as $a \neq 0$), and it is the smallest integer such that $a \in \text{fil}_n W_{m+1}(K)$. Moreover, we have $n \geq 1$ and $\text{ord}(tb'c) = -n$. We say that $(n, \nu(b), \nu(c))$ is the *conductor* of the triple (a, b, c) .

Remark 2.14. Under the assumptions of § 2.13, if moreover $n - \nu(c) = 2r$ is even, then the conditions of § 2.13 are equivalent to the conditions of 2.12.

2.15 Let $S = \text{Spec}(R)$, $\eta = \text{Spec}(K)$ be the generic point of S , let $b, c \in K$ and let \mathcal{G} be a $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over η trivialized by a cyclic extension of order p^{m+1} of η ($m \geq 0$). We denote by $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ the class such that $\psi_m^{-1} \circ \chi$ is the character associated to \mathcal{G} (see § 1.11). We say that (\mathcal{G}, b, c) is a *Legendre triple* if there exists $a \in W_{m+1}(K)$ such that $\delta_{m+1}(a) = \chi$ (see (2.5.2)), a, b and c are non-zero, and (a, b, c) is a Legendre triple.

DEFINITION 2.16. Let X be a smooth connected curve over k (respectively, the spectrum of a henselian discrete valuation ring of equal characteristic p and residue field k), let s be a closed point of X , let U be the open subscheme $X - \{s\}$ of X , $x, y \in \Gamma(U, \mathcal{O}_X)$, and let \mathcal{G} be a smooth $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over U . Let S be the spectrum of the completion of the local ring of X at s , let η be the generic point of S , and let $\tilde{h}: S \rightarrow X$ be the canonical map. We say that (\mathcal{G}, x, y) is a *Legendre triple* at s if there exist \mathcal{G}_t and \mathcal{G}_w two smooth $\overline{\mathbb{Q}}_\ell$ -sheaves of rank one over U satisfying the following conditions:

- (i) $\mathcal{G} \simeq \mathcal{G}_t \otimes \mathcal{G}_w$;
- (ii) \mathcal{G}_t is tamely ramified at s ;
- (iii) \mathcal{G}_w is trivialized by a cyclic extension of order p^{m+1} of U ($m \geq 0$);
- (iv) $(\tilde{h}_U^*(\mathcal{G}_w), \tilde{h}_U^*(x), \tilde{h}_U^*(y))$ is a Legendre triple in the sense of § 2.15.

3. Local Fourier transform

3.1 Lang's isogeny L of $\mathbb{G}_{a,k} = \text{Spec}(k[u])$, defined by $L^*(u) = u^p - u$, induces the Artin-Schreier exact sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{G}_{a,k} \xrightarrow{L} \mathbb{G}_{a,k} \rightarrow 0. \tag{3.1.1}$$

The push-forward of this extension by the character ψ_0^{-1} (see § 1.11) defines a $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one, \mathcal{L}_{ψ_0} , on $\mathbb{G}_{a,k}$. Following Deligne, if $f: X \rightarrow \mathbb{G}_{a,k}$ is a morphism of schemes, we put $\mathcal{L}_{\psi_0}(f) = f^* \mathcal{L}_{\psi_0}$.

3.2 Let $A = \text{Spec}(k[x])$ and $\check{A} = \text{Spec}(k[\check{x}])$ be two affine lines over k (equipped with coordinates x and \check{x}) which are in duality via the pairing $A \times_k \check{A} \rightarrow \mathbb{G}_{a,k}$ defined by $(x, \check{x}) \mapsto u = x\check{x}$. We denote by P and \check{P} the projective lines over k , completions of A and \check{A} , respectively, by $\infty \in P(k)$ and $\check{\infty} \in \check{P}(k)$ the points at infinity, by $j: A \rightarrow P$ and $\check{j}: \check{A} \rightarrow \check{P}$ the canonical injections and by pr and $\check{\text{pr}}$ the canonical projections of $A \times_k \check{A}$. We have the $\overline{\mathbb{Q}}_\ell$ -sheaf $\mathcal{L}_{\psi_0}(x\check{x})$ on $A \times_k \check{A}$; we put $\overline{\mathcal{L}}_{\psi_0}(x\check{x}) = (j \times \check{j})_* \mathcal{L}_{\psi_0}(x\check{x})$ on $P \times_k \check{P}$. For a complex K of $\mathbf{D}_c^b(A, \overline{\mathbb{Q}}_\ell)$, the *Fourier transform* of K is the complex $\mathfrak{F}_{\psi_0}(K)$ of $\mathbf{D}_c^b(\check{A}, \overline{\mathbb{Q}}_\ell)$ defined by

$$\mathfrak{F}_{\psi_0}(K) = \text{R}\check{\text{pr}}_1(\text{pr}^* K \otimes \mathcal{L}_{\psi_0}(x\check{x})). \tag{3.2.1}$$

In the following, we omit the subscript ψ_0 from the notation \mathfrak{F}_{ψ_0} and \mathcal{L}_{ψ_0} when there is no risk of confusion.

3.3 The Kummer covering of order two is the exact sequence

$$1 \rightarrow \mu_2(k) \rightarrow \mathbb{G}_{m,k} \xrightarrow{[2]} \mathbb{G}_{m,k} \rightarrow 1, \tag{3.3.1}$$

where $[2]$ is the square power map. We denote by \mathcal{K} the $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one on $\mathbb{G}_{m,k}$ obtained by push-forward of this extension by the unique non-trivial character $\mu_2(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$. For a morphism $f: X \rightarrow \mathbb{G}_{m,k}$, we put $\mathcal{K}(f) = f^* \mathcal{K}$.

Consider the open subschemes $U = A - \{0\}$ of A and $\check{U} = \check{A} - \{0\}$ of \check{A} , equipped with the isomorphisms $x: U \rightarrow \mathbb{G}_{m,k}$ and $\check{x}: \check{U} \rightarrow \mathbb{G}_{m,k}$. Let \mathcal{Q} be the $\overline{\mathbb{Q}}_\ell$ -sheaf on $\text{Spec}(k)$ defined by

$$\mathcal{Q} = H_c^1(U_{\bar{k}}, \mathcal{K}(x) \otimes \mathcal{L}_{\psi_0}(x)). \tag{3.3.2}$$

Then \mathcal{Q} has rank one and the $H_c^i(U_{\bar{k}}, \mathcal{K}(x) \otimes \mathcal{L}_{\psi_0}(x))$, for $i \neq 1$, vanish. Moreover, we have canonical isomorphisms

$$\mathfrak{F}(j_* \mathcal{K}(x))[1] \simeq \check{j}_* \mathcal{K}(\check{x}) \otimes \mathcal{Q}, \tag{3.3.3}$$

$$j^* \text{R}\check{\text{pr}}_1(\mathcal{L}_{\psi_0}(x^2\check{x}))[1] \simeq \mathcal{K}(\check{x}) \otimes \mathcal{Q}. \tag{3.3.4}$$

Indeed, the first assertion and the isomorphism (3.3.3) are proved in [Lau87, 1.4.3.1]. Consider the morphism $\pi: A \rightarrow A$ defined by $\pi(x) = x^2$. By the projection formula, we have a canonical isomorphism

$$\text{R}\check{\text{pr}}_1(\mathcal{L}_{\psi_0}(x^2\check{x})) \simeq \text{R}\check{\text{pr}}_1((\pi \times 1)^* \mathcal{L}_{\psi_0}(x\check{x})) \simeq \text{R}\check{\text{pr}}_1((\pi \times 1)_*(\overline{\mathbb{Q}}_\ell) \otimes \mathcal{L}_{\psi_0}(x\check{x})).$$

Since we have $\pi_*(\overline{\mathbb{Q}}_\ell) \simeq j_* \mathcal{K} \oplus \overline{\mathbb{Q}}_\ell$, the isomorphism (3.3.4) follows from (3.3.3) and [Lau87, 1.2.2.2].

3.4 Let z be a closed point of P , let \check{z} be a closed point of \check{P} , let T and \check{T} be the henselizations of P and \check{P} at z and \check{z} , respectively, let τ and $\check{\tau}$ be the generic points of T and \check{T} , respectively, let $h: T \rightarrow P$ and $\check{h}: \check{T} \rightarrow \check{P}$ be the canonical morphisms, and let pr and $\check{\text{pr}}$ be the canonical projections of $T \times_k \check{T}$. We denote also by x and \check{x} the pull-backs of the coordinates x and \check{x} of A and \check{A} over τ and $\check{\tau}$, respectively, and (abusively) by $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ the sheaf $(h \times \check{h})^* \mathcal{L}_{\psi_0}(x\check{x})$ over $T \times_k \check{T}$. Let \mathcal{F} be a $\overline{\mathbb{Q}}_\ell$ -sheaf over τ , \mathcal{F}_1 be its extension by zero to T . By [Lau87, 2.3.2.1 and 2.3.3.1], the complex of vanishing cycles $\Phi(\text{pr}^*(\mathcal{F}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$ in $\mathbf{D}_c^b(T \times_k \check{\tau}, \overline{\mathbb{Q}}_\ell)$, relatively to

the projection $\text{pr}: T \times_k \check{T} \rightarrow \check{T}$, is supported on $z \times_k \check{\tau}$ and has cohomology only in degree one. Following Laumon, we define the *local Fourier transform* of \mathcal{F} at (z, \check{z}) , denoted by $\mathfrak{F}^{(z, \check{z})}(\mathcal{F})$, to be the $\overline{\mathbb{Q}}_\ell$ -sheaf over $z \times_k \check{\tau}$ given as

$$\mathfrak{F}^{(z, \check{z})}(\mathcal{F}) = (i_z \times 1)^*(\Phi^1(\text{pr}^*(\mathcal{F}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))). \tag{3.4.1}$$

In fact, $\mathfrak{F}^{(z, \check{z})}(\mathcal{F})$ vanishes if $(z, \check{z}) \in A \times \check{A}$ (see [Lau87, 2.3.2.1 and 2.3.3.1]). Observe that if z or \check{z} is k -rational (which is the case if $(z, \check{z}) \notin A \times \check{A}$), then $z \times_k \check{T}$ is connected; more precisely, if z is k -rational, then $z \times_k \check{T} = \check{T}$, and if \check{z} is k -rational, then $z \times_k \check{T}$ is a finite étale covering of \check{T} .

3.5 We keep the notation of § 3.4, moreover, let k' be a finite extension of k , $u: k(z) \rightarrow k'$ and $\check{u}: k(\check{z}) \rightarrow k'$ be two k -homomorphisms, where $k(z)$ and $k(\check{z})$ are the residue fields of z and \check{z} , respectively. The pairs (z, u) and (\check{z}, \check{u}) define rational points $z' \in P_{k'}(k')$ and $\check{z}' \in \check{P}_{k'}(k')$. Let T' and \check{T}' be the henselizations of $P_{k'}$ and $\check{P}_{k'}$ at z' and \check{z}' , respectively, let τ' and $\check{\tau}'$ be the generic points of T' and \check{T}' , respectively, and let $f: T' \rightarrow T$ and $\check{f}: \check{T}' \rightarrow \check{T}$ be the canonical morphisms. The canonical morphism $\check{T}' \rightarrow k' \otimes_k \check{T}$ induces a morphism $\tilde{f}: \check{T}' \rightarrow z \times_k \check{T}$. For any $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F}' over τ' , we can consider the local Fourier transform $\mathfrak{F}^{(z', \check{z}')}(\mathcal{F}')$, of \mathcal{F}' at (z', \check{z}') , which is a $\overline{\mathbb{Q}}_\ell$ -sheaf over $\check{\tau}' = z' \times_{k'} \check{\tau}'$.

PROPOSITION 3.6. *We keep the notation of § 3.5.*

(i) *For any $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} over τ , we have a canonical functorial isomorphism over $\check{\tau}'$:*

$$\tilde{f}^*(\mathfrak{F}^{(z, \check{z})}(\mathcal{F})) \simeq \mathfrak{F}^{(z', \check{z}')}(\tilde{f}^* \mathcal{F}). \tag{3.6.1}$$

(ii) *Assume that \check{z} is k -rational. Then for any $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F}' over τ' , we have a canonical functorial isomorphism over $z \times_k \check{\tau}$:*

$$\tilde{f}_*(\mathfrak{F}^{(z', \check{z}')}(\mathcal{F}')) \simeq \mathfrak{F}^{(z, \check{z})}(\tilde{f}_* \mathcal{F}'). \tag{3.6.2}$$

Consider the following commutative diagram with Cartesian squares.

$$\begin{array}{ccccc} T' \times_{k'} \check{T}' & \xrightarrow{\gamma} & T' \times_k \check{T}' & \longrightarrow & T' \times_k \check{T} & \xrightarrow{\text{pr}_1} & T' \\ & \searrow & \downarrow & & \downarrow & & \downarrow f \\ & & & & T \times_k \check{T} & \xrightarrow{\text{pr}} & T \\ & & & & \downarrow \text{pr} & & \\ & & & & \check{T}' & \xrightarrow{\check{f}} & \check{T} \end{array}$$

Observe that γ is an open and a closed immersion, and hence is étale. Let \mathcal{G} be a $\overline{\mathbb{Q}}_\ell$ -sheaf over $T' \times_k \check{T}$, \mathcal{G}' be its pull-back over $T' \times_{k'} \check{T}'$. We consider the complexes of vanishing cycles $\Phi(\mathcal{G})$ in $\mathbf{D}_c^b(T' \times_k \check{\tau}, \overline{\mathbb{Q}}_\ell)$ and $\Phi(\mathcal{G}')$ in $\mathbf{D}_c^b(T' \times_{k'} \check{\tau}', \overline{\mathbb{Q}}_\ell)$, relatively to the second projections $T' \times_k \check{T} \rightarrow \check{T}$ and $T' \times_{k'} \check{T}' \rightarrow \check{T}'$, respectively. We denote by

$$p: T' \times_{k'} \check{\tau}' \rightarrow T' \times_k \check{\tau} \tag{3.6.3}$$

the canonical morphism. By [Del77, [Th. finitude] 3.7] and [DK73, XIII, 2.1.7.2], and since γ is étale, we have a canonical isomorphism

$$p^*(\Phi(\mathcal{G})) \xrightarrow{\sim} \Phi(\mathcal{G}'). \tag{3.6.4}$$

(i) We consider the sheaf $\mathcal{H} = \text{pr}^*(\mathcal{F}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $T \times_k \check{T}$, and its complex of vanishing cycles $\Phi(\mathcal{H})$ in $\mathbf{D}_c^b(T \times_k \check{\tau}, \overline{\mathbb{Q}}_\ell)$, relatively to the projection pr . We take for \mathcal{G} the inverse image of \mathcal{H} over $T' \times_k \check{T}$. By [DK73, XIII, 2.1.7.2], since f is étale, we have a canonical isomorphism

$$(f \times 1)^*(\Phi(\mathcal{H})) \xrightarrow{\sim} \Phi(\mathcal{G}). \tag{3.6.5}$$

Then the proposition follows from (3.6.4) and (3.6.5).

(ii) We consider the sheaf $\mathcal{H} = \text{pr}^*(f_*\mathcal{F}'_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $T \times_k \check{T}$, and its complex of vanishing cycles $\Phi(\mathcal{H})$ in $\mathbf{D}_c^b(T \times_k \check{\tau}, \overline{\mathbb{Q}}_\ell)$, relatively to the projection pr . We take $\mathcal{G} = \text{pr}_1^*(\mathcal{F}'_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $T' \times_k \check{T}$, where we have denoted (abusively) by $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ the pull-back of the sheaf $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $T' \times_k \check{T}$. By [DK73, XIII, 2.1.7.1] and the projection formula, since f is finite, we have a canonical isomorphism

$$(f \times 1)_*(\Phi(\mathcal{G})) \xrightarrow{\sim} \Phi(\mathcal{H}). \tag{3.6.6}$$

On the other hand, the canonical morphism $\check{T}' \rightarrow k' \otimes_k \check{T}$ is an isomorphism by assumption. Therefore, p is an isomorphism, and we deduce from (3.6.4) a functorial isomorphism

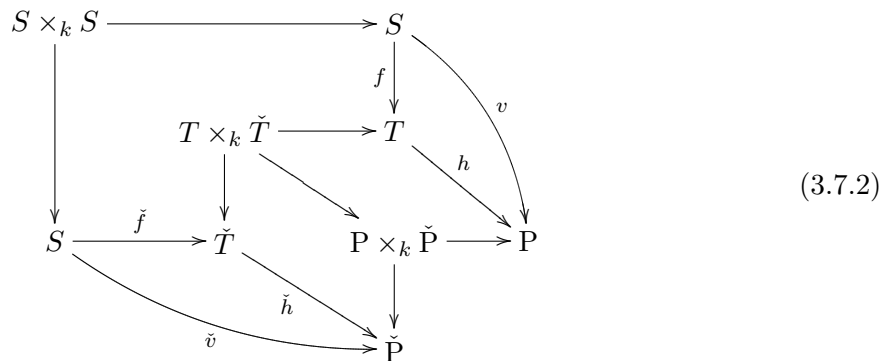
$$\Phi(\mathcal{G}) \xrightarrow{\sim} p_*(\Phi(\mathcal{G}')). \tag{3.6.7}$$

The proposition follows from (3.6.6) and (3.6.7).

THEOREM 3.7. *Let S be the spectrum of a henselian discrete valuation ring of equal characteristic p , with perfect residue field, let s (respectively, η) be the closed (respectively generic) point of S , let \mathcal{G} be a $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over η , and let $v: S \rightarrow A$ and $\check{v}: S \rightarrow \check{P}$ be two non-constant morphisms (with the notation of § 3.2). We put $z = v(s)$, $\check{z} = \check{v}(s)$, b and c the functions on η deduced by pull-back from the coordinates x and \check{x} of A and \check{A} respectively. We take again the notation of § 3.4 relatively to z and \check{z} , and denote by $f: S \rightarrow T$ and $\check{f}: S \rightarrow \check{T}$ the morphisms induced by v and \check{v} , respectively, by $q: T \rightarrow z$ the canonical morphism, and by $\check{f}: S \rightarrow z \times_k \check{T}$ the morphism $(q \circ f, \check{f})$. We assume that (\mathcal{G}, b, c) is a Legendre triple at s (cf. 2.16), and f and \check{f} are finite and étale at η . Then $\check{z} = \check{\infty}$ and we have a canonical isomorphism*

$$\mathfrak{F}^{(z, \check{z})}(f_*\mathcal{G}) \xrightarrow{\sim} \check{f}_* \left(\mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{H} \left(-\frac{1}{2} \frac{dc}{db} \right) \otimes \mathcal{Q} \right). \tag{3.7.1}$$

The following diagram summarizes the geometric picture of Theorem 3.7.



The proof of Theorem 3.7 is given in § 5. For $(z, \check{z}) = (\infty, \check{0})$ or $(\infty, \check{\infty})$, the result is also valid under an extra condition (Theorem 3.9).

Remark 3.8. We keep the notation of 3.7. It is clear that $\Omega_{T/k}^1$ is a free \mathcal{O}_T -module of rank one. So the \mathcal{O}_S -module $\Omega_{S/k}^1$ is of finite type, and hence free of rank one because its completion

along the closed point of S is free of rank one. In particular, dc/db is a well-defined function over η .

THEOREM 3.9. *Let S be the spectrum of a henselian discrete valuation ring of equal characteristic p , with perfect residue field, let s (respectively, η) be the closed (respectively, generic) point of S , let \mathcal{G} be a $\overline{\mathbb{Q}_\ell}$ -sheaf of rank one over η , and let $v: S \rightarrow \mathbb{P}$ and $\check{v}: S \rightarrow \check{\mathbb{P}}$ be two non-constant morphisms (with the notation of § 3.2). We put $z = v(s)$, $\check{z} = \check{v}(s)$, b and c the functions on η deduced by pull-back from the coordinates x and \check{x} of A and \check{A} respectively. We take again the notation of § 3.4 relatively to z and \check{z} , and denote by $f: S \rightarrow T$ and $\check{f}: S \rightarrow \check{T}$ the morphisms induced by v and \check{v} , respectively. We assume that the following conditions are satisfied:*

- (i) (\mathcal{G}, b, c) is a Legendre triple at s (cf. 2.16);
- (ii) f and \check{f} are finite and étale at η ;
- (iii) $z = \infty$.

Moreover, we assume that one of the following conditions is satisfied:

- (iv) $\check{z} = \check{0}$ and all of the slopes of $f_*(\mathcal{G})$ are < 1 ;
- (iv') $\check{z} = \check{\infty}$ and all of the slopes of $f_*(\mathcal{G})$ are > 1 .

Then we have a canonical isomorphism

$$\mathfrak{F}^{(z, \check{z})}(f_*\mathcal{G}) \xrightarrow{\sim} \check{f}_* \left(\mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K} \left(-\frac{1}{2} \frac{dc}{db} \right) \otimes \mathcal{Q} \right). \tag{3.9.1}$$

The proof of Theorem 3.9 is given in § 5.

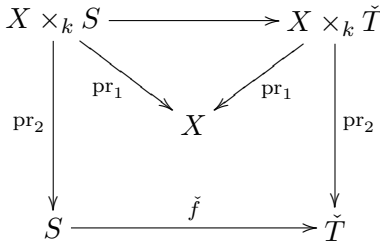
Remark 3.10. We keep the notation of 3.9, and put $\text{sw}(f_*\mathcal{G})$ and $\text{rk}(f_*\mathcal{G})$ the Swan conductor and the rank of $f_*(\mathcal{G})$. If conditions (i), (ii) and (iii) are satisfied, if f is tamely ramified and if $\text{sw}(f_*\mathcal{G}) \neq \text{rk}(f_*\mathcal{G})$, then one of the conditions (iv) or (iv') is satisfied (cf. (5.2.2) and [Katz88, 1.13]).

4. Nearby cycles and blow-up of the diagonal

4.1 We keep the notation of § 3.2. Let $X = \text{Spec}(B)$ be a smooth connected affine curve over k , $s \in X(k)$, U be the open subscheme $X - \{s\}$ of X , $t \in B$ be a local parameter at s which is invertible on U . Let $g: X \rightarrow \mathbb{P}$ and $\check{g}: X \rightarrow \check{\mathbb{P}}$ be two non-constant k -morphisms, $z = g(s)$, $\check{z} = \check{g}(s)$. We assume that g and \check{g} are generically étale and that $g(U) \subset A$ and $\check{g}(U) \subset \check{A}$. The coordinates x and \check{x} of A and \check{A} define two sections in $\Gamma(U, \mathcal{O}_X) = B_t$, that we denote also by x and \check{x} .

We take again the notation of § 3.4 relatively to the points z and \check{z} . Let $S = \text{Spec}(B^h)$ be the henselization of X at s , $h: S \rightarrow X$ be the canonical morphism, $f: S \rightarrow T$ and $\check{f}: S \rightarrow \check{T}$ be the morphisms induced by g and \check{g} , respectively. We denote (abusively) by t the uniformizer of B^h image of $t \in B$ and by s the closed point of S . Let η be the generic point of S , $\bar{\eta}$ (respectively, \bar{s}) be a geometric point of S above η (respectively, s). We denote by b and c the images of x and \check{x} , respectively, by the canonical homomorphism $B_t \rightarrow B_t^h$. Recall from Remark 3.8 that dc/db is a well-defined function over η .

We denote by pr_1 and pr_2 the canonical projections of $X \times_k S$ or $X \times_k \check{T}$.



We consider the sheaf $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $P \times_k \check{P}$, and denote also by $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ its pull-back by $g \times \check{h}$ over $X \times_k \check{T}$, and by $\overline{\mathcal{L}}_{\psi_0}(xc)$ its pull-back by $g \times (\check{h} \circ \check{f})$ over $X \times_k S$. These notation are coherent with our conventions, and do not lead to any ambiguity.

4.2 Let \mathcal{G} be a smooth $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one over U , $\mathcal{G}_!$ be the extension by 0 of \mathcal{G} to X , $\mathcal{G}_\eta = \check{h}_U^*(\mathcal{G})$. The purpose of this section is to study the complex of vanishing cycles

$$\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$$

in $\mathbf{D}_c^b(X \times_k \check{T}, \overline{\mathbb{Q}}_\ell)$, relatively to the projection $\text{pr}_2: X \times_k \check{T} \rightarrow \check{T}$. By [Del77, [Th. finitude] 3.7], $(1 \times \check{f})^*(\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})))$ is canonically isomorphic to the complex of vanishing cycles

$$\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(xc))$$

in $\mathbf{D}_c^b(X \times_k \eta, \overline{\mathbb{Q}}_\ell)$, relatively to the projection $\text{pr}_2: X \times_k S \rightarrow S$.

PROPOSITION 4.3. *Assume that $(\mathcal{G}, x, \check{x})$ is a Legendre triple at s (cf. 2.16).*

- (i) *The complex $\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(xc))$ is supported at $s \times_k \eta$ and has cohomology only in degree one, and the complex $\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$ is supported at $s \times_k \check{\tau}$ and has cohomology only in degree one.*
- (ii) *The sheaf $(i_s \times 1)^*(\Phi^1(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)))$ over η has a direct factor canonically isomorphic to*

$$\mathcal{D} = \mathcal{G}_\eta \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K} \left(-\frac{1}{2} \frac{dc}{db} \right) \otimes \mathcal{Q}, \tag{4.3.1}$$

where the sheaves \mathcal{K} and \mathcal{Q} are defined in § 3.3.

- (iii) *The morphism*

$$\check{f}_*(\mathcal{D}) \rightarrow (i_s \times 1)^*(\Phi^1(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) \tag{4.3.2}$$

induced by the trace morphism $\check{f}_* \check{f}^* \rightarrow \text{id}$, is injective.

Proposition 4.3(i) is due to Laumon [Lau87, 2.3.2.1 and 2.3.3.1]. We prove the first statement (which implies the second). By the t -exactness of the functor Φ (see [BBD82, 4.4.2] and [Ill94, 4.2]), it is enough to prove that the complex $\Phi(\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(xc))$ is supported on $s \times_k \eta$. Let ord be the valuation of $B_t^{\mathbb{h}}$ normalized by $\text{ord}(t) = 1$. If $\text{ord}(c) \geq 0$, the sheaf $\text{pr}_1^*(\mathcal{G}_!) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$ is smooth over $U \times_k S$, and the assertion follows from [DK73, XIII, 2.1.5]. If $\text{ord}(c) < 0$, the assertion is a consequence of [Lau87, 1.3.1.2].

Proposition 4.3(ii)–(iii) will be proved in §§ 4.15 and 4.18.

4.4 Let $X \times_k^{\text{log}} S$ be the logarithmic product of X and S over k , that is, the open subscheme of the blow-up of $X \times_k S$ along the closed point $s \times s$, obtained by removing the strict transforms

of the axes $X \times_k s$ and $s \times_k S$ (see [AS09, 4.3]). The parameter t identifies $X \times_k^{\log} S$ with the affine scheme defined by the k -algebra

$$B \otimes_k B^h[w, w^{-1}]/(t \otimes 1 - 1 \otimes t \cdot w). \tag{4.4.1}$$

We also denote by pr_1 and pr_2 the canonical projections from $X \times_k^{\log} S$ to X and S , respectively. Then pr_2 is smooth, and we have a canonical isomorphism $\text{pr}_2^{-1}(\eta) \simeq U \times_k \eta$. The strict transform of the graph $S \rightarrow X \times_k S$ of \hbar defines a closed embedding

$$\delta: S \rightarrow X \times_k^{\log} S, \tag{4.4.2}$$

whose ideal in the ring (4.4.1) is generated by $w - 1$. We put $Y = \text{pr}_2^{-1}(s)$ and $e = \delta(s) \in Y(k)$. Then Y is canonically isomorphic to the multiplicative group $\mathbb{G}_{m,k} = \text{Spec}(k[w, w^{-1}])$ and e is the neutral element 1.

4.5 We denote by \mathcal{H} the sheaf over $U \times_k \eta$ defined by

$$\mathcal{H} = \mathcal{H}om(\text{pr}_2^*(\mathcal{G}_\eta), \text{pr}_1^*(\mathcal{G})), \tag{4.5.1}$$

and consider the complex of nearby cycles

$$\Psi_\eta(\mathcal{H} \otimes \mathcal{L}_{\psi_0}(c(x - b)))$$

in $\mathbf{D}_c^b(Y \times_k \eta, \overline{\mathbb{Q}}_\ell)$, relatively to the projection $\text{pr}_2: X \times_k^{\log} S \rightarrow S$. The following proposition is a refinement of 4.3(i)–(ii). It will not be used in this article, and will be proved in § 4.19.

PROPOSITION 4.6. *Assume that $(\mathcal{G}, x, \tilde{x})$ is a Legendre triple at s (cf. 2.16) and $\tilde{z} \in \{\check{0}, \check{\infty}\}$.*

- (i) *The complex $\Psi_\eta(\mathcal{H} \otimes \mathcal{L}_{\psi_0}(c(x - b)))$ is supported on $\Sigma \times_k \eta$, where Σ is a finite subscheme of $Y = \mathbb{G}_{m,k}$, and has cohomology only in degree one.*
- (ii) *The sheaf $(i_e \times 1)^*(\Psi_\eta^1(\mathcal{H} \otimes \mathcal{L}_{\psi_0}(c(x - b))))$ over η has a direct factor canonically isomorphic to $\mathcal{K}((-1/2)(dc/db)) \otimes \mathcal{Q}$, where the sheaves \mathcal{K} and \mathcal{Q} are defined in § 3.3.*

Remark 4.7. Condition (2.13.3) contained in the definition of a Legendre triple is not necessary for Proposition 4.6(i).

4.8 We assume in the remainder of this section that $(\mathcal{G}, x, \tilde{x})$ is a Legendre triple at s . Let R be the completion of B^h , K be the fraction field of R . We identify R with the ring of power series $k[[t]]$. We denote also by b and c the images of b and c in K . By Definition 2.16, there exist \mathcal{G}_t and \mathcal{G}_w two smooth $\overline{\mathbb{Q}}_\ell$ -sheaves of rank one over U and $a \in W_{m+1}(K)$ ($m \geq 0$), satisfying the following conditions:

- (i) $\mathcal{G} \simeq \mathcal{G}_t \otimes \mathcal{G}_w$;
- (ii) \mathcal{G}_t is tamely ramified at s ;
- (iii) \mathcal{G}_w is trivialized by a cyclic extension of order p^{m+1} of U ;
- (iv) if we put $\chi = \delta_{m+1}(a) \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, then $\psi_m^{-1} \circ \chi$ is the character of the pull-back of \mathcal{G}_w to $\text{Spec}(K)$ (see § 1.11);
- (v) (a, b, c) is a Legendre triple. Let $(n, \nu(b), \nu(c))$ be its conductor (see § 2.13).

We consider the sheaves \mathcal{H}_t and \mathcal{H}_w over $U \times_k \eta$ defined by

$$\mathcal{H}_t = \mathcal{H}om(\mathrm{pr}_2^*(\tilde{h}_U^*(\mathcal{G}_t)), \mathrm{pr}_1^*(\mathcal{G}_t)), \tag{4.8.1}$$

$$\mathcal{H}_w = \mathcal{H}om(\mathrm{pr}_2^*(\tilde{h}_U^*(\mathcal{G}_w)), \mathrm{pr}_1^*(\mathcal{G}_w)), \tag{4.8.2}$$

so we have $\mathcal{H} \simeq \mathcal{H}_t \otimes \mathcal{H}_w$ (4.5.1). We take again the notation of § 2, and define $\alpha \in K$ by the equation $F^m d(a) = \alpha dt$. So we have $a \in \mathrm{fil}_n W_{m+1}(K)$, $\mathrm{ord}(t\alpha) = -n$, $2\mathrm{ord}(\alpha + cb') \geq -n + \nu(c)$ and $2\nu(b) + p\nu(c) < (p - 2)n$. In particular, we have $n - \nu(c) \geq 1$. We denote by γ the non-zero element of k defined by

$$\gamma = \frac{1}{2} \left(t^{n+1} \alpha \frac{t^{1-\nu(c)} c'}{c} \right) \pmod{tR}. \tag{4.8.3}$$

PROPOSITION 4.9. *The sheaf \mathcal{H}_t over $U \times_k \eta$ extends to a smooth sheaf over $X \times_k^{\mathrm{log}} S$ whose pull-back by δ over S is constant.*

We denote by κ_0 the generic point of Y (see § 4.4), by R_{L_0} the completion of the local ring of $X \times_k^{\mathrm{log}} S$ at κ_0 (which is a discrete valuation ring), by L_0 the fraction field of R_{L_0} , by I_K^t (respectively, $I_{L_0}^t$) the tame inertia group of K (respectively, L_0). The restriction of \mathcal{H}_t to $\mathrm{Spec}(L_0)$ is tamely ramified. Since the projections pr_1 and pr_2 of $X \times_k^{\mathrm{log}} S$ are smooth, they induce the same isomorphism $I_{L_0}^t \xrightarrow{\sim} I_K^t$. We deduce that the representation of $I_{L_0}^t$ defined by the sheaf \mathcal{H}_t is trivial, and hence the restriction of \mathcal{H}_t to $\mathrm{Spec}(L_0)$ is unramified. The first assertion follows by the Zariski–Nagata purity theorem [SGA2, X 3.4]. The second assertion is a consequence of the first and the fact that the restriction of \mathcal{H}_t to $\delta(\eta)$ is trivial.

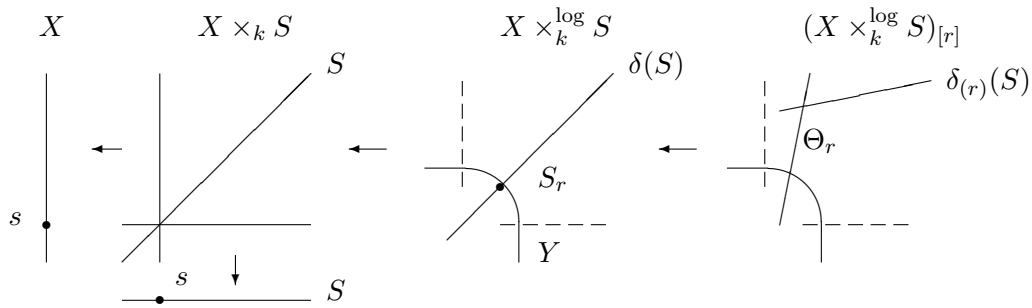
4.10 Let r be an integer greater than or equal to one, let S_r be the closed subscheme of S defined by t^r , let $(X \times_k^{\mathrm{log}} S)_{[r]}$ be the blow-up of $X \times_k^{\mathrm{log}} S$ along $\delta(S_r)$, and let $(X \times_k^{\mathrm{log}} S)_{(r)}$ be the dilatation of $X \times_k^{\mathrm{log}} S$ along δ of thickening r , that is, the open subscheme of $(X \times_k^{\mathrm{log}} S)_{[r]}$ obtained by removing the strict transform of Y or, equivalently, the maximal open subscheme of $(X \times_k^{\mathrm{log}} S)_{[r]}$ where the exceptional divisor is defined by $\mathrm{pr}_2^*(t^r)$ (see [AS09, 3.1]). We denote by Θ_r the exceptional divisor on $(X \times_k^{\mathrm{log}} S)_{(r)}$, by

$$\delta_{(r)} : S \rightarrow (X \times_k^{\mathrm{log}} S)_{(r)} \tag{4.10.1}$$

the unique lifting of δ (that is, the strict transform of δ), and abusively by pr_1 and pr_2 the projections from $(X \times_k^{\mathrm{log}} S)_{(r)}$ to X and S , respectively. Then pr_2 is smooth; the commutative diagram

$$\begin{array}{ccccc} \Theta_r & \longrightarrow & (X \times_k^{\mathrm{log}} S)_{(r)} & \longleftarrow & U \times_k \eta \\ \downarrow & & \downarrow \mathrm{pr}_2 & & \downarrow \\ S_r & \longrightarrow & S & \longleftarrow & \eta \end{array} \tag{4.10.2}$$

has Cartesian squares; and Θ_r is canonically isomorphic to the vector bundle $\mathbf{V}(\Omega_{X/k}^1(\log s) \otimes_X \mathcal{O}_{S_r}(S_r))$ over S_r (see [AS09, 4.6]).



It follows from (4.4.1) that $(X \times_k^{\log} S)_{(r)}$ is the affine scheme of ring

$$B \otimes_k B^h[\theta, (1 + 1 \otimes t^r \cdot \theta)^{-1}]/(t \otimes 1 - 1 \otimes t(1 + 1 \otimes t^r \cdot \theta)), \tag{4.10.3}$$

$\Theta_r \otimes_R k$ is the affine line $A = \text{Spec}(k[\theta])$ over k (with coordinate θ), and $\delta_{(r)}$ is defined by the equation θ . Let κ be the generic point of Θ_r , let R_L be the completion of the local ring of $(X \times_k^{\log} S)_{(r)}$ at κ (which is a discrete valuation ring), let L be the fraction field of R_L , and let

$$u: K \rightarrow L \tag{4.10.4}$$

$$v: K \rightarrow L \tag{4.10.5}$$

be the homomorphisms induced respectively by pr_1 and pr_2 . We consider L as an extension of K by v . By (4.10.3), we can identify R_L with the ring $k(\theta)[[t]]$. Then the k -homomorphisms u and v are defined by $u(t) = t(1 + t^r \theta)$ and $v(t) = t$.

PROPOSITION 4.11. Assume that $n - \nu(c) = 2r$ is even. Let Γ_η be the closed subscheme of the $U \times_k \eta$ inverse image by the morphism $\tilde{g} \times_k \eta$ of the section $\eta \rightarrow \check{A} \times_k \eta$ defined by c (which is also the closed subscheme of $U \times_k \eta$ defined by the equation $\check{x} - c$), let $\Gamma_{(r)}$ be the schematic closure of Γ_η in $(X \times_k^{\log} S)_{(r)}$. Then:

- (i) the scheme $\Gamma_{(r)}$ is quasi-finite over S , and $\delta_{(r)}(S)$ is the finite part of $\Gamma_{(r)}$;
- (ii) the sheaf $\mathcal{H} \otimes \mathcal{L}_{\psi_0}(c(x - b))$ over $U \times_k \eta$ extends to a smooth sheaf over $(X \times_k^{\log} S)_{(r)}$ whose restriction to $\Theta_r \otimes_R k = A$ is canonically isomorphic to the sheaf $\mathcal{L}_{\psi_0}(\gamma\theta^2)$.

(i) Recall first that any quasi-finite separated scheme Z over S can be decomposed canonically into a sum $Z^f \amalg Z^g$, where Z^f is finite over S (called the finite part of Z) and the special fiber of Z^g is empty. It follows from (2.9.1) and the inequality $\nu(c) < r(p - 1)$ that we have

$$\frac{u(c) - c}{t^{r+\nu(c)}c} \equiv \frac{t^{1-\nu(c)}c'}{c} \theta \pmod{tR_L}. \tag{4.11.1}$$

We deduce that the function $t^{-r-\nu(c)}(\check{x}/c - 1)$ on $U \times_k \eta$ extends to a regular function on a neighborhood of κ in $(X \times_k^{\log} S)_{(r)}$. Since $(X \times_k^{\log} S)_{(r)}$ is a smooth curve over S with an integral special fiber, $t^{-r-\nu(c)}(\check{x}/c - 1)$ extends to a regular function on $(X \times_k^{\log} S)_{(r)}$. The latter belongs to the ideal of $\Gamma_{(r)}$; so it defines a closed subscheme $\Gamma'_{(r)}$ of $(X \times_k^{\log} S)_{(r)}$ containing $\Gamma_{(r)}$. Moreover, it follows from (4.11.1) that the special fiber of $\Gamma'_{(r)}$ is the origin of $\Theta_r \otimes_R k = \text{Spec}(k[\theta])$. Therefore, the three closed subschemes $\delta_{(r)}(S) \subset \Gamma_{(r)} \subset \Gamma'_{(r)}$ of $(X \times_k^{\log} S)_{(r)}$ have the same special fiber, which implies the proposition.

(ii) It follows from 4.9 that \mathcal{H}_t extends to a smooth sheaf on $(X \times_k^{\log} S)_{(r)}$ whose restriction to $\Theta_r \otimes_R k$ is constant. The pull-back of $\mathcal{H}_w \otimes \mathcal{L}_{\psi_0}(c(x - b))$ to $\text{Spec}(L)$ corresponds to the Witt vector

$$u(a) - a + V^m(c(u(b) - b)) \in W_{m+1}(L).$$

By 2.12, the latter belongs to $W_{m+1}(R_L)$, and its residue class modulo t is equal to $V^m(\gamma\theta^2) \in W_{m+1}(k(\theta))$. Therefore, the pull-back of $\mathcal{H}_w \otimes \mathcal{L}_{\psi_0}(c(x - b))$ to $\text{Spec}(L)$ is canonically isomorphic to $\mathcal{L}_{\psi_0}(\gamma\theta^2)$. We deduce by the Zariski–Nagata purity theorem [SGA2, X 3.4] that $\mathcal{H}_w \otimes \mathcal{L}_{\psi_0}(c(x - b))$ extends to a smooth sheaf on $(X \times_k^{\log} S)_{(r)}$ whose restriction to $\Theta_r \otimes_R k = A$ is canonically isomorphic to $\mathcal{L}_{\psi_0}(\gamma\theta^2)$.

Remark 4.12. We keep the notation and assumptions of 4.11. Recall that we have the following commutative diagram.

$$\begin{array}{ccc} S & \xrightarrow{\tilde{h}} & X \\ \tilde{f} \downarrow & & \downarrow \tilde{g} \\ \tilde{T} & \xrightarrow{\tilde{h}} & \tilde{P} \end{array}$$

We consider η as a scheme over \tilde{A} by the composed morphism $\tilde{g} \circ \tilde{h} = \tilde{h} \circ \tilde{f}$. Then the closed subscheme Γ_η of $U \times_k \eta$ is canonically isomorphic to $U \times_{\tilde{A}} \eta$. The morphism δ (see (4.4.2)) induces a closed embedding $\eta \rightarrow U \times_k \eta$, which determines a connected component of $U \times_{\tilde{A}} \eta$. Proposition 4.11(i) says that only this connected component extends to a closed subscheme of $(X \times_k^{\log} S)_{(r)}$ which is finite and flat over S , namely $\delta_{(r)}(S)$; the other connected components of $U \times_{\tilde{A}} \eta$ are closed in $(X \times_k^{\log} S)_{(r)}$.

LEMMA 4.13. *Let \mathcal{X} and \mathcal{Q} be the sheaves defined in § 3.3, let γ be the element of k defined in (4.8.3), let \mathcal{Q}_γ be the $\overline{\mathbb{Q}}_\ell$ -sheaf over $\text{Spec}(k)$ defined by*

$$\mathcal{Q}_\gamma = H_c^1(A_{\bar{k}}, \mathcal{L}_{\psi_0}(\gamma\theta^2)). \tag{4.13.1}$$

- (i) *The sheaf \mathcal{Q}_γ has rank one, the $H_c^i(A_{\bar{k}}, \mathcal{L}_{\psi_0}(\gamma\theta^2))$, for $i \neq 1$, vanish, and the canonical morphism*

$$H_c^1(A_{\bar{k}}, \mathcal{L}_{\psi_0}(\gamma\theta^2)) \rightarrow H^1(A_{\bar{k}}, \mathcal{L}_{\psi_0}(\gamma\theta^2))$$

is an isomorphism.

- (ii) *If $n - \nu(c) = 2r$ is even, the sheaf $\mathcal{X}(-1/2)(dc/db) \otimes \mathcal{Q}$ over η is unramified and isomorphic to the geometrically constant sheaf \mathcal{Q}_γ .*
- (iii) *If $n - \nu(c) = 2r + 1$ is odd, the sheaf $\mathcal{X}(-1/2)(dc/db) \otimes \mathcal{Q}$ over η is tamely ramified, and its restriction to the quadratic extension $\underline{\eta} = \eta[\underline{t}]/(\underline{t}^2 - t)$ is unramified and isomorphic to the geometrically constant sheaf \mathcal{Q}_γ .*

Observe first that it is enough to prove the lemma after replacing η by $\text{Spec}(K)$.

(i) Since the sheaf $\mathcal{L}_{\psi_0}(\gamma\theta^2)$ is smooth on A and its Swan conductor at ∞ is 2, the assertion follows from the Grothendieck–Ogg–Shafarevich formula and [Del77, [Sommes trig.] 1.19 and 1.19.1].

(ii) Condition $2\text{ord}(\alpha + cb') \geq -n + \nu(c)$ implies that

$$\frac{1}{2} \left(t^{n+1} \alpha \frac{t^{1-\nu(c)} c'}{c} \right) \equiv -\frac{1}{2} (t^{2+2r} b' c') \pmod{t^{1+r+\nu(c)} R};$$

so $-(c'/(2b'))\gamma^{-1}$ is a square in K . Therefore, the sheaf $\mathcal{K}(-c'/(2b'))$ over $\text{Spec}(K)$ is unramified and isomorphic to the geometrically constant sheaf $\mathcal{K}(\gamma)$. The last assertion follows from (3.3.4).

(iii) Condition $2\text{ord}(\alpha + cb') \geq -n + \nu(c)$ implies that

$$\frac{1}{2} \left(t^{n+1} \alpha \frac{t^{1-\nu(c)} c'}{c} \right) \equiv -\frac{1}{2} (t^{3+2r} b' c') \pmod{t^{2+r+\nu(c)} R};$$

so $-(tc'/(2b'))\gamma^{-1}$ is a square in K . The proposition follows as in part (ii).

4.14 Let σ be a k -automorphism of S . We put $\delta(\sigma) : S \rightarrow X \times_k^{\text{log}} S$ the strict transform of the graph of $\hbar \circ \sigma$. The automorphism $1 \times \sigma$ of $X \times_k S$ lifts uniquely to $X \times_k^{\text{log}} S$ and the following diagram is commutative.

$$\begin{array}{ccc} S & \xrightarrow{\delta(\sigma)} & X \times_k^{\text{log}} S \\ \sigma \downarrow & & \downarrow 1 \times \sigma \\ S & \xrightarrow{\delta} & X \times_k^{\text{log}} S \end{array}$$

Hence, $1 \times \sigma$ induces an isomorphism between the dilatations of $X \times_k^{\text{log}} S$ along δ and $\delta(\sigma)$ with the same thickening.

Let σ be a k -automorphism of X such that $\sigma(x) = x$. We denote also by σ the k -automorphism of S induced by σ . The automorphism $\sigma \times 1$ of $X \times_k S$ lifts uniquely to $X \times_k^{\text{log}} S$ and the following diagram is commutative.

$$\begin{array}{ccc} S & \xrightarrow{\delta} & X \times_k^{\text{log}} S \\ & \searrow \delta(\sigma) & \downarrow \sigma \times 1 \\ & & X \times_k^{\text{log}} S \end{array}$$

Hence, $\sigma \times 1$ induces an isomorphism between the dilatations of $X \times_k^{\text{log}} S$ along δ and $\delta(\sigma)$ with the same thickening.

4.15 We can now prove Proposition 4.3(ii). Observe first that we have

$$(i_s \times 1)^*(\Psi(\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc))) = (i_s \times 1)^*(\Phi(\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc))).$$

Let $\overline{\mathcal{H}}$ be the extension by 0 of \mathcal{H} (4.5.1) to $X \times_k \eta$,

$$\mathcal{M} = \overline{\mathcal{H}} \otimes \overline{\mathcal{L}}_{\psi_0}(c(x-b)).$$

We identify \mathcal{M} with $\text{pr}_2^*(\mathcal{G}_\eta^\vee \otimes \mathcal{L}_{\psi_0}(-bc)) \otimes \text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$, where \mathcal{G}_η^\vee is the dual $\overline{\mathbb{Q}}_\ell$ -sheaf of \mathcal{G}_η over η . By [Ill94, 4.7] (applied with $Y = S$), we have a canonical isomorphism

$$\Psi_\eta(\mathcal{M}) \xrightarrow{\sim} \text{pr}_2^*(\mathcal{G}_\eta^\vee \otimes \mathcal{L}_{\psi_0}(-bc)) \otimes \Psi_\eta(\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)) \tag{4.15.1}$$

in $\mathbf{D}_c^b(X \times_k \eta, \overline{\mathbb{Q}}_\ell)$. So we are reduced to proving that the sheaf $(i_s \times 1)^*(\Psi_\eta^1(\mathcal{M}))$ over η has a direct factor isomorphic to $\mathcal{K}(-(1/2)(db/dc)) \otimes \mathcal{Q}$.

Assume first that $n - \nu(c) = 2r$ is even. It follows from 4.11(ii) and [DK73, 2.1.7.1] that we have canonical morphisms of representations of $\pi_1(\eta, \overline{\eta})$

$$\mathrm{H}_c^1(\mathbb{A}_k^1, \mathcal{L}_{\psi_0}(\gamma\theta^2)) \xrightarrow{u} \Psi_\eta^1(\mathcal{M})_{(\overline{s}, \overline{\eta})} \xrightarrow{v} \mathrm{H}^1(\mathbb{A}_k^1, \mathcal{L}_{\psi_0}(\gamma\theta^2)), \tag{4.15.2}$$

where the source of u and the target of v are considered as unramified representations of $\pi_1(\eta, \bar{\eta})$. Moreover, $v \circ u$ is the canonical morphism; so it is an isomorphism by 4.13(i). Hence, the required assertion follows by 4.13(ii).

Assume next that $n - \nu(c)$ is odd. We put $\underline{B} = B[t]/t^2 - t$, $\underline{X} = \text{Spec}(\underline{B})$, $\underline{s} \in \underline{X}(k)$ the unique point of \underline{X} above s , and denote by an underline the objects deduced from objects over X by the base change $\underline{X} \rightarrow X$. So \underline{S} is the spectrum of the henselization of the local ring of \underline{X} at \underline{s} and $\underline{\eta}$ is the generic point of \underline{S} . Let $\rho: \underline{\eta} \rightarrow \eta$ be the canonical morphism, G be the Galois group of $\underline{\eta}$ over η , that we identify with the group of X -automorphisms of \underline{X} . We consider the pull-back $\underline{\mathcal{M}}$ of \mathcal{M} over $\underline{X} \times_k \eta$, and the complex of nearby cycles $\Psi_{\eta}(\underline{\mathcal{M}})$ in $\mathbf{D}_c^b(\underline{X} \times_k \eta, \overline{\mathbb{Q}}_{\ell})$, relatively to the second projection $\underline{X} \times_k \underline{S} \rightarrow \underline{S}$. It follows from [DK73, XIII, 2.1.7.1], [Del77, [Th. finitude] 3.7] and the Hochschild–Serre spectral sequence applied to the following diagram

$$\begin{array}{ccccc} \underline{X} \times_k \underline{S} & \longrightarrow & X \times_k \underline{S} & \longrightarrow & X \times_k S \\ & \searrow & \downarrow & & \downarrow \\ & & \underline{S} & \longrightarrow & S \end{array}$$

that we have an isomorphism

$$(i_s \times 1)^*(\Psi_{\eta}(\underline{\mathcal{M}})) \simeq (\rho_*((i_{\underline{s}} \times 1)^*(\Psi_{\underline{\eta}}(\underline{\mathcal{M}}))))^{G \times G}, \tag{4.15.3}$$

where the group $G \times G$ acts on $\Psi_{\eta}(\underline{\mathcal{M}})$ via its action on $\underline{X} \times_k \underline{S}$.

We put $\underline{K} = K[t]/(t^2 - t)$. The image of (a, b, c) in $W_{m+1}(\underline{K}) \times \underline{K} \times \underline{K}$ is a Legendre triple of conductor $(2n, 2\nu(b), 2\nu(c))$ (see § 2.13). Hence, we can apply 4.11 over $(\underline{X} \times_k^{\text{log}} \underline{S})_{(r)}$ with $r = n - \nu(c)$. We deduce, as in the even case, that we have canonical morphisms of sheaves over $\underline{\eta}$

$$\mathcal{K} \left(-\frac{1}{2} \frac{db}{dc} \right) \otimes \mathcal{Q} \xrightarrow{u} (i_{\underline{s}} \times 1)^*(\Psi_{\underline{\eta}}^1(\underline{\mathcal{M}})) \xrightarrow{v} \mathcal{K} \left(-\frac{1}{2} \frac{db}{dc} \right) \otimes \mathcal{Q} \tag{4.15.4}$$

such that $v \circ u$ is the identity. We let $G \times G$ act on the sheaf $\mathcal{K}(-1/2)(db/dc) \otimes \mathcal{Q}$ over $\underline{\eta}$ through the action of the second factor G on $\underline{\eta}$. It follows from § 4.14, (4.10.3) and 4.11(ii) that u and v are $(G \times G)$ -equivariant. The required assertion follows by using (4.15.3).

4.16 Let \bar{S} be the integral closure of S in $\bar{\eta}$, \bar{s} be its closed point. We denote by $X \times_k^{\text{log}} \bar{S}$ the base change of $X \times_k^{\text{log}} S$ by the morphism $\bar{S} \rightarrow S$, which is also the logarithmic product of X and \bar{S} over k (see [AS09, 4.3]), by $\bar{h}: \bar{S} \rightarrow X$ the morphism induced by $h: S \rightarrow X$ and by

$$\bar{\delta}: \bar{S} \rightarrow X \times_k^{\text{log}} \bar{S} \tag{4.16.1}$$

the base change of δ (see (4.4.2)), which is also the strict transform of the graph of \bar{h} . For any k -automorphism σ of \bar{S} , we denote by $\bar{\delta}^{(\sigma)}: \bar{S} \rightarrow X \times_k^{\text{log}} \bar{S}$ the strict transform of the graph of $\bar{h} \circ \sigma$. Let r be an integer ≥ 1 , \bar{S}_r be the closed subscheme of \bar{S} defined by t^r . We denote by \mathcal{I}_{σ} the ideal of $\bar{\delta}^{(\sigma)}(\bar{S}_r)$ in $X \times_k^{\text{log}} \bar{S}$, by $(X \times_k^{\text{log}} \bar{S})_{[r]}^{(\sigma)}$ the blow-up of $X \times_k^{\text{log}} \bar{S}$ along \mathcal{I}_{σ} , and by $(X \times_k^{\text{log}} \bar{S})_{(r)}^{(\sigma)}$ the dilatation of $X \times_k^{\text{log}} \bar{S}$ along $\bar{\delta}^{(\sigma)}$ of thickening r , that is, the open subscheme of $(X \times_k^{\text{log}} \bar{S})_{[r]}^{(\sigma)}$ where the exceptional divisor is generated by $\text{pr}_2^*(t^r)$.

We consider η as an étale covering of $\bar{\tau}$ by \bar{f} . Observe that for any $\sigma \in \pi_1(\bar{\tau}, \bar{\eta})$, the morphism $\bar{\delta}^{(\sigma)}$ depends only on the class of σ in $\pi_1(\eta, \bar{\eta}) \backslash \pi_1(\bar{\tau}, \bar{\eta})$ (the quotient of $\pi_1(\bar{\tau}, \bar{\eta})$ by the subgroup $\pi_1(\eta, \bar{\eta})$ acting by translation on the left). Moreover, the natural action of $\pi_1(\bar{\tau}, \bar{\eta})$ on $X \times_k \bar{S}$

lifts to $X \times_k^{\log} \bar{S}$, and for any $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$, we have a commutative diagram.

$$\begin{array}{ccc}
 \bar{S} & \xrightarrow{\bar{\delta}^{(\sigma'\sigma)}} & X \times_k^{\log} \bar{S} \\
 \sigma \downarrow & & \downarrow 1 \times \sigma \\
 \bar{S} & \xrightarrow{\bar{\delta}^{(\sigma')}} & X \times_k^{\log} \bar{S}
 \end{array} \tag{4.16.2}$$

In particular, $1 \times \sigma$ induces an isomorphism from $(X \times_k^{\log} \bar{S})_{[r]}^{(\sigma'\sigma)}$ to $(X \times_k^{\log} \bar{S})_{[r]}^{(\sigma')}$, that transforms $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma'\sigma)}$ into $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma')}$.

LEMMA 4.17. *Let r be an integer greater than or equal to one and let \mathfrak{X} be the blow-up of $X \times_k^{\log} \bar{S}$ along the ideal $\mathcal{J} = \prod_{\sigma \in \pi_1(\eta, \bar{\eta}) \setminus \pi_1(\check{\tau}, \bar{\eta})} \mathcal{J}_\sigma$ (the quotient of $\pi_1(\check{\tau}, \bar{\eta})$ by the subgroup $\pi_1(\eta, \bar{\eta})$ acting by translation on the left).*

- (i) *The action of $\pi_1(\check{\tau}, \bar{\eta})$ on $X \times_k^{\log} \bar{S}$ lifts uniquely to \mathfrak{X} . For every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, there exists a unique morphism $\varphi_\sigma: \mathfrak{X} \rightarrow (X \times_k^{\log} \bar{S})_{[r]}^{(\sigma)}$ over $X \times_k^{\log} \bar{S}$.*
- (ii) *For every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, φ_σ induces an isomorphism above $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma)}$. We put $\mathfrak{X}_{(\sigma)} = \varphi_\sigma^{-1}((X \times_k^{\log} \bar{S})_{(r)}^{(\sigma)})$.*
- (iii) *For every $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$, we have $\sigma(\mathfrak{X}_{(\sigma'\sigma)}) = \mathfrak{X}_{(\sigma')}$.*
- (iv) *The scheme $\mathfrak{X}_{\bar{S}}$ is connected.*
- (v) *Assume that $n - \nu(c) = 2r$. Then for every $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$ such that $\sigma' \notin \pi_1(\eta, \bar{\eta})\sigma$, we have $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{(\sigma')} \cap \mathfrak{X}_{\bar{S}} = \emptyset$.*

(i) Since the action of $\pi_1(\check{\tau}, \bar{\eta})$ on $X \times_k^{\log} \bar{S}$ preserves the ideal \mathcal{J} , it lifts to an action on \mathfrak{X} . Since the ideal $\mathcal{J} \mathcal{O}_{\mathfrak{X}}$ is invertible, each ideal $\mathcal{J}_\sigma \mathcal{O}_{\mathfrak{X}}$ is invertible [Bou85, ch. II, § 5.6, théo. 4]. Hence, for every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, the canonical morphism $\mathfrak{X} \rightarrow X \times_k^{\log} \bar{S}$ lifts uniquely to a morphism $\varphi_\sigma: \mathfrak{X} \rightarrow (X \times_k^{\log} \bar{S})_{[r]}^{(\sigma)}$.

(ii) By the universal property of blow-ups, it is enough to prove that for any $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$, the inverse image of the ideal $\mathcal{J}_{\sigma'}$ over $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma)}$ is invertible. We denote by C the ring of the affine scheme $X \times_k^{\log} \bar{S}$ and (abusively) by t the function $\text{pr}_2^*(t) \in C$. The embedding δ (4.4.2) is defined by the equation $w - 1$ of the ring (4.4.1). Let W be the image of $w - 1$ in C . For every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, we put $W^{(\sigma)} = (1 \times \sigma)^*(W) \in C$. It follows from (4.16.2) that the closed embedding $\bar{\delta}^{(\sigma)}$ is defined by the equation $W^{(\sigma)}$. Hence, we have $\mathcal{J}_\sigma = (W^{(\sigma)}, t^r)$. For every $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$, we have by (4.4.1)

$$W^{(\sigma')} = \frac{\sigma^*(t)}{\sigma'^*(t)} (W^{(\sigma)} + 1) - 1.$$

By construction of $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma)}$, t is not a zero divisor and $W^{(\sigma)}$ is a multiple of t^r there. Therefore, the inverse image of $(W^{(\sigma')}, t^r)$ over $(X \times_k^{\log} \bar{S})_{(r)}^{(\sigma)}$ is an invertible ideal, equal to the inverse image of the invertible ideal $(t^r, (\sigma^*(t))/(\sigma'^*(t)) - 1)$ of $\mathcal{O}_{\bar{S}}$.

(iii) This follows from (4.16.2).

(iv) Let $\pi: \mathfrak{X} \rightarrow X \times_k^{\log} \bar{S}$ be the canonical morphism. Since π is proper and surjective, and since the special fiber of $X \times_k^{\log} \bar{S}$ is isomorphic to $\mathbb{G}_{m, \bar{S}}$ (see § 4.4), it is enough to prove that all

fibers of π are connected, or equivalently that the canonical morphism $\mathcal{O}_{X \times_k^{\text{log}} \bar{S}} \rightarrow \pi_*(\mathcal{O}_{\mathfrak{X}})$ is an isomorphism. We know that $\pi_*(\mathcal{O}_{\mathfrak{X}})$ is a coherent $(\mathcal{O}_{X \times_k^{\text{log}} \bar{S}})$ -algebra. Since $X \times_k^{\text{log}} \bar{S} = \text{Spec}(\mathcal{A})$ is affine, we are reduced to showing that $\mathcal{B} = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is isomorphic to \mathcal{A} . On the one hand, \mathcal{A} is a normal domain because $X \times_k^{\text{log}} \bar{S}$ is smooth over \bar{S} . On the other hand, \mathcal{B} is a domain with the same fraction field as \mathcal{A} because π is a blow-up. Since \mathcal{B} is finite over \mathcal{A} , we conclude that $\mathcal{B} \simeq \mathcal{A}$.

(v) We provide two proofs. The first uses rigid geometry. We keep the notation of part (iv). Recall that $\widehat{\mathfrak{X}}^{\text{rig}}$ is an annulus. Each open $\mathfrak{X}_{(\sigma)}$ of \mathfrak{X} defines a closed subdisk D_σ of $\widehat{\mathfrak{X}}^{\text{rig}}$. The action of $\pi_1(\check{\tau}, \bar{\eta})$ on \mathfrak{X} induces an action on $\widehat{\mathfrak{X}}^{\text{rig}}$. For every $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$, we have $\sigma(D_{\sigma'\sigma}) = D_{\sigma'}$. By 4.11(i), if $\sigma \notin \pi_1(\eta, \bar{\eta})$, then D_σ is not contained in D_{id} (cf. Remark 4.12). Hence, for $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$ such that $\sigma' \notin \pi_1(\eta, \bar{\eta})\sigma$, the disks D_σ and $D_{\sigma'}$ are disjoint, which implies the proposition.

The second proof is as follows. For every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, we know, by parts (ii), (iii) and §4.10, that $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{\bar{s}}$ is an affine line over \bar{s} . Let $\sigma, \sigma' \in \pi_1(\check{\tau}, \bar{\eta})$ be such that $\sigma' \notin \pi_1(\eta, \bar{\eta})\sigma$. If $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{(\sigma')} \cap \mathfrak{X}_{\bar{s}}$ is not empty, it is dense open in both $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{\bar{s}}$ and $\mathfrak{X}_{(\sigma')} \cap \mathfrak{X}_{\bar{s}}$. So the projective completions of these two affine lines are equal; we denote it by P . We claim that the strict transform of $\bar{\delta}^{(\sigma')}(\bar{S})$ in \mathfrak{X} is not contained in $\mathfrak{X}_{(\sigma)}$, while it is clearly contained in $\mathfrak{X}_{(\sigma')}$. Indeed, we are reduced by part (iii) and (4.16.2) to the case where $\sigma = \text{id}$ and $\sigma' \notin \pi_1(\eta, \bar{\eta})$; then the claim follows from part (ii) and 4.11(i) (cf. Remark 4.12). We conclude that $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{\bar{s}}$ and $\mathfrak{X}_{(\sigma')} \cap \mathfrak{X}_{\bar{s}}$ are different. Then $P = (\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{\bar{s}}) \cup (\mathfrak{X}_{(\sigma')} \cap \mathfrak{X}_{\bar{s}})$; in particular, P is open in $\mathfrak{X}_{\bar{s}}$. Since P is projective over \bar{s} , it is also closed in $\mathfrak{X}_{\bar{s}}$. Therefore P is a connected component of $\mathfrak{X}_{\bar{s}}$. Hence, $\mathfrak{X}_{\bar{s}}$ is not connected as it contains also the strict transform of the special fiber $Y \times_k \bar{s}$ of $X \times_k^{\text{log}} \bar{S}$ (see §4.4). We get a contradiction with part (iv).

4.18 We can now prove Proposition 4.3(iii). Assume first that $n - \nu(c) = 2r$ is even. We constructed in §4.15 canonical morphisms of representations of $\pi_1(\eta, \bar{\eta})$

$$\mathcal{D}_{\bar{\eta}} \xrightarrow{u} (\Phi^1(\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})))_{(\bar{s}, \bar{\eta})} \xrightarrow{v} \mathcal{D}_{\bar{\eta}},$$

such that $v \circ u$ is the identity. As a $\overline{\mathbb{Q}_\ell}$ -vector space, $\mathcal{D}_{\bar{\eta}}$ corresponds to the contribution of the nearby cycle complex of $\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$ over the dilatation $(X \times_k^{\text{log}} S)_{(r)}$. We consider the sheaf $\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$ over $X \times_k \bar{\eta}$, equipped with the action of $\pi_1(\check{\tau}, \bar{\eta})$ by transport of structure, and the blow-up \mathfrak{X} of $X \times_k^{\text{log}} \bar{S}$ defined in 4.17. Then $\mathcal{D}_{\bar{\eta}}$ corresponds also to the contribution of the nearby cycle complex of $\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$ over the open subscheme $\mathfrak{X}_{(\text{id})}$ of \mathfrak{X} (cf. 4.17(ii)). By 4.17(iii), for every $\sigma \in \pi_1(\check{\tau}, \bar{\eta})$, $\sigma \circ u$ is the contribution of the nearby cycle complex of $\text{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(xc)$ over $\mathfrak{X}_{(\sigma)}$. If $\sigma \notin \pi_1(\eta, \bar{\eta})$, then we have $\mathfrak{X}_{(\sigma)} \cap \mathfrak{X}_{(\text{id})} \cap \mathfrak{X}_{\bar{s}} = \emptyset$ by 4.17(v), and hence $v \circ \sigma \circ u = 0$, which implies the required assertion.

Assume next that $n - \nu(c)$ is odd, and consider as in §4.15 the base change $\underline{X} \rightarrow X$ obtained by taking a square root of t . We keep the same notation; moreover, we denote by $\underline{\mathcal{G}}$ the pull-back of \mathcal{G} to \underline{U} and by $\underline{\mathcal{G}}_1$ the extension by zero of $\underline{\mathcal{G}}$ to \underline{X} . We consider the complex of vanishing cycles $\Phi(\text{pr}_1^*(\underline{\mathcal{G}}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$ in $\mathbf{D}_c^b(\underline{X} \times_k \check{\tau}, \overline{\mathbb{Q}_\ell})$, relatively to the projection $\text{pr}_2: \underline{X} \times_k \check{T} \rightarrow \check{T}$. It follows from the even case that the morphism of sheaves over $\check{\tau}$

$$\check{f}_*(\rho_*(\rho^* \mathcal{D})) \rightarrow (i_{\check{s}} \times_k 1)^* \Phi^1(\text{pr}_1^*(\underline{\mathcal{G}}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$$

induced by the trace morphism $\check{f}_* \rho_* \rho^* \check{f}^* \rightarrow \text{id}$, is injective. Moreover, by the definition of \mathcal{D} in

this case (see § 4.15), we have a commutative diagram

$$\begin{array}{ccc}
 \rho_*(\rho^*\mathcal{D}) & \longrightarrow & \check{f}^*((i_s \times_k 1)^*\Phi^1(\mathrm{pr}_1^*(\mathcal{G}_1) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) \\
 \uparrow & & \uparrow \\
 \mathcal{D} & \longrightarrow & \check{f}^*((i_s \times_k 1)^*\Phi^1(\mathrm{pr}_1^*(\mathcal{G}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x})))
 \end{array}$$

where the vertical arrows are induced by the adjunction map $\mathrm{id} \rightarrow \rho_*\rho^*$ and [DK73, XIII, 2.1.7.1]. The required assertion follows since $\mathcal{D} \rightarrow \rho_*(\rho^*\mathcal{D})$ is injective.

4.19 We can now prove proposition 4.6, which will not be used in the remainder of this article.

(i) By the t -exactness of the functor Ψ (see [BBD82, 4.4.2] and [Ill94, 4.2]), it is enough to prove the first statement, which amounts to proving that $\Psi_\eta(\mathrm{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\psi_0}(cx))$ is supported on a finite subgroup of $Y \otimes_k \bar{k} = \mathbb{G}_{m,\bar{k}}$. We may assume k algebraically closed. By 4.9, we may reduce to the case where $\mathcal{G} = \mathcal{G}_w$. Since \mathcal{G} is trivialized by a cyclic extension of degree p^{m+1} of U , we may further reduce to the case where \mathcal{G} is a locally constant sheaf of Λ -modules of rank one over U , and Λ is a finite field of characteristic ℓ . We fix injective homomorphisms $\bar{\psi}_i: \mathbb{Z}/p^{i+1}\mathbb{Z} \rightarrow \Lambda^\times$ ($i \geq 0$) such that for any $z \in \mathbb{F}_p$, we have $\bar{\psi}_i(p^i z) = \bar{\psi}_0(z)$. For every point $y \in Y(k)$, we denote by H the henselization of $X \times_k^{\mathrm{log}} S$ at y , that we consider as an S -scheme by the morphism induced by pr_2 . We put

$$\rho(y) = \varphi_s(H, H_\eta, \mathrm{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(xc) \mid H_\eta) \tag{4.19.1}$$

the invariant defined in (A.12.2). Since the pull-back of $\mathrm{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(xc)$ to H_η is not constant, we have

$$\Psi_\eta^0(\mathrm{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(xc))_y = 0. \tag{4.19.2}$$

Then by A.13, it is enough to prove that the support of the function $\rho(y)$ is a finite subgroup of $Y = \mathbb{G}_{m,k}$.

Let κ_0 be the generic point of Y (see § 4.4), R_{L_0} be the completion of the local ring of $X \times_k^{\mathrm{log}} S$ at κ_0 (which is a discrete valuation ring), L_0 be the fraction field of R_{L_0} ,

$$\begin{aligned}
 u: K &\rightarrow L_0, \\
 v: K &\rightarrow L_0
 \end{aligned}$$

be the homomorphisms induced respectively by pr_1 and pr_2 . We consider L_0 as an extension of K by v . By (4.4.1), we can identify R_{L_0} with the ring $k(w)[[t]]$. Then the k -homomorphisms u and v are defined by $u(t) = tw$ and $v(t) = t$. We have $u(c)/c \equiv w^{\mathrm{ord}(c)} \pmod{tR_{L_0}}$ and $u(b')/b' \equiv w^{\mathrm{ord}(b')} \pmod{tR_{L_0}}$. Since $\mathrm{ord}(t^{n+1}cb') = 0$ (see § 2.13) and $\mathrm{ord}(c) \neq 0$ by assumption, then $t^{n+1}(c - u(c))u(b')$ is a unit of R_{L_0} . We denote by P the reduction of $t^{n+1}(c - u(c))u(b')$ in $k(w)$, and by λ the reduction of $t^{n+1}cb'$ in k . Then we have $P = \lambda(1 - w^{\mathrm{ord}(c)})w^{\mathrm{ord}(b')}$. It is enough to prove that for any $y \in Y(k)$, we have

$$\rho(y) = -\mathrm{ord}_y(P). \tag{4.19.3}$$

The pull-back of $\mathrm{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(cx)$ to $\mathrm{Spec}(L_0)$ corresponds to the Witt vector $u(a) + V^m(cu(b)) \in W_{m+1}(L_0)$. Since $\mathrm{ord}(bc) = -n - \nu(b)$ and $\mathrm{ord}(a + cb') \geq -n$, we have $u(a) + V^m(cu(b)) \in \mathrm{fil}_{n+\nu(b)}W_{m+1}(L_0)$ and

$$F^m d(u(a) + V^m(cu(b))) \equiv (c - u(c))d(u(b)) + u(b)dc \pmod{t^{-n+1}\Omega_{R_{L_0}}^1(\mathrm{log})}. \tag{4.19.4}$$

Let (S', η', s') be a finite covering of (S, η, s) such that $(H, H_\eta, \text{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(cx)|_{H_\eta})_{S'}$ is stable (Proposition A.9), R' be the completion of the local ring of S' , K' be the fraction field of R' , $R'_{L'_0} = R_{L_0} \otimes_R R'$, L'_0 be the fraction field of $R'_{L'_0}$. After replacing (S', η', s') by a finite covering (Proposition A.9(i)), we may assume that η' is inseparable over η and the image of dt by the canonical morphism $\Omega_{\eta'/k}^1 \rightarrow \Omega_{\eta/k}^1$ vanishes. Since we have $d(u(b)) = u(b')(w dt + t dw)$ and $dc = c' dt$ in $\Omega_{L'_0}^1$, then the following relation holds in $\Omega_{L'_0}^1$

$$(c - u(c)) d(u(b)) + u(b) dc = t(c - u(c))u(b') dw. \tag{4.19.5}$$

It follows that the Swan conductor of the pull-back of $\text{pr}_1^*(\mathcal{G}) \otimes \mathcal{L}_{\bar{\psi}_0}(cx)$ to $\text{Spec}(L'_0)$ is $[K' : K]n$, and its refined Swan is the residue class of $t(c - u(c))u(b') dw$ in $t^{-n}\Omega_{R'_{L'_0}}^1(\log) \otimes_{R'} k$, which is equal to $t^{-n}P dw$. Equation (4.19.3) is thus proved.

(ii) The proof is similar to that of Proposition 4.3(ii), given in § 4.15.

5. Proofs of Theorems 3.7 and 3.9

5.1 We observe first that, by 3.6(ii), we may reduce Theorems 3.7 and 3.9 to the case where the residue field of S at s is k (in particular, we have $\tilde{f} = \check{f}$ in Theorem 3.7). Hence, in this section, we denote by S the spectrum of a henselian discrete valuation ring of equal characteristic p , with residue field k , by s (respectively, η) the closed (respectively, generic) point of S , by \mathcal{G} a $\overline{\mathbb{Q}_\ell}$ -sheaf of rank one over η , and by $v: S \rightarrow \mathbb{P}$ and $\check{v}: S \rightarrow \check{\mathbb{P}}$ two non-constant morphisms (with the notation of § 3.2). We put $z = v(s)$, $\check{z} = \check{v}(s)$, b and c the functions on η deduced by pull-back from the coordinates x and \check{x} of \mathbb{A} and $\check{\mathbb{A}}$ respectively. We take again the notation of § 3.4 relatively to z and \check{z} , and denote by $f: S \rightarrow T$ and $\check{f}: S \rightarrow \check{T}$ the morphisms induced by v and \check{v} , respectively. We assume that (\mathcal{G}, b, c) is a Legendre triple at s (Definition 2.16), and f and \check{f} are finite and étale at η .

We denote by R the completion of the ring of S , by K the fraction field of R , and also by b and c the images of b and c in K . By Definition 2.16, there exist \mathcal{G}_t and \mathcal{G}_w two $\overline{\mathbb{Q}_\ell}$ -sheaves of rank one over η , satisfying the following conditions:

- (i) $\mathcal{G} \simeq \mathcal{G}_t \otimes \mathcal{G}_w$;
- (ii) \mathcal{G}_t is tamely ramified;
- (iii) \mathcal{G}_w is trivialized by a cyclic extension of order p^{m+1} of η ($m \geq 0$);
- (iv) the pull-back of (\mathcal{G}_w, b, c) over $\text{Spec}(K)$ is a Legendre triple in the sense of § 2.15.

PROPOSITION 5.2. *We keep the assumptions of § 5.1, and assume moreover that one of the following conditions is satisfied:*

- (i) $z \in \mathbb{A}$;
- (ii) $(z, \check{z}) = (\infty, \check{0})$ and all of the slopes of $f_*(\mathcal{G})$ are < 1 ;
- (iii) $(z, \check{z}) = (\infty, \check{\infty})$ and all of the slopes of $f_*(\mathcal{G})$ are > 1 .

Then the rank of $\mathfrak{F}^{(z, \check{z})}(f_*\mathcal{G})$ is equal to the degree of \check{f} , and in case (i), we have $\check{z} = \check{\infty}$.

Let t be a uniformizer, ord be the valuation of K normalized by $\text{ord}(t) = 1$. We put $b' = db/dt$. Since (\mathcal{G}, b, c) is a Legendre triple, we have by §§ 2.5 and 2.13,

$$-\text{ord}(c) = \text{sw}(\mathcal{G}) + \text{ord}(tb'/b) + \text{ord}(b). \tag{5.2.1}$$

If $z \in A(k)$, then we have $\check{z} = \infty$ since $\text{sw}(\mathcal{G}) \geq 1$. In this case, we may replace b by $b - x(z)$ in the equation above. In general, we deduce by [Ser68, VI, §2] that we have

$$-\text{ord}(c) = \begin{cases} \text{sw}(f_*\mathcal{G}) + \text{rk}(f_*\mathcal{G}) & \text{if } z \in A(k), \\ \text{sw}(f_*\mathcal{G}) - \text{rk}(f_*\mathcal{G}) & \text{if } z = \infty. \end{cases} \tag{5.2.2}$$

Hence, we have

$$\text{deg}(\check{f}) = \begin{cases} \text{sw}(f_*\mathcal{G}) + \text{rk}(f_*\mathcal{G}) & \text{if } (z, \check{z}) \in A \times \infty, \\ \text{sw}(f_*\mathcal{G}) - \text{rk}(f_*\mathcal{G}) & \text{if } (z, \check{z}) = (\infty, \infty), \\ \text{rk}(f_*\mathcal{G}) - \text{sw}(f_*\mathcal{G}) & \text{if } (z, \check{z}) = (\infty, \check{0}). \end{cases} \tag{5.2.3}$$

The proposition follows from (5.2.3) and [Lau87, 2.4.3], for which we give a new proof in Proposition B.6.

5.3 We identify S with the henselization of the affine line $\mathbb{A}_k^1 = \text{Spec}(k[u])$ at the origin 0 , and put $\mathbb{G}_{m,k} = \text{Spec}(k[u, u^{-1}])$ and $\check{h}: S \rightarrow \mathbb{A}_k^1$ the canonical morphism. By Kummer theory, \mathcal{G}_t is the pull-back of a smooth $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one, \mathcal{M} over $\mathbb{G}_{m,k}$, tamely ramified at 0 and ∞ (see [Lau87, 2.2.2.1]). On the other hand, there exists a connected affine elementary étale neighborhood $(X, s) \rightarrow (\mathbb{A}_k^1, 0)$ satisfying the following properties. Let $B = \Gamma(X, \mathcal{O}_X)$, U be the open $X - \{s\}$ of X , $\check{h}: S \rightarrow X$ be the unique morphism lifting \check{h} . Then:

- (a) \mathcal{G}_w is the pull-back of a smooth $\overline{\mathbb{Q}}_\ell$ -sheaf of rank one, $\tilde{\mathcal{G}}_w$ over U , trivialized by a cyclic extension of order p^{m+1} of U (by using Artin–Schreier–Witt theory);
- (b) there exist $\tilde{b}, \tilde{c} \in \Gamma(U, \mathcal{O}_X)$ such that $\check{h}_U^*(\tilde{b}) = b$ and $\check{h}_U^*(\tilde{c}) = c$;
- (c) there exists $t \in B$, which is a parameter at s and invertible on U .

We denote by $\tilde{\mathcal{G}}_t$ the pull-back of \mathcal{M} to U , by $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_t \otimes \tilde{\mathcal{G}}_w$, by $g: X \rightarrow \mathbb{P}$ and $\check{g}: X \rightarrow \check{\mathbb{P}}$ the k -morphisms such that $g_A^*(x) = \tilde{b}$ and $\check{g}_A^*(\check{x}) = \tilde{c}$.

$$\begin{array}{ccccc} \check{T} & \xleftarrow{\check{f}} & S & \xrightarrow{f} & T \\ \check{h} \downarrow & & \downarrow \check{h} & & \downarrow h \\ \check{\mathbb{P}} & \xleftarrow{\check{g}} & X & \xrightarrow{g} & \mathbb{P} \end{array} \tag{5.3.1}$$

By construction, $(\tilde{\mathcal{G}}, \tilde{b}, \tilde{c})$ is a Legendre triple at s , and we can apply 4.3.

5.4 With the notation of 4.1 and 4.2, we have a canonical isomorphism over $\check{\tau}$

$$(i_s \times 1)^*(\Phi^1(\text{pr}_1^*(\tilde{\mathcal{G}}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) \simeq \mathfrak{F}^{(z, \check{z})}(f_*\mathcal{G}). \tag{5.4.1}$$

It is a consequence of the functorial properties of the complex of nearby cycles and the fact that $\check{h}: S \rightarrow X$ is universally locally acyclic and $f: S \rightarrow T$ is finite. Then Propositions 4.3 and 5.2 imply Theorems 3.7 and 3.9.

6. Review of Stiefel–Whitney classes

6.1 In this section, K denotes a field of characteristic $\neq 2$, \overline{K} a separable closure and G_K the Galois group of \overline{K} over K . We denote by 1_K the trivial representation of G_K . By Kummer theory, $H^1(K, \mathbb{Z}/2\mathbb{Z})$ is identified with the group $K^\times/K^{\times 2}$. For $a \in K^\times/K^{\times 2}$ (or in K^\times), we denote by $\{a\}$ the associated element of $H^1(K, \mathbb{Z}/2\mathbb{Z})$, and by $\kappa_a: G_K \rightarrow \{\pm 1\}$ its image by the isomorphism $H^1(K, \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(G_K, \{\pm 1\})$ (which is also the character induced by the

quadratic extension $K(\sqrt{a})$ of K). For $a, b \in K^\times/K^{\times 2}$, we denote by $\{a, b\}$ the cup-product $\{a\} \cup \{b\}$ in $H^2(K, \mathbb{Z}/2\mathbb{Z})$.

6.2 For a non-degenerate quadratic form $\mathcal{Q} = \mathcal{Q}(X_1, \dots, X_n)$ of rank n over K , we denote by $w_m(\mathcal{Q}) \in H^m(K, \mathbb{Z}/2\mathbb{Z})$ ($m \geq 0$) its m th Stiefel–Whitney class and by

$$w(\mathcal{Q}) = 1 + w_1(\mathcal{Q}) + \dots \in H^*(K, \mathbb{Z}/2\mathbb{Z}) = \prod_m H^m(K, \mathbb{Z}/2\mathbb{Z})$$

its total Stiefel–Whitney class [Ser84, 1.2]. Recall that, if $\mathcal{Q} \sim a_1X_1^2 + \dots + a_nX_n^2$, where $a_i \in K^\times$, then we have $w(\mathcal{Q}) = \prod_i (1 + \{a_i\})$. If $d \in K^\times/K^{\times 2}$ is the discriminant of \mathcal{Q} , we have $w_1(\mathcal{Q}) = \{d\}$.

6.3 Let V be a finite-dimensional complex vector space, equipped with a non-degenerate quadratic form \mathcal{Q} , $\rho: G_K \rightarrow \mathbf{O}(V, \mathcal{Q})$ be a continuous orthogonal representation of G_K (i.e. the kernel of ρ is open). Deligne [Del76, 1.3 and 5.1] associated to (V, ρ) Stiefel–Whitney classes $w_m(V) \in H^m(K, \mathbb{Z}/2\mathbb{Z})$ ($m \geq 0$). The class $w_1(V)$ is identified via the isomorphism $H^1(K, \mathbb{Z}/2\mathbb{Z}) = \text{Hom}(G_K, \{\pm 1\})$ with the character $\det(V): G_K \rightarrow \{\pm 1\}$. The total Stiefel–Whitney class

$$w(V) = 1 + w_1(V) + \dots \in H^*(K, \mathbb{Z}/2\mathbb{Z})$$

satisfies the following properties.

- (i) If V is an orthogonal direct sum of two subrepresentations V' and V'' , then $w(V) = w(V')w(V'')$.
- (ii) If W is a totally isotropic invariant subspace of V , the quadratic form \mathcal{Q} on V induces a quadratic form on W^\perp/W , and a duality between W and V/W^\perp , and hence a quadratic form on $W \oplus V/W^\perp$; then we have

$$w(V) = w(W \oplus V/W^\perp)w(W^\perp/W) = (1 + \{-1\})^{\dim W} w(W^\perp/W). \tag{6.3.1}$$

6.4 Let L be a finite separable extension of K contained in \overline{K} and let G_L be the Galois group of \overline{K} over L . The discriminant of L over K , $d_{L/K} \in K^\times/K^{\times 2}$, is by definition the discriminant of the quadratic form $\text{Tr}_{L/K}(x^2)$, for $x \in L$. For $\alpha \in L^\times$, the quadratic form $\text{Tr}_{L/K}(\alpha x^2)$, for $x \in L$, has discriminant $d_{L/K} N_{L/K}(\alpha)$, where $N_{L/K}(\alpha)$ is the norm of α . We denote by $w(L, \text{Tr}_{L/K}(\alpha x^2))$ its total Stiefel–Whitney class.

For a complex character χ of G_L^{ab} (or of G_L), we denote by $N_{L/K}(\chi)$ the composition of χ with the transfer $G_K^{\text{ab}} \rightarrow G_L^{\text{ab}}$. For any finite-dimensional complex representation V of G_L , we have

$$\det(\text{Ind}_{G_L}^{G_K} V) = \kappa_{d_{L/K}}^{\dim(V)} \cdot N_{L/K}(\det V). \tag{6.4.1}$$

This follows from [Del73, 1.2] and the fact that $\det(\text{Ind}_{G_L}^{G_K} 1_L) = \kappa_{d_{L/K}}$ (see [Ser84, 1.4]).

PROPOSITION 6.5. *Let $L = K(t)$ be a finite separable extension of K contained in \overline{K} , of degree n , generated by an element $t \in L$, let G_L be the Galois group of \overline{K} over L , and let $f(X) \in K[X]$ be the minimal polynomial of t . We put $D = f'(t) \in L^\times$ and κ_D the associated quadratic character*

(see § 6.1). Then we have

$$d_{L/K} = (-1)^{\binom{n}{2}} N_{L/K}(D) \in K^\times / K^{\times 2}, \tag{6.5.1}$$

$$w_2(\text{Ind}_{G_L}^{G_K} \kappa_D) = \binom{n}{4} \{-1, -1\} + \{d_{L/K}, 2\}. \tag{6.5.2}$$

Recall [Ser68, III, Lemma 2] that we have

$$\text{Tr}_{L/K}(D^{-1}t^i) = \begin{cases} 0 & \text{if } 0 \leq i \leq n-2, \\ 1 & \text{if } i = n-1. \end{cases} \tag{6.5.3}$$

Therefore, the discriminant of the quadratic form $\text{Tr}_{L/K}(D^{-1}x^2)$ over K is $(-1)^{\binom{n}{2}} \in K^\times / K^{\times 2}$, which implies (6.5.1). By [Ser84, § 4, Theorem 1' and § 1, 1.5], we have

$$w_2(\text{Ind}_{G_L}^{G_K} \kappa_D) = w_2(L, \text{Tr}_{L/K}(D^{-1}x^2)) + \{d_{L/K}, 2\}. \tag{6.5.4}$$

Let m be the largest integer such that $2m \leq n$. We denote by W the sub- K -vector space of L generated by $1, t, \dots, t^{m-1}$, and by W^\perp the orthogonal subspace relatively to the quadratic form $\text{Tr}_{L/K}(D^{-1}x^2)$. By (6.5.3), W is totally isotropic; we have $W^\perp/W = Kt^m$ if $n = 2m + 1$, and $W = W^\perp$ otherwise. We deduce by (6.3.1) and (6.5.3) that we have

$$w(L, \text{Tr}_{L/K}(D^{-1}x^2)) = (1 + \{-1\})^m. \tag{6.5.5}$$

Equation (6.5.2) follows from (6.5.4), (6.5.5) and the fact that $\binom{m}{2} \equiv \binom{n}{4} \pmod 2$.

7. Refined logarithmic different

This short section is independent of the rest of the article, and does not use conventions 1.11.

7.1 Let K be a complete discrete valuation field, with residue field k , L be a finite separable extension of K . We denote by \mathcal{O}_K (respectively, \mathcal{O}_L) the valuation ring of K (respectively, L), by \mathfrak{m}_K (respectively, \mathfrak{m}_L) the maximal ideal of \mathcal{O}_K (respectively, \mathcal{O}_L), and by k_L the residue field of \mathcal{O}_L . Recall that the different $\mathcal{D}_{L/K}$ of L over K is the ideal of \mathcal{O}_L such that the inverse $\mathcal{D}_{L/K}^{-1}$ is the maximal fractional ideal \mathfrak{a} of L satisfying the condition $\text{Tr}_{L/K}(\mathfrak{a}) \subset \mathcal{O}_K$. Following Kato [Kato87b, 2.1], we define the *logarithmic different* of L over K , denoted by $\mathcal{D}_{L/K}^{\text{log}}$, to be the fractional ideal of L given as

$$\mathcal{D}_{L/K}^{\text{log}} = \mathfrak{m}_K^{-1} \mathfrak{m}_L \mathcal{D}_{L/K}. \tag{7.1.1}$$

In fact, the ideal $(\mathcal{D}_{L/K}^{\text{log}})^{-1}$ is the maximal fraction ideal \mathfrak{a} of L such that $\text{Tr}_{L/K}(\mathfrak{m}_L \mathfrak{a}) \subset \mathfrak{m}_K$, and also the minimal fraction ideal \mathfrak{a} of L such that $\text{Tr}_{L/K}(\mathfrak{a}) \supset \mathcal{O}_K$. We say that a generator δ of the \mathcal{O}_L -module $\mathcal{D}_{L/K}^{\text{log}}$ is a *refined logarithmic different* of L over K if, for any $a \in \mathcal{O}_L$, we have

$$\text{Tr}_{L/K}(\delta^{-1}a) = \text{Tr}_{k_L/k}(\bar{a}) \pmod{\mathfrak{m}_K}. \tag{7.1.2}$$

Observe that δ is unique in $L^\times / 1 + \mathfrak{m}_L$.

7.2 Let M be a finite separable extension of L , $\delta_{L/K}$ (respectively, $\delta_{M/L}$) be a refined logarithmic different of L over K (respectively, of M over L). Then $\delta_{M/K} = \delta_{M/L} \delta_{L/K}$ is a refined logarithmic different of M over K .

7.3 Assume that L is totally ramified over K of degree n . Let t be a uniformizer of L , $f(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of t . Then it follows from (6.5.3) and [Ser68, III, §6] that $\delta = t^{1-n}f'(t)$ is a refined logarithmic different of L over K .

7.4 Assume that k is perfect, K has characteristic p and L is totally ramified over K . Let x (respectively, t) be a uniformizer of K (respectively, L). Then $\delta = d \log(x)/d \log(t)$ is a refined logarithmic different of L over K . Observe first that the class of $d \log(x)/d \log(t)$ in $L^\times/1 + \mathfrak{m}_L$ does not depend on the choice of x . Let $f(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of t and let n be the degree of L over K . Since L is totally ramified over K , f is an Eisenstein polynomial, and we may assume that $x = -f(0)$. Therefore, we have $dx/dt \in f'(t)(1 + \mathfrak{m}_L)$ and $tx^{-1} \in t^{1-n}(1 + \mathfrak{m}_L)$, and the assertion follows from § 7.3.

8. Local epsilon factors

8.1 In this section, K denotes a complete discrete valuation field (of equal or unequal characteristics), with finite residue field k of order $q = p^f$, \bar{K} a separable closure of K , W_K the Weil group of \bar{K} over K and I the inertia subgroup of W_K . We denote by \mathcal{O}_K the valuation ring of K , by \mathfrak{m}_K the maximal ideal of \mathcal{O}_K , by ord the valuation of K normalized by $\text{ord}(K^\times) = \mathbb{Z}$, by $\mathcal{O}_{\bar{K}}$ the integral closure of \mathcal{O}_K in \bar{K} and by \bar{k} the residue field of $\mathcal{O}_{\bar{K}}$. In the following, a representation of W_K stands for a pair (V, ρ) , where V is a finite-dimensional $\bar{\mathbb{Q}}_\ell$ -vector space and ρ is a continuous homomorphism $W_K \rightarrow \text{GL}(V)$ (i.e. an open subgroup of I acts trivially).

The quotient group W_K/I is canonically isomorphic to \mathbb{Z} , generated by Frob, the geometric Frobenius of k (i.e. the inverse of the automorphism $x \mapsto x^q$ of \bar{k}). Class field theory provides an isomorphism

$$\text{Rec}_K: K^\times \xrightarrow{\sim} W_K^{\text{ab}}, \tag{8.1.1}$$

that we normalize by mapping uniformizers of K to liftings of Frob (see [Del73, 2.3]). We use Rec_K to identify the isomorphism classes of representations of dimension one of W_K with quasi-characters (that is, continuous homomorphisms) $K^\times \rightarrow \bar{\mathbb{Q}}_\ell^\times$. If the characteristic of K is $\neq 2$, we denote the Hilbert symbol over K by

$$(\ , \)_K: K^\times/K^{\times 2} \times K^\times/K^{\times 2} \rightarrow \{\pm 1\}. \tag{8.1.2}$$

8.2 We fix a non-trivial additive character $\psi: K \rightarrow \bar{\mathbb{Q}}_\ell^\times$, and the Haar measure dx on the additive group of K such that $\int_{\mathcal{O}_K} dx = 1$. We define the *conductor* of ψ , denoted by $\text{ord}(\psi)$, to be the biggest integer n such that $\psi|_{\mathfrak{m}_K^{-n}} = 1$. Let χ be a quasi-character of K^\times . The conductor of χ , denoted by $a(\chi)$, is 0 if χ is unramified, and the smallest integer m such that $\chi(1 + \mathfrak{m}_K^m) = 1$ if χ is ramified. The Swan conductor of χ , denoted by $\text{sw}(\chi)$, is 0 if χ is unramified, and $a(\chi) - 1$ if χ is ramified.

8.3 Deligne and Langlands attached to every representation V of W_K a *local ε -factor* $\varepsilon(V, \psi) \in \bar{\mathbb{Q}}_\ell^\times$, characterized by the following conditions.

- (i) For any exact sequence of representations $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$, we have

$$\varepsilon(V, \psi) = \varepsilon(V', \psi)\varepsilon(V'', \psi). \tag{8.3.1}$$

In particular, $\varepsilon(V, \psi)$ depends only on the class of V in the Grothendieck group of representations of W_K , and we can define $\varepsilon(V, \psi)$ when V is a virtual representation of W_K .

(ii) For every finite extension L of K contained in \overline{K} , there exists a constant

$$\lambda(L/K, \psi) \in \overline{\mathbb{Q}}_\ell^\times \tag{8.3.2}$$

such that, for any representation V_L of W_L and V_K the induced representation of W_K , we have

$$\varepsilon(V_K, \psi) = \lambda(L/K, \psi)^{\dim(V_L)} \varepsilon(V_L, \psi \circ \text{Tr}_{L/K}). \tag{8.3.3}$$

(iii) If V has dimension one, corresponding to a quasi-character $\chi: K^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, then $\varepsilon(V, \psi)$ is the constant $\varepsilon(\chi, \psi)$ of the local functional equation of Tate [Del73, § 3]. Recall that if χ is unramified and $\text{ord}(\psi) = 0$, then $\varepsilon(\chi, \psi) = 1$; and if χ is ramified, then

$$\varepsilon(\chi, \psi) = \int_{K^\times} \chi^{-1}(x)\psi(x) dx. \tag{8.3.4}$$

We omitted the Haar measure dx from the notation $\varepsilon(V, \psi, dx)$, as it has been fixed in § 8.2. Following Deligne [Del73, § 5], we put, for a representation V of W_K ,

$$\varepsilon_0(V, \psi) = \det(-\text{Frob}, V^I) \varepsilon(V, \psi). \tag{8.3.5}$$

The function ε_0 clearly satisfies properties (i) and (ii) with the same constant (8.3.2).

Remarks 8.4.

(i) For any $a \in K^\times$, we have

$$\varepsilon(V, \psi(ax)) = \det(V)(a) \cdot q^{\text{ord}(a) \dim(V)} \cdot \varepsilon(V, \psi); \tag{8.4.1}$$

and similarly for ε_0 .

(ii) If L is a finite, separable and unramified extension of K contained in \overline{K} and $\text{ord}(\psi) = 0$, then $\lambda(L/K, \psi) = 1$. This follows from [Del73, 5.5.3] and the fact that $\text{ord}(\psi \circ \text{Tr}_{L/K}) = 0$.

8.5 We fix a non-trivial additive character $\psi_k: k \rightarrow \overline{\mathbb{Q}}_\ell^\times$. For a character $\chi: k^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, we denote by $\tau(\chi, \psi_k)$ the Gauss sum

$$\tau(\chi, \psi_k) = - \sum_{x \in k^\times} \chi^{-1}(x)\psi_k(x). \tag{8.5.1}$$

We have $\tau(1, \psi_k) = 1$. If the characteristic p of k is odd, we denote by $\kappa_0: k^\times \rightarrow \{\pm 1\}$ the unique character of order two, and by G_{ψ_k} the quadratic Gauss sum associated to ψ_k , defined by

$$G_{\psi_k} = \sum_{x \in k} \psi_k(x^2). \tag{8.5.2}$$

Then we have $\tau(\kappa_0, \psi_k) = -G_{\psi_k}$ and, by [Del77, [Sommes trig.], 4.4],

$$q = \kappa_0(-1)G_{\psi_k}^2. \tag{8.5.3}$$

8.6 We call a ψ_k -gauge of ψ an element $\beta \in K^\times$ such that, for any $a \in \mathcal{O}_K$, with residue class \bar{a} in k , we have

$$\psi(\beta^{-1}a) = \psi_k(\bar{a}). \tag{8.6.1}$$

Such an element β exists, is unique in $K^\times/1 + \mathfrak{m}_K$, and we have $\text{ord}(\beta) = \text{ord}(\psi) + 1$.

PROPOSITION 8.7. *Let χ be a quasi-character of K^\times , $\beta \in K^\times$ be a ψ_k -gauge of ψ , π be a uniformizer of K .*

(i) Assume that χ is at most tamely ramified (that is, $a(\chi) \leq 1$), and let $\chi_k: k^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the character defined by χ . Then we have

$$\varepsilon_0(\chi, \psi) = -\chi(\beta) q^{\text{ord}(\psi)} \tau(\chi_k, \psi_k). \tag{8.7.1}$$

(ii) Assume $p \neq 2$ and that χ is wildly ramified (that is, $a(\chi) \geq 2$). We put $n = \text{sw}(\chi)$ and let r be the smallest integer such that $2r \geq n$; so we have $n = 2r$ or $n = 2r - 1$. Let c be an element of K^\times such that, for any $x \in \mathfrak{m}_K^r$, we have

$$\chi\left(1 + x + \frac{x^2}{2}\right) = \psi(cx). \tag{8.7.2}$$

Then $\text{ord}(\beta c) = -n$ and c is unique in $K^\times / 1 + \mathfrak{m}_K^{n-r+1}$. We have

$$\varepsilon_0(\chi, \psi) = \chi^{-1}(c) \psi(c) q^{-\text{ord}(c)} \kappa_0(-1) \binom{-n}{2} G_{\psi_k}^{-n-1} \times \begin{cases} 1 & \text{if } n \text{ is odd,} \\ (-2\beta c, \pi)_K & \text{if } n \text{ is even.} \end{cases} \tag{8.7.3}$$

(i) Assume that χ is unramified. By (8.4.1), we may assume that β is a uniformizer of K and $\text{ord}(\psi) = 0$. Then we have $\varepsilon(\chi, \psi) = 1$, and both sides of (8.7.1) are equal to $-\chi(\beta)$.

Assume that χ is tamely ramified. By (8.4.1), we may assume that $\beta = 1$ and $\text{ord}(\psi) = -1$. Then it follows from (8.3.4) that we have $\varepsilon(\chi, \psi) = \varepsilon_0(\chi, \psi) = -q^{-1} \tau(\chi_k, \psi_k)$.

(ii) Since $3r \geq a(\chi)$, for any $x, y \in \mathfrak{m}_K^r$, we have

$$\chi\left(1 + x + \frac{x^2}{2}\right) \chi\left(1 + y + \frac{y^2}{2}\right) = \chi\left(1 + x + y + \frac{(x + y)^2}{2}\right),$$

which implies easily the existence and the uniqueness of c ; the valuation of c is clear.

Let m be the smallest integer such that $2m > n = \text{sw}(\chi)$; so $m = r$ if $n = 2r - 1$, and $m = r + 1$ if $n = 2r$. In both cases, we have $n + 1 = m + r$. For any $x \in \mathfrak{m}_K^m$, we have $\chi(1 + x) = \psi(cx)$. We compute the integral

$$\varepsilon_0(\chi, \psi) = \varepsilon(\chi, \psi) = \int_{K^\times} \chi^{-1}(x) \psi(x) dx$$

by splitting it according to the classes $K^\times / 1 + \mathfrak{m}_K^m$. Only the classes contained in $c(1 + \mathfrak{m}_K^r)$ remain:

$$\varepsilon_0(\chi, \psi) = \int_{c(1 + \mathfrak{m}_K^r)} \chi^{-1}(x) \psi(x) dx = q^{-\text{ord}(c)} \chi^{-1}(c) \psi(c) \int_{\mathfrak{m}_K^r} \chi^{-1}(1 + x) \psi(cx) dx.$$

For any $x \in \mathfrak{m}_K^r$, we have

$$\chi^{-1}(1 + x) \psi(cx) = \chi^{-1}(1 + x) \chi\left(1 + x + \frac{x^2}{2}\right) = \chi\left(1 + \frac{x^2}{2}\right) = \psi\left(\frac{cx^2}{2}\right).$$

We deduce that

$$\varepsilon_0(\chi, \psi) = \chi^{-1}(c) \psi(c) q^{-\text{ord}(c)} \int_{\mathfrak{m}_K^r} \psi\left(\frac{cx^2}{2}\right) dx. \tag{8.7.4}$$

If $n = 2r - 1$ is odd, then $\text{ord}(c) + 2r = -\text{ord}(\psi)$ and $\int_{\mathfrak{m}_K^r} \psi((cx^2)/2) dx = q^{-r}$; so (8.7.3) follows by (8.5.3).

Assume that $n = 2r$ is even, so $\text{ord}(c) + 2r = -\text{ord}(\beta)$. By (8.6.1), we have

$$\int_{\mathfrak{m}_K^r} \psi\left(\frac{cx^2}{2}\right) dx = q^{-r-1} \sum_{x \in \mathfrak{m}_K^r / \mathfrak{m}_K^{r+1}} \psi_k\left(\frac{\beta cx^2}{2}\right) = q^{-r-1} G_{\psi_k}(2\beta c, \pi)_K.$$

So (8.7.3) follows by (8.5.3) and the relation $(-1, \pi)_K = \kappa_0(-1)$.

PROPOSITION 8.8. Assume $p \neq 2$ and $\text{ord}(\psi) = 0$. Let $\beta \in K^\times$ be a ψ_k -gauge of ψ (see § 8.6), let L be a finite, separable, totally ramified extension of K of degree n , let π_L be a uniformizer of L , let $\mathcal{D}_{L/K}$ be the different of L over K , let δ be a refined logarithmic different of L over K (see § 7.1), and let $m = \text{ord}_L(\mathcal{D}_{L/K})$, where ord_L is the valuation of L normalized by $\text{ord}_L(\pi_L) = 1$. Then we have

$$\lambda(L/K, \psi) = \kappa_0(-1)^{\binom{m+1}{2}} G_{\psi_k}^{-m} \times \begin{cases} 1 & \text{if } m \text{ is even,} \\ (-2\beta\delta, \pi_L)_L & \text{if } m \text{ is odd.} \end{cases} \tag{8.8.1}$$

Let $f(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of π_L , $D = f'(\pi_L)$ (which is a generator of $\mathcal{D}_{L/K}$), $\kappa_D: W_L \rightarrow \{\pm 1\}$ be the character defined by the class of D in $L^\times/L^{\times 2}$ (see § 6.1), $V_K = \text{Ind}_{W_L}^{W_K} \kappa_D$, $\psi_L = \psi \circ \text{Tr}_{L/K}$. We have

$$\lambda(L/K, \psi) = \frac{\varepsilon(V_K, \psi)}{\varepsilon(\kappa_D, \psi_L)}. \tag{8.8.2}$$

It is clear that V_K is an orthogonal representation of W_K . By (6.4.1) and (6.5.1), the determinant of V_K is the unramified character $\kappa_{-1}^{\binom{n}{2}}$. Therefore, by [Ser71, Theorem 1], the Artin conductor $a(V_K)$ of V_K is even. Let r be the smallest integer such that $2r \geq m = \text{ord}_L(D)$. Since $a(V_K) = m + a(\kappa_D)$, we have $a(V_K) = 2r$; moreover, κ_D is unramified if and only if $m = 2r$ is even.

We identify $H^2(K, \mathbb{Z}/2\mathbb{Z})$ with $\{\pm 1\}$ by the isomorphism inv_K , and the Hilbert symbol $(\ , \)_K$ with the pairing $\{ \ , \ }$ induced by the cup-product (see § 6.1). By [Del76, 1.5], since $\det(V_K)$ is unramified and $\text{ord}(\psi) = 0$, we have

$$\varepsilon(V_K, \psi) = w_2(V_K)q^r, \tag{8.8.3}$$

where $w_2(V_K) \in \{\pm 1\}$ is the second Stiefel–Whitney class of V_K (see § 6.3). Since $(-1, -1)_K = (-1, 2)_K = 1$, we deduce from 6.5 that we have

$$w_2(V_K) = (d_{L/K}, 2)_K = (D, 2)_L = \begin{cases} 1 & \text{if } m \text{ is even,} \\ (2, \pi_L)_L & \text{if } m \text{ is odd.} \end{cases} \tag{8.8.4}$$

To prove (8.8.4), we expressed the Hilbert symbol in terms of the tame symbol. By (8.5.3), we have

$$q^r = \kappa_0(-1)^r G_{\psi_k}^{2r} = \kappa_0(-1)^{\binom{m+1}{2}} \times \begin{cases} G_{\psi_k}^m & \text{if } m = 2r, \\ G_{\psi_k}^{m+1} & \text{if } m = 2r - 1. \end{cases} \tag{8.8.5}$$

We put $\psi'_L(x) = \psi_L(D^{-1}x)$. Then we have $\text{ord}_L(\psi'_L) = 0$ and $\beta\pi_L^{1-n}$ is a ψ_k -gauge of ψ'_L . Indeed, since $D\pi_L^{1-n}$ is a refined logarithmic different of L over K (see § 7.3), for any $a \in \mathcal{O}_L$, with residue class \bar{a} in $\mathcal{O}_L/\pi_L\mathcal{O}_L$, we have by (7.1.2)

$$\psi'_L(\beta^{-1}\pi_L^{n-1}a) = \psi(\beta^{-1}\text{Tr}_{L/K}(D^{-1}\pi_L^{n-1}a)) = \psi_k(\bar{a}).$$

Since $\kappa_D(D) = (D, D)_L = (D, -1)_L = \kappa_0(-1)^m$, we have by (8.4.1)

$$\varepsilon(\kappa_D, \psi_L) = \kappa_D(D)q^m\varepsilon(\kappa_D, \psi'_L) = G_{\psi_k}^{2m}\varepsilon(\kappa_D, \psi'_L). \tag{8.8.6}$$

If $m = 2r$ is even, then κ_D is unramified and we have $\varepsilon(\kappa_D, \psi'_L) = 1$, which implies (8.8.1) in this case. Assume that $m = 2r - 1$ is odd, so κ_D is tamely ramified; in particular, the character $k^\times \rightarrow \{\pm 1\}$ defined by κ_D is non-trivial, and hence is equal to κ_0 . By 8.7(i), we have

$$\varepsilon(\kappa_D, \psi'_L) = \varepsilon_0(\kappa_D, \psi'_L) = -\kappa_D(\beta\pi_L^{1-n})\tau(\kappa_0, \psi_k) = \kappa_D(\beta\pi_L^{1-n})G_{\psi_k}. \tag{8.8.7}$$

Since $\beta\pi_L^{1-n}$ is a uniformizer of L , we have $\text{ord}_L(D\beta\pi_L^{1-n}) = 2r$ and

$$\kappa_D(\beta\pi_L^{1-n}) = (D, \beta\pi_L^{1-n})_L = (-D\beta\pi_L^{1-n}, \beta\pi_L^{1-n})_L = (-\beta D\pi_L^{1-n}, \varpi_L)_L,$$

where ϖ_L is any uniformizer of L . Moreover, a refined logarithmic different of L over K being unique in $L^\times/1 + \pi_L\mathcal{O}_L$, we have $(D\pi_L^{1-n}, \varpi_L)_L = (\delta, \varpi_L)_L$, which proves (8.8.1) in this case.

8.9 Assume that K is of equal characteristic p . Recall that we fixed a non-trivial additive character $\psi_k: k \rightarrow \overline{\mathbb{Q}_\ell}^\times$. We denote by $\text{res}: \Omega_K^1 \rightarrow k$ the residue homomorphism and by $\text{ord}: \Omega_K^1 - \{0\} \rightarrow \mathbb{Z}$ the valuation defined by $\text{ord}(x dy) = \text{ord}(x)$, if $x, y \in K^\times$ and $\text{ord}(y) = 1$. For a non-zero element ω of Ω_K^1 , we denote by $\psi_\omega: K \rightarrow \overline{\mathbb{Q}_\ell}^\times$ the non-trivial additive character defined, for any $a \in K$, by

$$\psi_\omega(a) = \psi_k(\text{res}(a\omega)). \tag{8.9.1}$$

Let x be a uniformizer of K , β be the element of K^\times such that $\omega = \beta x^{-1} dx$. Then β is a ψ_k -gauge of ψ_ω and we have $\text{ord}(\psi_\omega) = \text{ord}(\omega) = \text{ord}(\beta) - 1$.

COROLLARY 8.10. *Assume that K is of equal characteristic $p \neq 2$. Let L be a finite, separable, totally ramified extension of K of degree n , let x be a uniformizer of K , let t be a uniformizer of L , and let $d_{L/K} \in K^\times/K^{\times 2}$ be the discriminant of L over K (see § 6.4). We put $x' = dx/dt$ and $m = \text{ord}_L(x')$, where ord_L is the valuation of L normalized by $\text{ord}_L(t) = 1$. Then we have*

$$d_{L/K} = (-1)^{\binom{n}{2}} N_{L/K}(t^n x'/x), \tag{8.10.1}$$

$$\lambda(L/K, \psi_{dx}) = \kappa_0(-1)^{\binom{m+1}{2}} G_{\psi_k}^{-m} \times \begin{cases} 1 & \text{if } m \text{ is even,} \\ (2x', t)_L & \text{if } m \text{ is odd.} \end{cases} \tag{8.10.2}$$

Let $f(X) \in \mathcal{O}_K[X]$ be the minimal polynomial of t . Since $t^{1-n} f'(t)$ and tx'/x are refined logarithmic differentials of L over K (see § 7.1), the quotient $f'(t)(t^n x'/x)^{-1}$ belongs to $1 + t\mathcal{O}_L$. Hence, (8.10.1) follows from (6.5.1). On the other hand, x is a ψ_k -gauge of ψ_{dx} and we have $\text{ord}(\psi_{dx}) = 0$. Then (8.10.2) follows from 8.8.

PROPOSITION 8.11. *Assume that K is of equal characteristic p . Let $\chi: K^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ be a wildly ramified quasi-character of Swan conductor $n = \text{sw}(\chi) \geq 1$, $c \in K^\times$, ω be a non-zero element of Ω_K^1 , $\psi_m: \mathbb{Z}/p^{m+1}\mathbb{Z} \rightarrow \overline{\mathbb{Q}_\ell}^\times$ ($m \geq 0$) be injective homomorphisms. We assume that the following conditions are satisfied.*

- (i) *There exist a character $\chi_w: K^\times \rightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$ ($m \geq 0$) and a tamely ramified quasi-character $\chi_t: K^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ such that $\chi = \chi_t \cdot (\psi_m^{-1} \circ \chi_w)$. We denote by $\gamma \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ the cohomology class corresponding to χ_w by the reciprocity isomorphism (8.1.1).*
- (ii) *There exists $a \in \text{fil}_n W_{m+1}(K)$ such that $\delta_{m+1}(a) = \gamma$ (see (2.5.2)) and*

$$2\text{ord}(F^m da + c\omega) \geq -n, \tag{8.11.1}$$

where $F^m d$ is the homomorphism defined in (2.4.1).

- (iii) *We have $\psi_k = \psi_0 \circ \text{Tr}_{k/\mathbb{F}_p}$ and $\psi_m(p^m a) = \psi_0(a)$ for any $a \in \mathbb{F}_p$, where $p^m a$ denotes the embedding $\mathbb{F}_p \rightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$ induced by the multiplication by p^m on \mathbb{Z} .*

Let $\psi_\omega: K \rightarrow \overline{\mathbb{Q}_\ell}^\times$ be the additive character defined in (8.9.1) and let r be the smallest integer such that $2r \geq n$. Then, for any $x \in \mathfrak{m}_K^r$, we have

$$\chi\left(1 + x + \frac{x^2}{2}\right) = \psi_\omega(cx). \tag{8.11.2}$$

We deduce the proposition from Witt’s explicit reciprocity law according to Fontaine [Fon90, 2.4.3]. Let $W = W(k)$, $W_{m+1} = W_{m+1}(k)$, let $\mathcal{O}_\mathcal{E}$ be the p -adic completion of the ring $W((t))$ of Laurent power series over W in the variable t (which is an absolutely unramified, complete, discrete valuation ring), and let \mathcal{E} be the fraction field of $\mathcal{O}_\mathcal{E}$. We identify the residue field of $\mathcal{O}_\mathcal{E}$ with K by mapping the residue class of t to a uniformizer of K . We denote by $\widehat{\Omega}_{\mathcal{O}_\mathcal{E}/W}^1$ the module of continuous differential forms of $\mathcal{O}_\mathcal{E}$ over W and by $\text{res}_t: \widehat{\Omega}_{\mathcal{O}_\mathcal{E}/W}^1 \rightarrow W$ the residue homomorphism (cf. [Fon90, 2.2]). For $z \in W_{m+1}(K)$ and $u \in K^\times$, we denote by $[z, u]_m$ the element of $\mathbb{Z}/p^{m+1}\mathbb{Z} \subset W_{m+1}(\overline{K})$ defined by

$$[z, u]_m = g_u(\xi) - \xi,$$

where ξ is an element of $W_{m+1}(\overline{K})$ such that $F(\xi) - \xi = z$, F is the Frobenius homomorphism, and $g_u \in G_K^{\text{ab}}$ is the image of u by the reciprocity isomorphism (8.1.1). If we put $\mathcal{O}_{\mathcal{E},m} = \mathcal{O}_\mathcal{E}/p^{m+1}\mathcal{O}_\mathcal{E}$, we have a homomorphism

$$w_m: W_{m+1}(K) \rightarrow \mathcal{O}_{\mathcal{E},m}$$

defined for an element $z = (z_0, z_1, \dots, z_m)$ of $W_{m+1}(K)$, by $w_m(z) = \sum_{0 \leq j \leq m} p^j (\tilde{z}_j)^{p^{m-j}}$, where \tilde{z}_j is any lifting of z_j in $\mathcal{O}_{\mathcal{E},m}$. Then, if $z \in W_{m+1}(K)$ and if \tilde{u} is a unit of $\mathcal{O}_{\mathcal{E},m}$ lifting an element u of K^\times , we have

$$[z, u]_m = -\text{Tr}_m(\text{res}_m(w_m(z)d \log \tilde{u})), \tag{8.11.3}$$

where Tr_m (respectively, res_m) is the reduction modulo p^{m+1} of the trace homomorphism of W over \mathbb{Z}_p (respectively, res_t). Note that the minus sign in (8.11.3) does not appear in [Fon90, 2.4.3] because the reciprocity map used there is the inverse of (8.1.1) (the reciprocity map used by Fontaine sends uniformizers to arithmetic Frobenius).

Multiplication by p^m on $\widehat{\Omega}_{\mathcal{O}_\mathcal{E}/W}^1$ induces a homomorphism

$$\Omega_K^1 = \widehat{\Omega}_{\mathcal{O}_\mathcal{E}/W}^1 \otimes_{\mathcal{O}_\mathcal{E}} K \rightarrow \widehat{\Omega}_{\mathcal{O}_\mathcal{E}/W}^1 \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{E},m},$$

that we abusively denote by a multiplication by p^m . For any $z \in W_{m+1}(K)$, we have

$$dw_m(z) = p^m F^m d(z). \tag{8.11.4}$$

We can now prove the proposition. Since $r \geq 1$, we may assume $\chi_t = 1$. We put $a = (a_0, \dots, a_m) \in W_{m+1}(K)$ and $\nu_i = \text{ord}(a_i)$. Let \tilde{a}_i be a lifting of a_i in $t^{\nu_i}W_{m+1}[[t]]$ ($0 \leq i \leq m$), $x \in \mathfrak{m}_K^r$, and let \tilde{x} be a lifting of x in $t^rW_{m+1}[[t]]$. It follows from the choice of the \tilde{a}_i and the fact that $3r \geq n + 1$, that we have $w_m(a) \in t^{-n}W_{m+1}[[t]]$, and $\text{res}_m(w_m(a)\tilde{x}^i d\tilde{x}) = 0$ for $i \geq 2$. Therefore, we have

$$\begin{aligned} \left[a, 1 + x + \frac{x^2}{2} \right]_m &= -\text{Tr}_m \left(\text{res}_m \left(w_m(a) \frac{1 + \tilde{x}}{1 + \tilde{x} + (\tilde{x}^2/2)} d\tilde{x} \right) \right) \\ &= -\text{Tr}_m(\text{res}_m(w_m(a) d\tilde{x})) \\ &= \text{Tr}_m(\text{res}_m(\tilde{x} d(w_m(a)))) \\ &= p^m \text{Tr}_{k/\mathbb{F}_p}(\text{res}_K(xF^m d(a))), \end{aligned}$$

where, on the right-hand side, p^m denotes the embedding $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}$ induced by the multiplication by p^m on \mathbb{Z} . We conclude by (8.11.1) that for any $x \in \mathfrak{m}_K^r$, we have

$$\begin{aligned} \chi\left(1 + x + \frac{x^2}{2}\right) &= \psi_m^{-1}\left(\left[a, 1 + x + \frac{x^2}{2}\right]_m\right) \\ &= \psi_0(-\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{res}_K(xF^m d(a)))) \\ &= \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{res}_K(xcw))) \\ &= \psi_\omega(cx). \end{aligned}$$

9. Laumon’s formula for local epsilon factors

9.1 Let T be the spectrum of a henselian discrete valuation field of equal characteristic p , with finite residue field k of order $q = p^f$, τ (respectively, $\bar{\tau}$) be the generic point (respectively, a geometric generic point) of T , $G = \pi_1(\tau, \bar{\tau})$. We denote by K the completion of the function field $k(\tau)$ of T and by

$$\mathrm{Rec}_T: K^\times \rightarrow G^{\mathrm{ab}} \tag{9.1.1}$$

the reciprocity homomorphism, normalized as in (8.1.1).

Recall that we fixed a non-trivial additive character $\psi_0: \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$ (see § 1.11). Let $\psi_k: k \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be the additive character $\psi_0 \circ \mathrm{Tr}_{k/\mathbb{F}_p}$. For a complex C of $\mathbf{D}_c^b(T, \overline{\mathbb{Q}}_\ell)$ and a non-zero meromorphic differential form ω on T (that is, $\omega \in \Omega_{k(\tau)}^1 - \{0\}$), Laumon attached a local ε -factor $\varepsilon(T, C, \omega) \in \overline{\mathbb{Q}}_\ell^\times$ (see [Lau87, 3.1.5.4]). For any $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} over τ , we have (with the notation of § 8.3)

$$\varepsilon(T, j_*\mathcal{F}, \omega) = \varepsilon(\mathcal{F}_{\bar{\tau}}, \psi_\omega), \tag{9.1.2}$$

$$\varepsilon(T, j_!\mathcal{F}, \omega) = \varepsilon_0(\mathcal{F}_{\bar{\tau}}, \psi_\omega), \tag{9.1.3}$$

where $j: \tau \rightarrow T$ is the canonical injection, ψ_ω is the additive character defined in (8.9.1).

In the situation of § 3.4, we use the notation above for T and \check{T} . We equip with a \vee the objects relative to \check{T} .

THEOREM 9.2 [Lau87, 3.6.2]. *The assumptions are those of 3.7; moreover, we assume that k is finite, $z = 0$ and $\check{z} = \infty$. We denote by $\mathcal{G}_!$ the extension by 0 of \mathcal{G} to S , and by d the dimension of $\mathfrak{F}^{(0, \infty)}(f_*(\mathcal{G}))$ over $\overline{\mathbb{Q}}_\ell$. Then we have*

$$(-1)^d \det(\mathrm{Rec}_{\check{T}}(\check{x}^{-1}), \mathfrak{F}^{(0, \infty)}(f_*(\mathcal{G}))) = \varepsilon(T, f_*(\mathcal{G}_!), dx). \tag{9.2.1}$$

Let k' be the residue field of S at s . First, we reduce the theorem to the case where $k' = k$. We denote by $0' \in P_{k'}(k')$ and $\infty' \in \check{P}_{k'}(k')$ the points induced by $0 \in P(k)$ and $\infty \in \check{P}(k)$, by $v': S \rightarrow P_{k'}$ and $\check{v}': S \rightarrow \check{P}_{k'}$ the morphisms induced by v and \check{v} , by T' and \check{T}' the henselizations of $P_{k'}$ and $\check{P}_{k'}$ at $0'$ and ∞' respectively, by $f': S \rightarrow T'$ and $\check{f}': S \rightarrow \check{T}'$ the morphisms induced by v' and \check{v}' (or by f and \check{f}) and by $w: \check{T}' \rightarrow \check{T}$ the canonical morphism. We have $T' = T \otimes_k k'$ and $\check{T}' = \check{T} \otimes_k k'$. By 3.6(ii), we have

$$w_*(\mathfrak{F}^{(0', \infty')}(f'_*(\mathcal{G}))) \simeq \mathfrak{F}^{(0, \infty)}(f_*(\mathcal{G})). \tag{9.2.2}$$

Let d' be the dimension of $\mathfrak{F}^{(0', \infty')}(f'_*(\mathcal{G}))$ over $\overline{\mathbb{Q}}_\ell$. Since \check{T}' is a finite étale covering of \check{T} of

degree d/d' , we deduce by [Del73, 1.2] that we have

$$\begin{aligned} & \det(\text{Rec}_{\check{T}}(\check{x}^{-1}), \mathfrak{F}^{(0,\infty)}(f_*(\mathcal{G}))) \\ &= \det(\text{Rec}_{\check{T}}(\check{x}^{-1}), w_*(1))^{d'} \cdot \det(\text{Rec}_{\check{T}'}(\check{x}^{-1}), \mathfrak{F}^{(0',\infty')}(f'_*(\mathcal{G}))) \\ &= (-1)^{d-d'} \det(\text{Rec}_{\check{T}'}(\check{x}^{-1}), \mathfrak{F}^{(0',\infty')}(f'_*(\mathcal{G}))). \end{aligned} \tag{9.2.3}$$

Since T' is a finite étale covering of T , we have, by 8.4(ii) and 8.9,

$$\varepsilon(T', f'_*(\mathcal{G}_1), dx) = \varepsilon(T, f_*(\mathcal{G}_1), dx). \tag{9.2.4}$$

Equations (9.2.3) and (9.2.4) show that we may assume $k' = k$. We denote by L the completion of the function field $k(\eta)$ of S , by \mathcal{O}_L its valuation ring, by t a uniformizer of $k(\eta)$, and by ord_L the valuation of L normalized by $\text{ord}_L(t) = 1$. For any $y \in L$, we put $y' = dy/dt$; if $y \in k(\eta)$, then y' is well defined in $k(\eta)$ (see 3.8). We consider L as a finite, separable, totally ramified extension of both K and \check{K} , the completions of the function fields of T and \check{T} , respectively. Since x is a uniformizer of K and \check{x}^{-1} is a uniformizer of \check{K} , we have $[L : K] = \text{ord}_L(b)$, $[L : \check{K}] = -\text{ord}_L(c)$, and tb'/b (respectively, $-tc'/c$) is a refined logarithmic different of L over K (respectively, of L over \check{K}) (see § 7.4). We put $m = \text{ord}_L(b')$, $i = -\text{ord}(c)$, and denote by $d_{L/\check{K}}$ the discriminant of L over \check{K} (see § 6.4).

We denote by $\chi : L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ the quasi-character defined by the sheaf \mathcal{G} over η , by $(\ , \)_L$ the Hilbert symbol over L , by $\kappa_0 : k^\times \rightarrow \{\pm 1\}$ the unique character of order two, by $\psi_k : k \rightarrow \overline{\mathbb{Q}}_\ell^\times$ the additive character $\psi_0 \circ \text{Tr}_{k/\mathbb{F}_p}$, and by G_{ψ_k} the quadratic Gauss sum associated to ψ_k (see (8.5.2)). Observe that we have the following equality of additive characters of L

$$\psi_{db} = \psi_{dx} \circ \text{Tr}_{L/K}. \tag{9.2.5}$$

On the one hand, by 3.7, we have a canonical isomorphism of sheaves over $\check{\tau}$

$$\mathfrak{F}^{(0,\infty)}(f_*(\mathcal{G})) \simeq \check{f}_*(\mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K}(-\frac{1}{2}b'c') \otimes \mathcal{Q}).$$

We deduce by (6.4.1) that we have

$$\begin{aligned} & \det(-\text{Rec}_{\check{T}}(\check{x}), \mathfrak{F}^{(0,\infty)}(f_*(\mathcal{G}))) \\ &= (-1)^i (\check{x}, d_{L/\check{K}})_{\check{K}} \det(\text{Rec}_S(c), \mathcal{G} \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K}(-\frac{1}{2}b'c') \otimes \mathcal{Q}). \end{aligned} \tag{9.2.6}$$

Moreover, we have

$$\begin{aligned} (\check{x}, d_{L/\check{K}})_{\check{K}} &= (\check{x}, (-1)^{\binom{i}{2}} N_{L/\check{K}}(-t^i c'/c))_{\check{K}} \\ &= \kappa_0(-1)^{\binom{i}{2}} (c, t^i c')_L, \end{aligned} \tag{9.2.7}$$

$$\det(\text{Rec}_S(c), \mathcal{G}) = \chi(c), \tag{9.2.8}$$

$$\begin{aligned} \det(\text{Rec}_S(c), \mathcal{L}_{\psi_0}(bc)) &= \psi_0^{-1} \left(-\text{Tr}_{k/\mathbb{F}_p} \left(\text{res}_L \left(bc \frac{dc}{c} \right) \right) \right) \\ &= (\psi_k(\text{res}_L(cdb)))^{-1} = (\psi_{db}(c))^{-1}, \end{aligned} \tag{9.2.9}$$

$$\det(\text{Rec}_S(c), \mathcal{K}(-\frac{1}{2}b'c')) = (c, -2b'c')_L, \tag{9.2.10}$$

$$\det(\text{Rec}_S(c), \mathcal{Q}) = (-1)^i G_{\psi_k}^{-i}. \tag{9.2.11}$$

Equation (9.2.10) is obvious from the definitions, equation (9.2.7) follows from (8.10.1), equation (9.2.11) follows from [Lau87, 1.4.3.2(ii)], and equation (9.2.9) is a consequence of [Ser68, XIV, § 5, Proposition 15]: the power -1 above ψ_0 is due to the convention in § 3.1, and the minus sign before $\text{Tr}_{k/\mathbb{F}_p}$ is due to the convention in the definition of the reciprocity law (8.1.1). We deduce

that we have

$$\det(-\text{Rec}_{\tilde{T}}(\tilde{x}), \mathfrak{F}^{(0, \infty)}(f_*(\mathcal{G}))) = \kappa_0(-1)^{\binom{i}{2}} \chi(c) (\psi_{db}(c))^{-1} G_{\psi_k}^{-i}(c, -2t^i b')_L. \tag{9.2.12}$$

On the other hand, we have

$$\varepsilon(T, f_*(\mathcal{G}), dx) = \lambda(L/K, \psi_{dx}) \varepsilon(S, \mathcal{G}, db) = \lambda(L/K, \psi_{dx}) \varepsilon_0(\chi, \psi_{db}). \tag{9.2.13}$$

By assumption, (\mathcal{G}, b, c) is a Legendre triple (cf. 2.16). So χ is wildly ramified. We put $n = \text{sw}(\chi)$ and r the smallest integer such that $2r \geq n$. By 8.11, for any $y \in t^r \mathcal{O}_L$, we have

$$\chi\left(1 + y + \frac{y^2}{2}\right) = \psi_{db}(cy).$$

For any $y \in \mathcal{O}_L$, with residue class \bar{y} in k , we have by (7.1.2)

$$\psi_{db}((tb')^{-1}y) = \psi_{dx}(x^{-1} \text{Tr}_{L/K}((tb'/b)^{-1}y)) = \psi_k(\bar{y}).$$

We deduce from 8.7(ii) that we have $1 + m - i = \text{ord}(tb'c) = -n$ and

$$\varepsilon_0(\chi, \psi_{db}) = \chi^{-1}(c) \psi_{db}(c) q^i \kappa_0(-1)^{\binom{-n}{2}} G_{\psi_k}^{-n-1} \times \begin{cases} 1 & \text{if } n \text{ is odd} \\ (2b'c, t)_L & \text{if } n \text{ is even} \end{cases} \tag{9.2.14}$$

By (8.10.2), we have

$$\lambda(L/K, \psi_{dx}) = \kappa_0(-1)^{\binom{m+1}{2}} G_{\psi_k}^{-m} \times \begin{cases} 1 & \text{if } m \text{ is even,} \\ (2b', t)_L & \text{if } m \text{ is odd.} \end{cases} \tag{9.2.15}$$

To conclude the proof, it remains to check that the product of the right-hand sides of equations (9.2.12), (9.2.14) and (9.2.15) is equal to 1. Since we have $q = \kappa(-1)G_{\psi_k}^2$ (see (8.5.3)), $1 + m = i - n$ and hence $\binom{m+1}{2} = \binom{i}{2} + \binom{-n}{2} - in$, we are reduced to checking that

$$1 = \kappa_0(-1)^{i(n+1)} (c, -2t^i b')_L \times \begin{cases} 1 & \text{if } m \text{ is even and } n \text{ is odd,} \\ (c, t)_L & \text{if } m \text{ is odd and } n \text{ is even,} \\ (2b'c, t)_L & \text{if } m \text{ and } n \text{ are even,} \\ (2b', t)_L & \text{if } m \text{ and } n \text{ are odd.} \end{cases} \tag{9.2.16}$$

Since we have $\kappa_0(-1) = (-1, t) = (t, t)$, we are further reduced to checking that

$$1 = \begin{cases} (c, -2b')_L & \text{if } m \text{ is even and } n \text{ is odd,} \\ (c, -2b't)_L & \text{if } m \text{ is odd and } n \text{ is even,} \\ (-2b', tc)_L & \text{if } m \text{ and } n \text{ are even,} \\ (-2tb', tc)_L & \text{if } m \text{ and } n \text{ are odd.} \end{cases} \tag{9.2.17}$$

In each case, the valuations of both terms of the Hilbert symbol are even, which proves the required result.

Appendix A. Semi-continuity of the Swan conductor

A.1 In this appendix, (S, η, s) denotes an excellent henselian trait, of equal characteristic $p > 0$, with algebraically closed residue field k , that is, $S = \text{Spec}(V)$ is the spectrum of an excellent henselian discrete valuation ring, of equal characteristic $p > 0$, η and s are the generic and the closed points of S . We denote by K the fraction field of V . We fix a geometric generic point $\bar{\eta}$ of S , and a finite field Λ of characteristic $\neq p$. A finite covering of (S, η, s) stands for a trait (S', η', s') equipped with a finite covering $S' \rightarrow S$.

A.2 Let R be a complete discrete valuation ring, let L be the fraction field of R , let \mathfrak{m} be the maximal ideal of R , let L' be a finite separable extension of L , and let R' be the integral closure of R in L' . We say that L' is a *stable* extension of L if $\mathfrak{m}R'$ is the maximal ideal of R' .

A.3 Let R be a complete discrete valuation ring which is a k -algebra, let F be the residue field of R , let L be the fraction field of R , and let \mathfrak{m} be the maximal ideal of R . We assume that F is an extension of finite type of k . Then the R -module of absolute 1-differential forms Ω_R^1 is complete, separated, and hence free of finite rank over R . We denote by $\Omega_R^1(\log)$ the sub- R -module of Ω_L^1 generated by Ω_R^1 and $d \log(x)$ where x is a uniformizer of R (cf. [AS09, 5.4]). We put $\Omega_F^1(\log) = \Omega_R^1(\log) \otimes_R F$. We have a canonical exact sequence

$$0 \longrightarrow \Omega_F^1 \longrightarrow \Omega_F^1(\log) \xrightarrow{\text{res}} F \longrightarrow 0. \tag{A.3.1}$$

Let \mathcal{F} be a Λ -sheaf of rank one over $\text{Spec}(L)$. Kato [Kato89] associates to \mathcal{F} a Swan conductor and a refined Swan conductor, that can also be defined using our ramification theory [AS09]. The Swan conductor $n = \text{sw}(\mathcal{F})$ is an integer ≥ 0 , that vanishes if and only if \mathcal{F} is tamely ramified. The refined Swan conductor $\text{rsw}(\mathcal{F})$ is an element of the F -vector space

$$\Omega_F^1(\log) \otimes_F (\mathfrak{m}^{-n}/\mathfrak{m}^{-n+1}). \tag{A.3.2}$$

If \mathcal{F} is trivialized by a stable extension of L , then we have [Kato89, remark after 6.8]

$$(\text{res} \otimes 1)(\text{rsw}(\mathcal{F})) = 0 \in \mathfrak{m}^{-n}/\mathfrak{m}^{-n+1}. \tag{A.3.3}$$

A.4 We denote by \mathcal{C}_S the following category. Objects of \mathcal{C}_S are normal affine S -schemes H for which there exist an S -curve X (that is, a flat S -scheme of finite type and relative dimension one) and a closed point x of X_s , such that X is smooth over S outside x , and H is S -isomorphic to the henselization of X at x . Let H, H' be two objects of \mathcal{C}_S . A morphism $f: H' \rightarrow H$ of \mathcal{C}_S is a finite morphism of S -schemes which is étale at the generic point of H' .

A.5 Let H be an object of \mathcal{C}_S , (S', η', s') be a finite covering of (S, η, s) . Then $H \times_S S'$ is an object of $\mathcal{C}_{S'}$ (see [Kato87a, 5.4]).

A.6 Let H be an object of \mathcal{C}_S . We denote by H° the set of height 1 points of H , $H_\eta^\circ = H_\eta \cap H^\circ$ and $H_s^\circ = H_s \cap H^\circ$. Then:

- (i) H_η is geometrically regular over η , of dimension one, and for any $\mathfrak{p} \in H_\eta^\circ$, the residue field $\kappa(\mathfrak{p})$ of H at \mathfrak{p} is a finite extension of K ;
- (ii) H_s is a reduced henselian local scheme, of dimension one.

Indeed, let X be an S -curve and let x be a closed point of X_s such that X is smooth over S outside x , and H is S -isomorphic to the henselization of X at x . The set of geometric points of H is canonically isomorphic to the set of geometric points of X which are generizations of x ; moreover, the strict henselizations of X and H at associated geometric points are isomorphic [SGA4, VIII 7.3]. We deduce that H_η is regular of dimension one and, hence, geometrically regular over η by A.5. The second assertion of part (i) is a consequence of [EGA4, 8.2.9 and its corollaries]. The scheme H_s is the henselization of X_s at x , which implies part (ii).

We denote by \widetilde{H}_s the normalization of H_s , which is a finite disjoint union of strictly local traits (indexed by H_s°). We put

$$\delta(H) = \dim_k(\mathcal{O}_{\widetilde{H}_s} / \mathcal{O}_{H_s}). \tag{A.6.1}$$

A.7 We denote by \mathfrak{F}_S the following category. Objects of \mathfrak{F}_S are triples (H, U, \mathcal{F}) , where H is an object of \mathcal{C}_S , U is a non-empty open subscheme of H_η , and \mathcal{F} is a locally constant constructible étale sheaf of Λ -modules over U . Let $(H, U, \mathcal{F}), (H', U', \mathcal{F}')$ be two objects of \mathfrak{F}_S . A morphism $(H', U', \mathcal{F}') \rightarrow (H, U, \mathcal{F})$ of \mathfrak{F}_S is a pair (f, g) made of a morphism $f: H' \rightarrow H$ of \mathcal{C}_S such that $f(U') \subset U$, and a morphism $g: \mathcal{F}' \rightarrow f_U^* \mathcal{F}$, where $f_U: U' \rightarrow U$ is the restriction of f .

Let (S', η', s') be a finite covering of (S, η, s) . By **A.5**, the base change $S' \rightarrow S$ induces a natural functor $\mathfrak{F}_S \rightarrow \mathfrak{F}_{S'}$, that we denote by

$$(H, U, \mathcal{F}) \mapsto (H, U, \mathcal{F})_{S'}. \tag{A.7.1}$$

A.8 An object (H, U, \mathcal{F}) of \mathfrak{F}_S is said to be *stable* if there exists a finite étale connected covering U' of U satisfying the following conditions.

- (i) The pull-back of \mathcal{F} to U' is constant.
- (ii) The normalization H' of H in U' belongs to \mathcal{C}_S , and the residue fields of H' at all points of $H'_\eta - U'_\eta$ are finite separable extensions of K .

PROPOSITION A.9 [**Kato87a**, 6.3]. *Let (H, U, \mathcal{F}) be an object of \mathfrak{F}_S .*

- (i) *Let (S', η', s') be a finite covering of (S, η, s) . If (H, U, \mathcal{F}) is a stable object of \mathfrak{F}_S , then $(H, U, \mathcal{F})_{S'}$ is a stable object of $\mathfrak{F}_{S'}$.*
- (ii) *There exists (S', η', s') a finite covering of (S, η, s) , such that $(H, U, \mathcal{F})_{S'}$ is a stable object of $\mathfrak{F}_{S'}$.*

Proposition (i) follows from **A.5** and proposition (ii) follows from [**Epp73**].

A.10 Let (H, U, \mathcal{F}) be a stable object of \mathfrak{F}_S such that \mathcal{F} has *rank* one over U . For $\mathfrak{p} \in H^\circ$, we denote by $R_{\mathfrak{p}}$ the completion of the local ring of H at \mathfrak{p} (which is a discrete valuation ring), and by $\kappa(\mathfrak{p})$ its residue field. Following [**DK73**, XVI], [**Lau81**] and [**Kato87a**, 6.4], we define the *total dimension* of \mathcal{F} at a point \mathfrak{p} , denoted by $\dim_{\text{tot}}_{\mathfrak{p}}(\mathcal{F})$, to be the integer given as follows. For $\mathfrak{p} \in H_\eta^\circ$, we put

$$\dim_{\text{tot}}_{\mathfrak{p}}(\mathcal{F}) = [\kappa(\mathfrak{p}) : K](\text{sw}_{\mathfrak{p}}(\mathcal{F}) + 1), \tag{A.10.1}$$

where $\text{sw}_{\mathfrak{p}}(\mathcal{F})$ is the Swan conductor of the pull-back to \mathcal{F} over $\text{Spec}(R_{\mathfrak{p}}) \times_H U$.

For $\mathfrak{p} \in H_s^\circ$, we denote by $\widetilde{H}_{s,\mathfrak{p}}$ the integral closure of H_s in $\kappa(\mathfrak{p})$ (which is a strictly local trait) and by $\text{ord}_{s,\mathfrak{p}}$ the associated valuation of $\kappa(\mathfrak{p})$, normalized by $\text{ord}_{s,\mathfrak{p}}(\kappa(\mathfrak{p})^\times) = \mathbb{Z}$. We denote also by $\text{ord}_{s,\mathfrak{p}}: \Omega_{\kappa(\mathfrak{p})}^1 - \{0\} \rightarrow \mathbb{Z}$ the valuation defined by $\text{ord}_{s,\mathfrak{p}}(\alpha d\beta) = \text{ord}_{s,\mathfrak{p}}(\alpha)$, if $\alpha, \beta \in \kappa(\mathfrak{p})^\times$ and $\text{ord}_{s,\mathfrak{p}}(\beta) = 1$. We distinguish two cases.

- (i) Assume that \mathcal{F} extends to a locally constant constructible sheaf of Λ -modules $\widetilde{\mathcal{F}}$ over an open subscheme \widetilde{U} of H that contains \mathfrak{p} . We denote by $\text{sw}_{s,\mathfrak{p}}(\mathcal{F})$ the Swan conductor of the pull-back of $\widetilde{\mathcal{F}}$ to $\widetilde{H}_{s,\mathfrak{p}} \times_H \widetilde{U}$. We put

$$\dim_{\text{tot}}_{\mathfrak{p}}(\mathcal{F}) = \text{sw}_{s,\mathfrak{p}}(\mathcal{F}) + 1. \tag{A.10.2}$$

(ii) Assume that \mathcal{F} is ramified at \mathfrak{p} . We denote by $n = \text{sw}_{\mathfrak{p}}(\mathcal{F})$ and $\text{rsw}_{\mathfrak{p}}(\mathcal{F})$ the Swan and the refined Swan conductors of the pull-back of \mathcal{F} to $\text{Spec}(R_{\mathfrak{p}}) \times_H U$ (see A.3). Let π be a uniformizer of V . Since (H, U, \mathcal{F}) is stable, we have $\text{res}(\text{rsw}_{\mathfrak{p}}(\mathcal{F}) \otimes [\pi^n]) = 0$ (see A.3.3). Hence, we can identify $\text{rsw}_{\mathfrak{p}}(\mathcal{F}) \otimes [\pi^n]$ with the image of an element $\omega \in \Omega_{\kappa(\mathfrak{p})}^1$, which does not depend on the choice of π up to a multiplication by an element of k^\times . We put

$$\text{dimtot}_{\mathfrak{p}}(\mathcal{F}) = -\text{ord}_{s,\mathfrak{p}}(\omega). \tag{A.10.3}$$

We put

$$\varphi_{\eta}(H, U, \mathcal{F}) = \sum_{\mathfrak{p} \in H_{\eta} - U} \text{dimtot}_{\mathfrak{p}}(\mathcal{F}), \tag{A.10.4}$$

$$\varphi_s(H, U, \mathcal{F}) = \sum_{\mathfrak{p} \in H_s^{\circ}} \text{dimtot}_{\mathfrak{p}}(\mathcal{F}). \tag{A.10.5}$$

LEMMA A.11 [Kato87a, 6.5]. Let (H, U, \mathcal{F}) be a stable object of \mathfrak{F}_S such that \mathcal{F} has rank one over U and let (S', η', s') be a finite covering of (S, η, s) . We put $(H', U', \mathcal{F}') = (H, U, \mathcal{F})_{S'}$.

(i) For any $\mathfrak{p}' \in H_s'^{\circ}$ with image \mathfrak{p} in H_s° , we have $\text{dimtot}_{\mathfrak{p}}(\mathcal{F}) = \text{dimtot}_{\mathfrak{p}'}(\mathcal{F}')$.

(ii) For any $\mathfrak{p} \in H_{\eta} - U$, we have

$$\text{dimtot}_{\mathfrak{p}}(\mathcal{F}) = \sum_{\mathfrak{p}'} \text{dimtot}_{\mathfrak{p}'}(\mathcal{F}'), \tag{A.11.1}$$

where \mathfrak{p}' runs over the points of H' above \mathfrak{p} .

A.12 Let (H, U, \mathcal{F}) be an object of \mathfrak{F}_S such that \mathcal{F} has rank one over U . By A.9(ii), there exists a finite covering (S', η', s') of (S, η, s) such that $(H, U, \mathcal{F})_{S'}$ is a stable object of $\mathfrak{F}_{S'}$. We put

$$\varphi_{\eta}(H, U, \mathcal{F}) = \varphi_{\eta'}((H, U, \mathcal{F})_{S'}), \tag{A.12.1}$$

$$\varphi_s(H, U, \mathcal{F}) = \varphi_{s'}((H, U, \mathcal{F})_{S'}). \tag{A.12.2}$$

By A.11, these numbers do not depend on the choice of (S', η', s') .

THEOREM A.13 (Deligne, Kato). Let (H, U, \mathcal{F}) be an object of \mathfrak{F}_S such that \mathcal{F} has rank one over U , let x be the closed point of H , and let $u: U \rightarrow H_{\eta}$ be the canonical injection. Then we have

$$\dim(\Psi_x^0(u_! \mathcal{F})) - \dim(\Psi_x^1(u_! \mathcal{F})) = \varphi_s(H, U, \mathcal{F}) - \varphi_{\eta}(H, U, \mathcal{F}) - 2\delta(H). \tag{A.13.1}$$

If \mathcal{F} is unramified at every point of H_s° , Deligne [Lau81, 5.1.1] proved the theorem for sheaves of any rank. In the general case, Kato [Kato87a, 6.7] proved the theorem for sheaves of any rank, with another definition of the invariant $\varphi_s(H, U, \mathcal{F})$. One of the present authors, Saito [Sai87], gave another proof for sheaves of any rank, with yet another definition of the invariant $\varphi_s(H, U, \mathcal{F})$. The latter corresponds to a second formula announced by Kato [Kato87b, 4.5]. If \mathcal{F} has rank one, the invariant $\varphi_s(H, U, \mathcal{F})$ in Kato's latter formula coincides with our definition [Kato89, remark after 6.8].

Note that formula (A.13.1) also holds in the case where S has unequal characteristic.

Appendix B. Dimension of the local Fourier transform

B.1 We fix an algebraically closed field k of characteristic $p > 0$, a finite field Λ of characteristic $\neq p$ and a non-trivial additive character $\psi_0: \mathbb{F}_p \rightarrow \Lambda^\times$. We denote by \mathcal{L}_{ψ_0} the Artin–Schreier locally constant sheaf of Λ -modules of rank one over $\mathbb{G}_{a,k}$ associated to ψ_0 . Apart from this change of convention, we keep the same notation as in § 3. In particular, we consider the sheaves $\mathcal{L}_{\psi_0}(x\check{x})$ and $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$ over $A \times_k \check{A}$ and $P \times_k \check{P}$, respectively. For a scheme W over $A \times_k \check{A}$ (respectively, $P \times_k \check{P}$), we denote also by $\mathcal{L}_{\psi_0}(x\check{x})$ (respectively, $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$) the pull-back of $\mathcal{L}_{\psi_0}(x\check{x})$ (respectively, $\overline{\mathcal{L}}_{\psi_0}(x\check{x})$) to W .

B.2 Let X be a smooth connected curve over k , $f: X \rightarrow P$ be a k -morphism, étale over a dense open subscheme of X , $Y = f^{-1}(A)$, $s \in X(k)$, $z = f(s)$, and $\check{z} \in \check{P}(k)$. We denote by \check{T} and H the henselizations of \check{P} and $X \times_k \check{P}$ at \check{z} and (s, \check{z}) , respectively, by $\check{\tau}$ the generic point of \check{T} and (abusively) by \check{z} the closed point of \check{T} . We consider H as a \check{T} -scheme by the morphism $H \rightarrow \check{T}$ induced by the canonical projection $X \times_k \check{P} \rightarrow \check{P}$. We denote by U the inverse image of $Y \times_k \check{A}$ in $H_{\check{\tau}} = H \times_{\check{T}} \check{\tau}$, and let

$$\rho(s, \check{z}) = \varphi_{\check{z}}(H, U, \mathcal{L}_{\psi_0}(x\check{x})), \tag{B.2.1}$$

where $\varphi_{\check{z}}$ is the invariant defined in (A.12.2).

LEMMA B.3. *Under the assumptions of B.2, we have*

$$\rho(s, \check{z}) = \begin{cases} -\text{ord}_s(f^* dx) & \text{if } (z, \check{z}) \in P \times \check{\infty}, \\ 1 & \text{if } (z, \check{z}) = (\infty, \check{0}), \end{cases} \tag{B.3.1}$$

where $\text{ord}_s(f^* dx)$ is the order of the non-zero meromorphic differential form $f^* dx$ over X .

The case where $(z, \check{z}) = (\infty, \check{0})$ follows directly from the definition. Assume that $\check{z} = \check{\infty}$. We put $y = \check{x}^{-1}$ and consider the base change $\check{T}_1 \rightarrow \check{T}$ given by $\check{T}[y_1]/(y_1^p - y)$. We denote by \mathfrak{p} the generic point of the special fiber of the canonical projection $X \times_k \check{T}_1 \rightarrow \check{T}_1$, by R the completion of the local ring of $X \times_k \check{T}_1$ at \mathfrak{p} , by K the fraction field of R , and by b the image of x in R (which is a unit). Since $f^* dx \neq 0$, b is not a p th power in R . By [AS09, § 10], the Swan conductor of $\mathcal{L}_{\psi_0}(x\check{x})$ at \mathfrak{p} is p , and its refined Swan conductor is the class of the differential form

$$db \otimes [y_1^{-p}] \in \Omega_R^1(\log) \otimes_R (y_1 R)^{-p}/(y_1 R)^{-p+1}.$$

Moreover, $\mathcal{L}_{\psi_0}(x\check{x})$ is trivialized by a stable extension of K , namely the extension L of K defined by the equation $t^p - t = b/y_1^p$. Indeed, the integral closure of R in L is generated over R by $t_1 = y_1 t$ which satisfies the equation $t_1^p - y_1^{p-1} t_1 = b$. The lemma follows.

B.4 We keep the assumptions of B.2. Moreover, let Y_0 be a dense open subscheme of X contained in Y , let $j: Y_0 \rightarrow X$ be the canonical injection, and let \mathcal{G} be a locally constant sheaf of Λ -modules of rank one over Y_0 . We denote by $\text{pr}_1: X \times_k \check{T} \rightarrow X$ the first projection, and consider the complex of nearby cycles

$$\Psi(\text{pr}_1^*(j_! \mathcal{G}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))$$

in $\mathbf{D}_c^b(X, \Lambda)$, relatively to the second projection $\text{pr}_2: X \times_k \check{T} \rightarrow \check{T}$. We fix an algebraic closure of $k(\check{\tau})$ and denote by θ the associated geometric point of \check{T} . We consider the sheaf $\mathcal{G}_\theta \otimes \mathcal{L}_{\psi_0}(\check{x}f_\theta)$ over $X_\theta = X \times_k \theta$ (cf. § 3.1 for the notation).

PROPOSITION B.5. *We keep the assumptions of B.2 and B.4.*

(i) If $s \in Y - Y_0$ and $\check{z} = \check{\infty}$, then we have

$$\dim(\Psi_s^1(\text{pr}_1^*(j_!\mathcal{G}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) = \text{sw}_s(\mathcal{G}) + 1 + \text{ord}_s(f^* dx).$$

(ii) If $(z, \check{z}) = (\infty, \check{\infty})$, then we have

$$\dim(\Psi_s^1(\text{pr}_1^*(j_!\mathcal{G}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) = \text{sw}_{s \times \theta}(\mathcal{G}_\theta \otimes \mathcal{L}_{\psi_0}(\check{x}f_\theta)) + 1 + \text{ord}_s(f^* dx).$$

(iii) If $(z, \check{z}) = (\infty, \check{0})$, then we have

$$\dim(\Psi_s^1(\text{pr}_1^*(j_!\mathcal{G}) \otimes \overline{\mathcal{L}}_{\psi_0}(x\check{x}))) = \text{sw}_{s \times \theta}(\mathcal{G}_\theta \otimes \mathcal{L}_{\psi_0}(\check{x}f_\theta)) - \text{sw}_s(\mathcal{G}).$$

This follows from A.13 and B.3.

PROPOSITION B.6 [Lau87, 2.4.3]. Let $z \in P(k)$, $\check{z} \in \check{P}(k)$, let T and \check{T} be the henselizations of P and \check{P} at z and \check{z} , respectively, and let τ and $\check{\tau}$ be the generic points of T and \check{T} , respectively. Let \mathcal{F} be a constructible sheaf of Λ -modules over τ , of rank $\text{rk}(\mathcal{F})$ and Swan conductor $\text{sw}(\mathcal{F})$, and let $\Theta(\mathcal{F})$ be the set of slopes of \mathcal{F} . Then the rank of the local Fourier transform of \mathcal{F} at (z, \check{z}) (see § 3.4) is given by

$$\text{rk}(\mathfrak{F}^{(z, \check{z})}(\mathcal{F})) = \begin{cases} \text{sw}(\mathcal{F}) + \text{rk}(\mathcal{F}) & \text{if } (z, \check{z}) \in A \times \check{\infty}, \\ \text{sw}(\mathcal{F}) - \text{rk}(\mathcal{F}) & \text{if } (z, \check{z}) = (\infty, \check{\infty}) \text{ and } \Theta(\mathcal{F}) \subset]1, \infty[, \\ 0 & \text{if } (z, \check{z}) = (\infty, \check{\infty}) \text{ and } \Theta(\mathcal{F}) \subset [0, 1[, \\ \text{rk}(\mathcal{F}) - \text{sw}(\mathcal{F}) & \text{if } (z, \check{z}) = (\infty, \check{0}) \text{ and } \Theta(\mathcal{F}) \subset [0, 1[, \\ 0 & \text{if } (z, \check{z}) = (\infty, \check{0}) \text{ and } \Theta(\mathcal{F}) \subset [1, \infty[. \end{cases} \tag{B.6.1}$$

By Brauer induction, we may reduce to the case where $\mathcal{F} = f_*(\mathcal{G})$, $f: S \rightarrow T$ is a finite morphism, étale above τ , S is the spectrum of a henselian discrete valuation ring, with generic point η , and \mathcal{G} is a constructible sheaf of Λ -modules of rank one over η . There exist a connected smooth curve X over k , a k -morphism $\tilde{f}: X \rightarrow P$, a point $s \in X(k)$, a dense open subscheme Y_0 of X , and a locally constant constructible sheaf of Λ -modules of rank one, $\tilde{\mathcal{G}}$, over Y_0 , such that S is isomorphic to the henselization of X at s , $z = \tilde{f}(s)$, f is induced by \tilde{f} , $\tilde{f}(Y_0) \subset A$, and \mathcal{G} is isomorphic to the pull-back of $\tilde{\mathcal{G}}$ to η . We take again the notation of B.2 and B.4 (applied to \tilde{f} and $\tilde{\mathcal{G}}$). It follows from [DK73, XIII, 2.1.7.1 and 2.1.7.2] that we have a canonical isomorphism

$$\mathfrak{F}^{(z, \check{z})}(\mathcal{F}) \simeq \Psi_s^1(\text{pr}_1^*(j_!\tilde{\mathcal{G}}) \otimes \overline{\mathcal{L}}(x\check{x})). \tag{B.6.2}$$

Let R be the completion of the local ring of S , let K be its fraction field, let t be a uniformizer of R , let b be the image of x in K , and let ord be the valuation of K normalized by $\text{ord}(t) = 1$. For $v \in K$, we put $v' = dv/dt \in K$. If $(z, \check{z}) \in A \times \check{\infty}$, then by B.5 and [Ser68, VI, § 2], we have

$$\begin{aligned} \text{rk}(\mathfrak{F}^{(z, \check{z})}(\mathcal{F})) &= \text{sw}(\mathcal{G}) + 1 + \text{ord}(b') \\ &= \text{sw}(\mathcal{G}) + \text{ord}\left(\frac{tb'}{b - x(z)}\right) + \text{ord}(b - x(z)) \\ &= \text{sw}(\mathcal{F}) + \text{rk}(\mathcal{F}). \end{aligned} \tag{B.6.3}$$

We fix an algebraic closure of $k(\check{\tau})$ and let θ be the associated geometric point of \check{T} , and $T_{\{\theta\}}$ be the henselization of $T \times_k \theta$ at $z \times_k \theta$. We denote by a subscript $\{\theta\}$ the objects deduced from objects over T by the base change $T_{\{\theta\}} \rightarrow T$. Similarly as for (B.6.3), we have

$$\begin{aligned} \text{rk}(\mathfrak{F}^{(\infty, \check{\infty})}(\mathcal{F})) &= \text{sw}(\mathcal{G}_{\{\theta\}} \otimes f_{\{\theta\}}^*(\mathcal{L}_{\psi_0}(x\check{x}))) + 1 + \text{ord}(b') \\ &= \text{sw}(\mathcal{G}_{\{\theta\}} \otimes f_{\{\theta\}}^*(\mathcal{L}_{\psi_0}(x\check{x}))) + \text{ord}(tb(b^{-1})') + \text{ord}(b) \\ &= \text{sw}(\mathcal{F}_{\{\theta\}} \otimes \mathcal{L}_{\psi_0}(x\check{x})) - \text{rk}(\mathcal{F}), \end{aligned} \tag{B.6.4}$$

$$\begin{aligned} \text{rk}(\mathfrak{F}^{(\infty, \check{0})}(\mathcal{F})) &= \text{sw}(\mathcal{G}_{\{\theta\}} \otimes f_{\{\theta\}}^*(\mathcal{L}_{\psi_0}(x\check{x}))) - \text{sw}(\mathcal{G}) \\ &= \text{sw}(\mathcal{F}_{\{\theta\}} \otimes \mathcal{L}_{\psi_0}(x\check{x})) - \text{sw}(\mathcal{F}). \end{aligned} \tag{B.6.5}$$

Let κ be a geometric generic point of $T_{\{\theta\}}$, $I = \pi_1(\tau, \kappa)$, $I_{\{\theta\}} = \pi_1(\tau_{\{\theta\}}, \kappa)$, $I^{(a)}$ and $I_{\{\theta\}}^{(a)}$ ($a \in \mathbb{Q}_{\geq 0}$) be the classical logarithmic ramification filtrations of I and $I_{\{\theta\}}$, respectively ([Ser68, IV]; cf. [AS02, AS03] for the notation). For every $a \in \mathbb{Q}_{\geq 0}$, the canonical surjective homomorphism $I_{\{\theta\}} \rightarrow I$ identifies I^a with the image of $I_{\{\theta\}}^a$. We consider the slope decomposition of the representation \mathcal{F}_κ of I

$$\mathcal{F}_\kappa = \bigoplus_{\lambda \in \Theta(\mathcal{F})} \mathcal{F}_{\kappa, \lambda}.$$

By [Lau87, 2.1.2.7], to conclude the proof of the proposition, it is enough to show that

$$(\mathcal{F}_{\kappa, 1} \otimes \mathcal{L}_{\psi_0}(x\check{x})_\kappa)^{I_{\{\theta\}}^{(1)}} = 0. \tag{B.6.6}$$

Recall that $I^{(1)}/I^{(1+)}$ is an \mathbb{F}_p -vector space. By [AS09, 14.3 and 14.4], we have an isomorphism

$$\text{Hom}_{\mathbb{Z}}(I^{(1)}/I^{(1+)}, \mathbb{F}_p) \simeq k, \tag{B.6.7}$$

and similarly for $I_{\{\theta\}}^{(1)}/I_{\{\theta\}}^{(1+)}$ (in [AS09], we trivialize the line N_{-1} by x^{-1}). Since we fixed a non-trivial character $\psi_0: \mathbb{F}_p \rightarrow \Lambda^\times$, the action of $I^{(1)}/I^{(1+)}$ on $\mathcal{F}_{\kappa, 1}$ determines a finite set of characters $I^{(1)}/I^{(1+)} \rightarrow \mathbb{F}_p$, and hence a finite set of points $\Sigma \subset k$. Similarly, the action of $I_{\{\theta\}}^{(1)}/I_{\{\theta\}}^{(1+)}$ on $\mathcal{L}_{\psi_0}(x\check{x})_\kappa$ determines the point $\check{x} \in k(\theta)$ (cf. [AS09, 9.13]). Since $\check{x} \notin \Sigma$, equation (B.6.6) follows.

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