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Extension theorems for differential forms and Bogomolov–Sommese vanishing on log canonical varieties

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Abstract

Given a normal variety Z, a p-form σ defined on the smooth locus of Z and a resolution of singularities $\pi : \widetilde{Z} \to Z$, we study the problem of extending the pull-back $\pi^*(\sigma)$ over the π-exceptional set $E \subset \widetilde{Z}$. For log canonical pairs and for certain values of p, we show that an extension always exists, possibly with logarithmic poles along E. As a corollary, it is shown that sheaves of reflexive differentials enjoy good pull-back properties. A natural generalization of the well-known Bogomolov–Sommese vanishing theorem to log canonical threefold pairs follows.

Contents

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1. Introduction and statement of main result

1.1 Introduction

Let Z be a normal projective variety and $\sigma \in H^0(Z, \Omega_Z^{[p]})$ a p-form which is defined away from the singularities. A natural question to ask is: if $\pi : \widetilde{Z} \to Z$ is a resolution of singularities, can one extend $\pi^*(\sigma)$ as a differential form to all of $\widetilde{Z}$, perhaps allowing logarithmic poles along the π-exceptional set?

If $p = \dim Z$ and if the pair $(Z, \emptyset)$ is log canonical, the answer is 'yes', almost by definition. For other values of p, the problem has been studied by Hodge-theoretic methods; see the papers of Steenbrink [Ste85], Steenbrink–van Straten [vSS85], Flenner [Fle88] and the references therein. In a nutshell, the answer is 'yes' if the codimension of the singular set is large.

In this paper, we consider logarithmic varieties with log canonical singularities. We show that for these varieties and certain values of p, the answer is 'yes', irrespective of the codimension of the singular set.

As a corollary, we show that sheaves of reflexive differentials enjoy good pull-back properties and prove a version of the well-known Bogomolov–Sommese vanishing theorem for log canonical threefold pairs.

1.2 Main results

The following is the main result of this paper. In essence, it asserts that a (logarithmic) p-form defined away from the singular set of a log canonical threefold pair gives rise to p-forms on any log resolution.

Theorem 1.1 (Extension theorem for log canonical pairs). *Let Z be a normal variety of dimension n and $\Delta \subset Z$ a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\pi : \widetilde{Z} \to Z$ be a log resolution, and set*

$$\widetilde{\Delta}_{\mathrm{lc}} := \text{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \text{non-klt locus of } (Z, \Delta)),$$

where the non-klt locus is the minimal closed subset $W \subset Z$ such that that pair (Z, Δ) is klt away from W. If $p \in \{n, n-1, 1\}$, then the sheaf $\pi_* \Omega^p_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive.

Remark 1.1.1. Logarithmic differentials are introduced and discussed in [Iit82, ch. 11c] or [Del70, ch. 3]. The notion of log resolution is recalled in Definition 2.6 below. We refer the reader to [KM98, § 2.3] for the definition of log canonical and klt singularities.

Remark 1.1.2. Since the coefficients of its components are equal to 1 (cf. Definition 2.4), the boundary divisor Δ is contained in the non-klt locus of (X, Δ). We have nevertheless chosen to explicitly include it in the definition of $\widetilde{\Delta}_{\mathrm{lc}}$ for reasons of clarity.

The name 'extension theorem' is justified by the following remark.

Remark 1.2. Theorem 1.1 asserts precisely that for any open set $U \subset Z$ and any number $p \in \{n, n-1, 1\}$, the restriction morphism

$$H^0(\pi^{-1}(U),\ \Omega^p_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})) \to H^0(\pi^{-1}(U) \setminus \mathrm{Exc}(\pi),\ \Omega^p_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})) \tag{1.2.1}$$

is surjective, where $\mathrm{Exc}(\pi) \subset \widetilde{Z}$ denotes the π-exceptional set,

Remark 1.3. After this paper appeared in preprint form, we learned that more general results had been claimed in Langer [Lan03, Theorems 4.9 and 4.11]. However, in discussions with Langer we found that the proof of [Lan03, Theorem 4.9] contains a gap that at present has

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still not been filled: in the last paragraph of the proof, it is not clear that the prerequisites of [Lan03, Lemma 4.8] are satisfied. For a special case of the statement for surfaces, see [Lan01, Theorem 4.2].

For an application of Theorem 1.1, recall the well-known Bogomolov–Sommese vanishing theorem for snc pairs, cf. [EV92, Corollary 6.9]: if Z is a smooth projective variety, $\Delta \subset Z$ a divisor with simple normal crossings and $\mathscr{A} \subset \Omega^p_Z(\log \Delta)$ an invertible subsheaf, then the Kodaira–Iitaka dimension of $\mathscr{A}$ is not larger than p, i.e., $\kappa(\mathscr{A}) \leqslant p$. As a corollary to Theorem 1.1, we will show in § 8 that a similar result holds for threefold pairs with log canonical singularities. We refer to Definition 2.3 for the definition of the Kodaira–Iitaka dimension for sheaves that are not necessarily locally free.

Theorem 1.4 (Bogomolov–Sommese vanishing for log canonical threefolds and surfaces). *Let Z be a normal variety of dimension* $\dim Z \leqslant 3$ *and let $\Delta \subset Z$ be a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\mathscr{A} \subset \Omega^{[p]}_Z(\log \Delta)$ be a reflexive subsheaf of rank one. If $\mathscr{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathscr{A}) \leqslant p$.*

In fact, a stronger result holds, see Theorem 8.3.

1.3 Outline of the paper

We introduce notation and recall standard facts in § 2. In § 3 we prepare for the proof of Theorem 1.1 by showing how extension properties of a given space Z can often be deduced from extension properties of finite covers of Z. This already gives extension results for an important class of surface singularities that appears naturally within the minimal model program. Because of their importance in applications, we briefly discuss these singularities in § 3.2.

Theorem 1.1 is shown in §§ 5–7 for n-forms, $(n-1)$-forms and 1-forms, respectively. The proof of the extension result for $(n-1)$-forms relies on universal properties of the functorial resolution of singularities and on liftings of local group actions. The extension for 1-forms is shown using results of Steenbrink and Namikawa that are Hodge theoretic in nature.

Section 8 discusses pull-back properties of sheaves of differentials and gives a proof of the Bogomolov–Sommese vanishing theorem for log canonical threefolds and surfaces, Theorem 1.4. For the reader's convenience, an appendix recalling the variant of Hartshorne's formal duality theorem for cohomology with supports that is required in our context is included, cf. § 7.3.

Part I. Tools

2. Notation and standard facts

2.1 Reflexive tensor operations

When dealing with sheaves that are not necessarily locally free, we frequently use square brackets to indicate taking the reflexive hull.

Notation 2.1. Let Z be a normal variety and $\mathscr{A}$ a coherent sheaf of $\mathscr{O}_Z$-modules. Let $n \in \mathbb{N}$ and set $\mathscr{A}^{[n]} := \otimes^{[n]}\mathscr{A} := (\mathscr{A}^{\otimes n})^{**}$, $\mathrm{Sym}^{[n]}\,\mathscr{A} := (\mathrm{Sym}^n\,\mathscr{A})^{**}$, etc. Likewise, for a morphism $\gamma : X \to Z$ of normal varieties, set $\gamma^{[*]}\mathscr{A} := (\gamma^*\mathscr{A})^{**}$. If $\mathscr{A}$ is reflexive of rank one, we say that $\mathscr{A}$ is $\mathbb{Q}$-Cartier if there exists an $n \in \mathbb{N}$ such that $\mathscr{A}^{[n]}$ is invertible.

In the sequel, we will frequently state and prove results that hold for the sheaf of differentials $\Omega^{[1]}_Z$, the reflexive hull of its symmetric products, exterior products, tensor products or any combination of these tensor operations. The following shorthand notation is therefore useful.

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Notation 2.2. A reflexive tensor operation is any combination of the reflexive tensor product $\otimes^{[k]}$, the symmetric product $\mathrm{Sym}^{[l]}$ or the exterior product $\bigwedge^{[m]}$. If T is a tensor operation, such as $\mathsf{T} = \otimes^{[2]}\,\mathrm{Sym}^{[3]}$, and $\mathscr{F}$ is a sheaf of $\mathscr{O}_Z$-modules on a scheme Z, we often write $\mathsf{T}\mathscr{F}$ instead of $\otimes^{[2]}_{\mathscr{O}_Z}\,\mathrm{Sym}^{[3]}_{\mathscr{O}_Z}\,\mathscr{F}$.

We will be working with the Kodaira–Iitaka dimension of reflexive sheaves on normal spaces. Since this is perhaps not quite standard, we recall the definition here.

Definition 2.3 (Kodaira–Iitaka dimension). Let Z be a normal projective variety and $\mathscr{A}$ a reflexive sheaf of rank one on Z. If $h^0(Z, \mathscr{A}^{[n]}) = 0$ for all $n \in \mathbb{N}$, then we say that $\mathscr{A}$ has Kodaira–Iitaka dimension $\kappa(\mathscr{A}) := -\infty$. Otherwise, set

$$M := \{n \in \mathbb{N} \mid h^0(Z, \mathscr{A}^{[n]}) > 0\}.$$

Recall that the restriction of $\mathscr{A}$ to the smooth locus of Z is locally free and consider the rational mapping

$$\phi_n : Z \dashrightarrow \mathbb{P}(H^0(Z, \mathscr{A}^{[n]})^*) \quad \text{for each } n \in M.$$

The Kodaira–Iitaka dimension of $\mathscr{A}$ is then defined as

$$\kappa(\mathscr{A}) := \max_{n\in M}(\dim \overline{\phi_n(Z)}).$$

2.2 Logarithmic pairs and the extension theorem

For the reader's convenience, we recall a few definitions of logarithmic geometry. Although not quite standard, the following notion of a morphism of logarithmic pairs is useful for our purposes.

Definition 2.4 (Logarithmic pair). A *logarithmic pair* (Z, Δ) consists of a normal variety or complex space Z and a reduced, but not necessarily irreducible, Weil divisor $\Delta \subset Z$. A *morphism of logarithmic pairs* $\gamma : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ is a morphism $\gamma : \widetilde{Z} \to Z$ such that $\gamma^{-1}(\Delta) = \widetilde{\Delta}$ set-theoretically.

Definition 2.5 (snc pairs). Let (Z, Δ) be a logarithmic pair and $z \in Z$ a point. We say that (Z, Δ) is *snc at* z if there exists a Zariski-open neighborhood U of z such that U is smooth and $\Delta \cap U$ has only simple normal crossings. The pair (Z, Δ) is *snc* if it is snc at all $z \in Z$.

Given a logarithmic pair (Z, Δ), let $(Z, \Delta)_{\mathrm{reg}}$ be the maximal open set of Z where (Z, Δ) is snc, and let $(Z, \Delta)_{\mathrm{sing}}$ be its complement, with the induced reduced subscheme structure.

Remark 2.5.1. If a logarithmic pair (Z, Δ) is snc at a point z, this implies that all components of Δ are smooth at z. Without the condition that U is Zariski open this would no longer be true, and Definition 2.5 would define normal crossing pairs rather than pairs with simple normal crossing.

Definition 2.6 (Log resolution). A *log resolution* of (Z, Δ) is a birational morphism of pairs $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ such that the π-exceptional set $\mathrm{Exc}(\pi)$ is of pure codimension one, such that $(\widetilde{Z}, \mathrm{supp}(\widetilde{\Delta} \cup \mathrm{Exc}(\pi)))$ is snc and such that π is isomorphic over $(Z, \Delta)_{\mathrm{reg}}$.

The following definitions will be helpful in the proof of Theorem 1.1 and its corollaries.

Notation 2.7. If (Z, Δ) is a logarithmic pair and T a reflexive tensor operation, the sheaf $\mathsf{T}\Omega^1_Z(\log \Delta)$ will be called the sheaf of T-forms.

Definition 2.8 (Extension theorem). If (Z, Δ) is a logarithmic pair and T a reflexive tensor operation, we say that *the extension theorem holds for* T*-forms on* (Z, Δ) if the following holds:

let $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a log resolution and E_Δ the union of all π-exceptional components not contained in $\widetilde{\Delta}$. Then the push-forward sheaf

$$\pi_* \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_\Delta))$$

is reflexive. Equivalently, the restriction morphism

$$H^0(\pi^{-1}(U),\ \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_\Delta))) \to H^0(\pi^{-1}(U) \setminus \mathrm{Exc}(\pi),\ \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta})) \tag{2.8.1}$$

is surjective for any open set $U \subseteq Z$.

2.3 Pull-back properties of logarithmic and regular differentials

Morphisms of snc pairs give rise to pull-back morphisms of logarithmic differentials. In this section, we briefly recall the standard fact that the pull-back morphism associated with a finite map is isomorphic if the branch locus is contained in the boundary. We refer to [Iit82, ch. 11] for details.

Fact 2.9. Let $\gamma : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a morphism of snc pairs, $U \subseteq Z$ an open set and $\widetilde{U} = \gamma^{-1}(U)$. Then there exists a natural pull-back map of forms

$$\gamma^* : H^0(U,\ \Omega^1_Z(\log \Delta)) \to H^0(\widetilde{U},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}))$$

and an associated sheaf morphism

$$d\gamma : \gamma^* \Omega^1_Z(\log \Delta) \to \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}).$$

If γ is finite and unramified over $Z \setminus \Delta$, then $d\gamma$ is an isomorphism. □

Remark 2.10. If T is any reflexive tensor operation, then the pull-back morphism also gives a pull-back of T-forms, $\gamma^* : H^0(Z, \mathsf{T}\Omega^1_Z(\log \Delta)) \to H^0(\widetilde{Z}, \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}))$, that obviously extends to a pull-back of rational T-forms.

We state one immediate consequence for future reference. The following notation is useful in the formulation.

Notation 2.11. Let X be a normal variety, $\Gamma \subset X$ a reduced Weil divisor and $\mathscr{F}$ a reflexive coherent sheaf of $\mathscr{O}_X$-modules. We will often consider sections of $\mathscr{F}|_{X \setminus \Gamma}$. Equivalently, we consider rational sections of $\mathscr{F}$ with poles of arbitrary order along Γ, and let $\mathscr{F}(*\Gamma)$ be the associated sheaf of these sections on X. More precisely, we define

$$\mathscr{F}(*\Gamma) := \varinjlim_m ((\mathscr{F} \otimes \mathscr{O}_X(m \cdot \Gamma))^{**}).$$

With this notation, we have $H^0(X, \mathscr{F}(*\Gamma)) = H^0(X \setminus \Gamma, \mathscr{F})$.

Corollary 2.12. *Under the conditions of Fact 2.9, let T be any reflexive tensor operation and assume that γ is a finite morphism. Let $\Gamma \subset Z$ be a reduced divisor and $\sigma \in H^0(Z,\ \mathsf{T}\Omega^1_Z(\log \Delta)(*\Gamma))$ a T-form that might have poles along Γ.*

(2.12.i) *If γ is unramified over $Z \setminus \Delta$, then the form σ has only logarithmic poles along Γ if and only if $\gamma^*(\sigma)$ has only logarithmic poles along $\mathrm{supp}(\gamma^{-1}(\Gamma))$, i.e.,*

$$\sigma \in H^0(Z,\ \mathsf{T}\Omega^1_Z(\log \Delta)) \Leftrightarrow \gamma^*(\sigma) \in H^0(\widetilde{Z},\ \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta})).$$

(2.12.ii) *If $\mathsf{T} = \bigwedge^{[p]}$, then σ is a regular form if and only if $\gamma^*(\sigma)$ is regular, i.e.,*

$$\sigma \in H^0(Z,\ \Omega^p_Z) \Leftrightarrow \gamma^*(\sigma) \in H^0(\widetilde{Z}, \Omega^p_{\widetilde{Z}}).$$

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Proof. Assertion (2.12.i) follows immediately from Fact 2.9. The proof of (2.12.ii) is left to the reader. □

2.4 Comparing log resolutions

Reflexivity of the push-forward of sheaves of differentials from an arbitrary birational model of a given pair can often be concluded if we know the reflexivity of the push-forward from a particular log resolution. This is summarized in the following elementary lemma.

Lemma 2.13. *Let (Z, Δ) be a logarithmic pair and $W \subset Z$ a subvariety. For $i \in \{1, 2\}$, let $\pi_i : (Z_i, \Delta_i) \to (Z, \Delta)$ be a birational morphism of logarithmic pairs and*

$$\Gamma_i := \text{largest reduced divisor contained in } \pi_i^{-1}(\Delta \cup W).$$

If T is a reflexive tensor operation, (Z_2, Γ_2) is snc and $(\pi_2)_* \mathsf{T}\Omega^1_{Z_2}(\log \Gamma_2)$ is reflexive, then $(\pi_1)_* \mathsf{T}\Omega^1_{Z_1}(\log \Gamma_1)$ is reflexive as well.

Remark 2.13.1. In the setup of Lemma 2.13, the sheaves $(\pi_1)_* \mathsf{T}\Omega^1_{Z_1}(\log \Gamma_1)$ and $(\pi_2)_* \mathsf{T}\Omega^1_{Z_2}(\log \Gamma_2)$ are isomorphic away from a set of codimension at least two. If the sheaves are reflexive, this implies that they are in fact isomorphic.

Proof of Lemma 2.13. Choose an snc logarithmic pair $(\widetilde{Z}, \widetilde{\Delta})$, together with birational morphisms of pairs $\varphi_i : (\widetilde{Z}, \widetilde{\Delta}) \to (Z_i, \Delta_i)$ such that $\widetilde{\Gamma}_2 := \operatorname{supp}(\varphi_2^{-1}(\Gamma_2))$ is a divisor with snc support and such that the following diagram commutes.

$$\begin{array}{ccc} (\widetilde{Z}, \widetilde{\Delta}) & \xrightarrow{\varphi_2} & (Z_2, \Delta_2) \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ (Z_1, \Delta_1) & \xrightarrow[\pi_1]{} & (Z, \Delta) \end{array}$$

Let $U \subseteq Z$ be open and $\sigma \in H^0(U, \mathsf{T}\Omega^1_Z(\log \Delta))$ a T-form on U. For convenience, set $\psi := \pi_1 \circ \varphi_1 = \pi_2 \circ \varphi_2$ and denote the preimages of U on Z_1, Z_2 and $\widetilde{Z}$ by U_1, U_2 and $\widetilde{U}$, respectively.

By assumption, $\pi_2^*(\sigma)$ extends to a T-form on (Z_2, Γ_2) without poles along the exceptional set $\operatorname{Exc}(\pi_2)$, i.e., $\pi_2^*(\sigma) \in H^0(U_2,\ \mathsf{T}\Omega^1_{Z_2}(\log \Gamma_2))$. If we set

$$\widetilde{\Gamma} := \text{largest reduced divisor contained in } \psi^{-1}(\Delta \cup W),$$

then $\widetilde{\Gamma}$ contains $\widetilde{\Gamma}_2$ and Fact 2.9 implies that $\psi^*(\sigma)$ extends to a T-form on $(\widetilde{U}, \widetilde{\Gamma}_2)$. In particular,

$$\psi^*(\sigma) \in H^0(\widetilde{U}, \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \widetilde{\Gamma}_2)) \subseteq H^0(\widetilde{U}, \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \widetilde{\Gamma})).$$

Now, if $\Gamma'_1 \subset \operatorname{Exc}(\pi_1)$ is any irreducible component with strict transform $\widetilde{\Gamma}'_1 \subset \widetilde{Z}$, it is clear that the T-form $\pi_1^*(\sigma)$ has (logarithmic) poles along Γ'_1 if and only if $\varphi_1^* \pi_1^*(\sigma) = \psi^*(\sigma)$ has (logarithmic) poles along $\widetilde{\Gamma}'_1$. The proof is then finished once we observe that $\Gamma'_1 \subseteq \pi_1^{-1}(\Delta \cup W)$ if and only if $\widetilde{\Gamma}'_1 \subseteq \psi^{-1}(\Delta \cup W)$. □

3. Finite covering tricks and log canonical singularities

3.1 The finite covering trick

In order to prove the extension theorem for a given pair (Z, Δ), it is often convenient to go to a cover of Z and argue there. For instance, if (Z, Δ) is log canonical one might want to consider local index-one covers where singularities are generally easier to describe.

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Proposition 3.1 (Finite covering trick). *Consider a commutative diagram of surjective morphisms of logarithmic pairs as follows,*

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{D}) & \xrightarrow{\widetilde{\gamma},\ \text{finite}} & (\widetilde{Z}, \widetilde{\Delta}) \\ {\scriptstyle \widetilde{\pi}}\ {\scriptstyle \text{contracts } E_X} \downarrow & & \downarrow {\scriptstyle \pi}\ {\scriptstyle \text{log resolution contracts } E_Z} \\ (X, D) & \xrightarrow[\gamma,\ \text{finite}]{} & (Z, \Delta) \end{array}$$

where $\widetilde{X}$ is the normalization of the fiber product $\widetilde{Z} \times_Z X$. Let T be a reflexive tensor operation, $\sigma \in H^0(Z,\ \mathsf{T}\Omega^1_Z(\log \Delta))$ a T-form and $E_Z \subset \operatorname{Exc}(\pi) \subset \widetilde{Z}$ a π-exceptional divisor. Assume that either:

(3.1.1) *E_Z is the union of all π-exceptional components not contained in $\widetilde{\Delta}$; or*

(3.1.2) *$\mathsf{T} = \bigwedge^{[p]}$, and no component of $E_Z \subset \widetilde{Z}$ is contained in $\widetilde{\Delta}$.*

Then

$$\widetilde{\pi}^* \gamma^{[*]}(\sigma) \in H^0(\widetilde{X}, \mathsf{T}\Omega^1_{\widetilde{X}}(\log(\widetilde{D} + E_X))) \Longrightarrow \pi^*(\sigma) \in H^0(\widetilde{Z}, \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_Z))),$$

where $E_X := \operatorname{supp}(\widetilde{\gamma}^{-1}(E_Z))$ is the reduced preimage of E_Z.

Example 3.1.3. If T is not of the form $\bigwedge^{[p]}$, the assumption made in (3.1.1) is indeed necessary. For an example in the simple case where $\mathsf{T} = \operatorname{Sym}^{[2]}$ and $\Delta = \emptyset$, let $\widetilde{Z}$ be the total space of $\mathscr{O}_{\mathbb{P}^1}(-2)$ and E_Z the zero section. It is reasonably easy to write down a form

$$\sigma \in H^0(\widetilde{Z}, \operatorname{Sym}^2 \Omega^1_{\widetilde{Z}}(\log E_Z)) \setminus H^0(\widetilde{Z}, \operatorname{Sym}^2 \Omega^1_{\widetilde{Z}}).$$

Because E_Z contracts to a quotient singularity that has a smooth 2:1 cover, this example shows that the conclusion of Proposition 3.1 holds only for differentials with logarithmic poles along E_Z, and that the boundary given there is indeed the smallest possible.

In order to give an explicit example for σ, consider the standard coordinate cover of $\widetilde{Z}$ with open sets $U_1, U_2 \simeq \mathbb{A}^2$, where U_i carries coordinates x_i, y_i and coordinate change is given as

$$\phi_{1,2} : (x_1, y_1) \mapsto (x_2, y_2) = (x_1^{-1}, x_1^2 y_1).$$

In these coordinates the bundle map $U_i \to \mathbb{P}^1$ is given as $(x_i, y_i) \to x_i$ and the zero section E_Z is given as $E_Z \cap U_i = \{y_i = 0\}$. Now take

$$\sigma_2 := y_2^{-1}(dy_2)^2 \in H^0(U_2, \operatorname{Sym}^2(\Omega^1_{\widetilde{Z}}(\log E_Z)))$$

and observe that $\phi^*_{1,2}(\sigma_2)$ extends to a form in $H^0(U_1,\ \operatorname{Sym}^2(\Omega^1_{\widetilde{Z}}(\log E_Z)))$.

Proof of Proposition 3.1. Suppose that we are given a T-form $\sigma \in H^0(Z, \mathsf{T}\Omega^1_Z(\log \Delta))$ such that

$$\widetilde{\pi}^* \gamma^{[*]}(\sigma) \in H^0(\widetilde{X},\ \mathsf{T}\Omega^1_{\widetilde{X}}(\log(\widetilde{D} + E_X))). \tag{3.1.4}$$

We need to show that σ extends to all of $\widetilde{Z}$ as a T-form, i.e., that

$$\pi^*(\sigma) \in H^0(\widetilde{Z}, \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_Z))). \tag{3.1.5}$$

Since (3.1.5) holds outside of $\operatorname{Exc}(\pi)$, and since $\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_Z))$ is locally free, it suffices to show (3.1.5) near general points of components of $\operatorname{Exc}(\pi)$. Thus, let $E'_Z \subset \operatorname{Exc}(\pi)$ be an irreducible component and $x \in E'_Z$ a general point. Over a suitably small neighborhood of x, the morphism $\widetilde{\gamma}$ is branched only along E'_Z, if at all.

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We will apply Corollary 2.12 for this small neighborhood of x. If $E'_Z \subseteq \widetilde{\Delta} + E_Z$, then (3.1.5) follows from (3.1.4) by (2.12.i). This proves the statement in case (3.1.1). If $E'_Z \not\subseteq \widetilde{\Delta} + E_Z$, we are in case (3.1.2), so $\mathsf{T} = \bigwedge^{[p]}$. Then inclusion (3.1.5) follows from (3.1.4) by (2.12.ii). This proves the statement in case (3.1.2). □

The following are two immediate consequences of Proposition 3.1.

COROLLARY 3.2. *Let (Z, Δ) be a logarithmic pair, T a reflexive tensor operation and assume that there exists a finite morphism of pairs $\gamma : (X, D) \to (Z, \Delta)$ such that the extension theorem holds for T-forms on (X, D), in the sense of Definition 2.8. Then the extension theorem holds for T-forms (Z, Δ).*

Proof. Let $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a log resolution and consider the snc divisor

$$\Gamma_Z := \operatorname{supp}(\widetilde{\Delta} \cup \operatorname{Exc}(\pi)).$$

Further, let $U \subseteq Z$ be an open set and

$$\sigma \in H^0(U \setminus (Z, \Delta)_{\mathrm{sing}}, \mathsf{T}\Omega^1_Z(\log \Delta)) = H^0(\pi^{-1}(U) \setminus \operatorname{Exc}(\pi), \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \Gamma_Z))$$

a T-form defined away from the singularities. We need to show that its pull-back extends to a T-form on $(\pi^{-1}(U),\ \Gamma_Z)$, i.e.,

$$\pi^*(\sigma) \in H^0(\pi^{-1}(U),\ \mathsf{T}\Omega^1_{\widetilde{Z}}(\log \Gamma_Z)). \tag{3.2.1}$$

For convenience of notation, we shrink Z and assume without loss of generality that $U = Z$. In order to prove (3.2.1), consider a commutative diagram of surjective morphisms of pairs,

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{D}) & \xrightarrow{\ \widetilde{\gamma},\ \text{finite}\ } & (\widetilde{Z}, \widetilde{\Delta}) \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow {\scriptstyle \pi,\ \text{log resolution}} \\ (X, D) & \xrightarrow[\ \gamma,\ \text{finite}\ ]{} & (Z, \Delta), \end{array} \tag{3.2.2}$$

where $\widetilde{X}$ is the normalization of the fiber product. Let

$$\Gamma_X := \operatorname{supp}(\widetilde{\gamma}^{-1}(\Gamma_Z)) = \operatorname{supp}(\widetilde{D} \cup \operatorname{Exc}(\widetilde{\pi})).$$

Then it follows from Lemma 2.13 that $\widetilde{\pi}^*\gamma^*(\sigma)$ extends to a T-form on $(\widetilde{X}, \Gamma_X)$, i.e.,

$$\widetilde{\gamma}^*\pi^*(\sigma) = \widetilde{\pi}^*\gamma^*(\sigma) \in H^0(\widetilde{X}, \mathsf{T}\Omega^1_{\widetilde{X}}(\log \Gamma_X)).$$

Since $\operatorname{Exc}(\pi) \subseteq \Gamma_Z$, (3.2.1) follows from case (3.1.1) of Proposition 3.1 with $E_Z := \operatorname{Exc}(\pi) \setminus \widetilde{\Delta}$. □

COROLLARY 3.3. *In order to prove the Theorem 1.1 in full generality, it suffices to show it under the additional assumption that $K_Z + \Delta$ is Cartier.*

Proof. Assume that Theorem 1.1 has been shown for all log canonical logarithmic pairs whose log canonical divisor is Cartier. Let (Z, Δ) be an arbitrary logarithmic pair that is log canonical with log resolution $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ and assume that we are given an open subset $U \subseteq Z$ and a form $\sigma \in H^0(U,\ \Omega^{[p]}_Z(\log \Delta))$, with $p \in \{\dim Z, \dim Z - 1, 1\}$. We need to show that

$$\pi^*(\sigma) \in H^0(\widetilde{U},\ \Omega^{[p]}_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})), \tag{3.3.1}$$

where $\widetilde{U} := \pi^{-1}(U)$ and

$$\widetilde{\Delta}_{\mathrm{lc}} := \text{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \text{non-klt locus of } (Z, \Delta)).$$

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Since the assertion of Theorem 1.1 is local on Z in the Zariski topology, we can shrink Z and assume without loss of generality that $U = Z$, and that $K_Z + \Delta$ is $\mathbb{Q}$-torsion, i.e., that there exists a number $m \in \mathbb{N}^+$ such that $\mathscr{O}_Z(m(K_Z + \Delta)) \cong \mathscr{O}_Z$. Let $\gamma : (X, D) \to (Z, \Delta)$ be the associated index-one-cover, as described in [KM98, Definition 2.52] or [Rei87, § 3.6f]. By the inductive assumption, the statement of Theorem 1.1 holds for the pair (X, D).

Since γ branches only over the singular points of (Z, Δ), if at all, [KM98, Proposition 5.20] immediately gives that (X, D) is again log canonical. Better still, [KM98, Proposition 5.20] implies that

$$\text{non-klt locus of } (X, D) \subseteq \gamma^{-1}(\text{non-klt locus of } (Z, \Delta)).$$

Thus, defining $\widetilde{X}$ as the normalization of $X \times_Z \widetilde{Z}$, $\widetilde{\pi} : \widetilde{X} \to X$ the natural morphism and setting

$$\widetilde{D}_{\mathrm{lc}} := \text{largest reduced divisor contained in } \widetilde{\pi}^{-1}(D \cup \text{non-klt locus of } (X, D)),$$

gives that $\widetilde{D}_{\mathrm{lc}} \subseteq \gamma^{-1}(\widetilde{\Delta}_{\mathrm{lc}})$. Now, applying the argument from the proof of Corollary 3.2 along with case (3.1.2) of Proposition 3.1 implies (3.3.1), as desired. □

3.2 Finitely dominated and boundary-lc pairs

It follows from Corollary 3.2 that the extension theorem holds for pairs with quotient singularities, or in fact for pairs that can be locally finitely dominated by snc pairs. Surface singularities that appear in minimal model theory often have this property. Because of their importance in the applications, we discuss one class of examples in more detail here.

Definition 3.4 (Finitely dominated pair). A logarithmic pair (Z, Δ) is said to be *finitely dominated by analytic snc pairs* if, for any point $z \in Z$, there exists an analytic neighborhood U of z and a finite, surjective morphism of logarithmic pairs $(\widetilde{U}, D) \to (U, \Delta \cap U)$, where $\widetilde{U}$ is smooth and the divisor D has only simple normal crossings.

Remark 3.5. By Corollary 3.2, if T is any reflexive tensor operation, then the extension theorem holds for T-forms on any pair (Z, Δ) that is finitely dominated by analytic snc pairs.

Definition 3.6 (boundary-lc). A logarithmic pair (Z, Δ) is called *boundary-lc* if (Z, Δ) is log canonical and $(Z \setminus \Delta, \emptyset)$ is log terminal.

Example 3.7. It follows immediately from the definition that dlt pairs are boundary-lc, cf. [KM98, Definition 2.37]. For a less obvious example, let Z be the cone over a conic and Δ the union of two rays through the vertex. Then (Z, Δ) is boundary-lc, but not dlt.

The next example shows how boundary-lc pairs appear as limits of dlt pairs. These limits play an important role in Keel–McKernan's proof of the Miyanishi conjecture for surfaces, [KM99, § 6], and in the last two authors' recent attempts to generalize Shafarevich hyperbolicity to families over higher-dimensional base manifolds, [KK07, KK08b], see also [KS06].

Example 3.8. Let (Z, Δ) be a log canonical logarithmic pair. Suppose that Δ is $\mathbb{Q}$-Cartier and that for any positive, sufficiently small rational number $\varepsilon \in \mathbb{Q}^+$, the non-reduced pair $(Z, (1 - \varepsilon)\Delta)$ is dlt, or equivalently klt. Then (Z, Δ) is boundary-lc.

Lemma 3.9. *Let (Z, Δ) be a boundary-lc pair of dimension two. Then Z is $\mathbb{Q}$-factorial and (Z, Δ) is finitely dominated by analytic snc pairs. In particular, dlt surface pairs are finitely dominated by analytic snc pairs.*

The proof of Lemma 3.9 uses the notion of discrepancy, which we recall for the reader's convenience.

Definition 3.10 (Discrepancy, cf. [KM98, § 2.3]). Let (Z, Δ) be a logarithmic pair and let $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a log resolution. If $\widetilde{\Delta}' \subset \widetilde{\Delta}$ is the strict transform of Δ, the $\mathbb{Q}$-divisors $K_{\widetilde{Z}} + \widetilde{\Delta}'$ and $\pi^*(K_Z + \Delta)$ differ only by a $\mathbb{Q}$-linear combination of exceptional divisors. We can therefore write

$$K_{\widetilde{Z}} + \widetilde{\Delta}' = \pi^*(K_Z + \Delta) + \sum_{\substack{E_i \subset \widetilde{Z} \\ \pi\text{-exceptional divisors}}} a(E_i, Z, \Delta) \cdot E_i.$$

The rational number $a(E_i, Z, \Delta)$ is called the *discrepancy of the divisor* E_i.

Proof of Lemma 3.9. Let $z \in (Z, \Delta)_{\text{sing}}$ be an arbitrary singular point. If $z \notin \Delta$, then the statement follows from [KM98, Proposition 4.18]. We can thus assume without loss of generality for the remainder of the proof that $z \in \Delta$.

Next observe that for any rational number $0 < \varepsilon < 1$, the non-reduced pair $(Z, (1 - \varepsilon)\Delta)$ is *numerically dlt*; see [KM98, Notation 4.1] for the definition and use [KM98, Lemma 3.41] for an explicit discrepancy computation. By [KM98, Proposition 4.11], Z is then $\mathbb{Q}$-factorial. Using $\mathbb{Q}$-factoriality, we can then choose a sufficiently small Zariski neighborhood U of z and consider the index-one cover for $\Delta \cap U$. This gives a finite morphism of pairs $\gamma : (\widetilde{U}, \widetilde{\Delta}) \to (U, \Delta \cap U)$, where the morphism γ is branched only over the singularities of U, where $\gamma^{-1}(z) = \{\widetilde{z}\}$ is a single point and where $\widetilde{\Delta} = \gamma^*(\Delta \cap U)$ is Cartier; see [KM98, Definition 5.19] for the construction. Since discrepancies only increase under taking finite covers [KM98, Proposition 5.20], the pair $(\widetilde{U}, \widetilde{\Delta})$ will again be boundary-lc. In particular, it suffices to prove the claim for a neighborhood of $\widetilde{z}$ in $(\widetilde{U}, \widetilde{\Delta})$. We can thus assume without loss of generality that $z \in \Delta$ and that Δ is Cartier in our original setup.

Next, we claim that $(Z, \emptyset)$ is canonical at z. In fact, let E be any divisor centered above z, as in [KM98, Definition 2.24]. Since $z \in \Delta$, and since Δ is Cartier, the pull-back of Δ to any resolution where E appears will contain E with multiplicity at least one. In particular, we have the following inequality of discrepancies: $0 \leqslant a(E, Z, \Delta) + 1 \leqslant a(E, Z, \emptyset)$. This shows that $(Z, \emptyset)$ is canonical at z, as claimed.

By [KM98, Theorem 4.20], it is then clear that Z has a Du Val quotient singularity at z. Again replacing Z by a finite cover of a suitable neighborhood of z, and replacing z by its preimage in the covering space, we can henceforth assume without loss of generality that Z is smooth. But then the claim follows from [KM98, Theorem 4.15]. □

Remark 3.11. It follows from a result of Brieskorn [Bri68] that any two-dimensional pair (X, Δ) that is finitely dominated by analytic snc pairs has quotient singularities in the following sense: for every point $x \in X$ there exists a finite subgroup $G \subset GL_2(\mathbb{C})$ without quasi-reflections, an analytic neighborhood U in X and a biholomorphic map $\varphi : U \to V$ to an analytic neighborhood V of $\pi(0, 0)$ in $\mathbb{C}^2/G$, where $\pi : \mathbb{C}^2 \to \mathbb{C}^2/G$ denotes the quotient map. Furthermore, the preimage $\pi^{-1}(\varphi(\Delta \cap V))$ coincides with the intersection of $a_1 D_1 + a_2 D_2$ with $\pi^{-1}(V)$, where $a_j \in \{0, 1\}$ and $D_j = \{z_j = 0\} \subset \mathbb{C}^2$.

4. Vector fields and local group actions on singular spaces

In this section, we discuss vector fields on singular complex spaces and their relation to local Lie group actions. We will then show that local group actions and vector fields lift to functorial resolutions. This will be used in the proof of the extension theorem for $(n - 1)$-forms in § 6.

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4.1 Local actions and logarithmic vector fields

For the reader's convenience, we recall the standard definition of a local group action.

Definition 4.1 (Local group action, cf. [Kau65, § 4]). Let G be a connected complex Lie group and Z a reduced complex space. A *local G-action* is given by a holomorphic map $\Phi : \Theta \to Z$, where Θ is an open neighborhood of the neutral section $\{e\} \times Z$ in $G \times Z$ such that:

(4.1.1) for all $z \in Z$ the subset $\Theta(z) := \{g \in G \mid (g, z) \in \Theta\}$ is connected;

(4.1.2) setting $\Phi(g, z) =: g{\bullet}z$, we have $e{\bullet}z = z$ for all $z \in Z$ and, if $(gh, z) \in \Theta$, if $(h, z) \in \Theta$ and if $(g, h{\bullet}z) \in \Theta$, then $(gh){\bullet}z = g{\bullet}(h{\bullet}z)$ holds.

There is a natural notion of equivalence of local G-actions on Z given by shrinking Θ to a smaller neighborhood of $\{e\} \times Z$ in $G \times Z$. To an equivalence class of actions one assigns a linear map λ from the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $H^0(Z, \mathscr{T}_Z)$ of vector fields on Z, as follows. If $\xi \in \mathfrak{g}$ is any element of the Lie algebra, its image $\xi_Z = \lambda(\xi)$ is defined by the equation

$$\xi_Z(f)(z) = \frac{d}{dt}\Big|_{t=0} f(\exp_G(-t\xi){\bullet}z),$$

where f is an arbitrary holomorphic function defined near z and $\exp_G : \mathfrak{g} \to G$ is the exponential map of G. If we consider $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G, the map λ is a homomorphism of Lie algebras. The converse statement is a classical result of complex analysis.

Fact 4.2 (Vector fields and local group actions [Kau65, Satz 3]). If $\lambda : \mathfrak{g} \to H^0(Z, \mathscr{T}_Z)$ is a homomorphism of Lie algebras, then, up to equivalence, there exists a unique local G-action on Z that induces the given λ. In particular, any vector field $\eta \in H^0(Z, \mathscr{T}_Z)$ induces a local $\mathbb{C}$-action Φ_η on Z. □

We also note that if (Z, Δ) is a logarithmic pair, then the local $\mathbb{C}$-actions stabilizing Δ are precisely the ones that correspond to logarithmic vector fields, i.e., global sections of $\mathscr{T}_Z(-\log \Delta)$.

The next result is crucial for the lifting property of local group actions.

Lemma 4.3 (Smoothness of the action map). *The action map $\Phi : \Theta \to Z$ of a local G-action is smooth, i.e., a flat submersion.*

Proof. Since the map Φ is locally equivariant, it suffices to show that it is smooth at points of the form $(e, z) \in \Theta$. Given such a point $(e, z) \in \Theta$, there exist an open neighborhood $\Xi = \Xi(e)$ of the identity $e \in G$ and two open neighborhoods U, U' of z in Z such that

$$\Psi : \Xi \times U \to \Xi \times U', (g, z) \mapsto (g, \Phi(g, z))$$

is well defined. The map Ψ is an open embedding; in particular, it is smooth. If we denote the canonical (smooth) projection by $\pi_2 : \Xi \times U' \to U'$, the claim follows from the observation that $\Phi|_{\Xi \times U} = \pi_2 \circ \Psi$ is the composition of smooth morphisms. □

4.2 Lifting vector fields to functorial resolutions

Unlike in the surface case, there is no notion of a 'minimal resolution of singularities' in higher dimensions. There is, however, a canonical resolution procedure that has certain universal properties. We briefly recall the relevant facts.

Theorem 4.4 (Functorial resolution of singularities, cf. [Kol07, Theorems 3.35 and 3.45]). *There exists a resolution functor $\mathcal{R} : (Z, \Delta) \to (\pi_{Z,\Delta} : R(Z, \Delta) \to (Z, \Delta))$ that assigns to any*

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logarithmic pair (Z, Δ) *a new pair* $R(Z, \Delta)$ *and a morphism* $\pi_{Z,\Delta} : R(Z, \Delta) \to (Z, \Delta)$*, with the following properties.*

(4.4.1) *The morphism* $\pi := \pi_{Z,\Delta} : R(Z, \Delta) \to (Z, \Delta)$ *is a log resolution of* (Z, Δ)*.*

(4.4.2) *The morphism* π *is projective over any compact subset of* Z*.*

(4.4.3) *The functor* $\mathcal{R}$ *commutes with smooth holomorphic maps. That is to say, for any smooth morphism* $f : (X, D) \to (Z, \Delta)$ *of logarithmic pairs there exists a unique smooth morphism* $\mathcal{R}(f) : R(X, D) \to R(Z, \Delta)$ *giving a fiber product square as follows.*

$$\begin{array}{ccc} R(X,D) & \xrightarrow{\mathcal{R}(f)} & R(Z,\Delta) \\ {\scriptstyle \pi_{X,D}}\downarrow & & \downarrow{\scriptstyle \pi_{Z,\Delta}} \\ (X,D) & \xrightarrow{f} & (Z,\Delta) \end{array}$$

□

Notation 4.5. We call a log resolution $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ *functorial* if it is of the form $\mathcal{R}(Z, \Delta)$.

Proposition 4.6 (Lifting of local actions to the functorial resolution). *Let* $\Phi : \Theta \to Z$ *be a local* G*-action on a complex space* Z*. Let* $\pi : (\widetilde{Z}, \emptyset) \to (Z, \emptyset)$ *be a functorial log resolution. Then* Φ *lifts to a local* G*-action on* $\widetilde{Z}$*. More precisely, if* $\widetilde{\Theta} := (\mathrm{Id}_G \times \pi)^{-1}(\Theta) \subset G \times \widetilde{Z}$*, then there exists a local action* $\widetilde{\Phi} : \widetilde{\Theta} \to \widetilde{Z}$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \widetilde{\Theta} & \xrightarrow{\widetilde{\Phi}} & \widetilde{Z} \\ {\scriptstyle \mathrm{Id}_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Theta & \xrightarrow{\Phi} & Z \end{array}$$

Furthermore, if (Z, Δ) *is a logarithmic pair, if* $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ *is a functorial log resolution, if* $\Phi = \Phi_\xi$ *for some* $\xi \in H^0(Z, \mathscr{T}_Z(-\log \Delta))$ *and if* W *is any* Φ*-invariant subvariety of* Z*, we set*

$$\widetilde{\Delta}_W := \text{largest reduced divisor contained in } \pi^{-1}(\Delta \cup W).$$

Then $\widetilde{\Phi}$ *stabilizes* $\widetilde{\Delta}_W$*.*

Proof. Using Lemma 4.3 and the fact that $\mathcal{R}$ commutes with smooth holomorphic maps, we see that the application of $\mathcal{R}$ to the diagram

$$G \times Z \hookleftarrow \Theta \to Z$$

induces a holomorphic map $\widetilde{\Phi} : (\mathrm{Id}_G \times \pi)^{-1}(\Theta) =: \widetilde{\Theta} \to \widetilde{Z}$ such that the following diagram commutes.

$$\begin{array}{ccccc} G \times \widetilde{Z} & \xleftarrow{\text{inclusion}} & \widetilde{\Theta} & \xrightarrow{\widetilde{\Phi}} & \widetilde{Z} \\ {\scriptstyle \mathrm{Id}_G \times \pi}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ G \times Z & \xleftarrow[\text{inclusion}]{} & \Theta & \xrightarrow{\Phi} & Z \end{array}$$

It remains to check that $\widetilde{\Phi} : \widetilde{\Theta} \to \widetilde{Z}$ defines a local G-action. First, notice that $\widetilde{\Theta}$ is an open neighborhood of the neutral section $\{e\} \times \widetilde{Z}$ in $G \times \widetilde{Z}$. By construction, for a point $\widetilde{z} \in \widetilde{Z}$ we have

$$\widetilde{\Theta}(\widetilde{z}) = \Theta(\pi(\widetilde{z})). \tag{4.6.1}$$

Furthermore, we have $g \bullet \pi(\widetilde{z}) = \pi(g \bullet \widetilde{z})$ for all $\widetilde{z} \in \widetilde{Z}$ and for all $g \in \widetilde{\Theta}(\widetilde{z})$. It immediately follows that $\widetilde{\Theta}(\widetilde{z})$ is connected for all $\widetilde{z} \in \widetilde{Z}$. Since the biholomorphic map $\widetilde{\Phi}_e : \widetilde{Z} \to \widetilde{Z}$ fixes any point

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in $\widetilde{Z} \setminus \mathrm{Exc}(\pi)$, it coincides with $\mathrm{Id}_{\widetilde{Z}}$. Given $\widetilde{z} \in \widetilde{Z}$, let $g, h \in G$ be such that the assumptions of (4.1.2) are fulfilled. By (4.6.1) there exists an open neighborhood U of $\pi(\widetilde{z})$ in Z such that both $\widetilde{\Phi}_{gh}$ and $\widetilde{\Phi}_g \circ \widetilde{\Phi}_h$ are defined on $\pi^{-1}(U)$. Since they coincide on $\pi^{-1}(U) \setminus \mathrm{Exc}(\pi)$, they coincide at $\widetilde{z}$. Hence, we have shown that $\widetilde{\Phi} : \widetilde{\Theta} \to \widetilde{Z}$ is a local G-action.

If (Z, Δ) is a logarithmic pair, if $\xi \in H^0(Z, \mathscr{T}_Z(-\log \Delta))$ is a logarithmic vector field and if W is a Φ_ξ-invariant subvariety of Z, for all $\widetilde{z} \in \widetilde{\Delta}_W$ and for all $g \in \widetilde{\Theta}(\widetilde{z}) = \Theta(\pi(\widetilde{z}))$ we have $\pi(g\bullet\widetilde{z}) = g\bullet\pi(\widetilde{z}) \in \Delta \cup W$. Since $\widetilde{\Theta}(\widetilde{z})$ is connected, this shows the claim. □

Corollary 4.7. *Let (Z, Δ) be a logarithmic pair, $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ a functorial log resolution and W a subvariety of Z that is invariant under any local automorphism of (Z, Δ). Set*

$$\widetilde{\Delta}_W := \text{largest reduced divisor contained in } \pi^{-1}(\Delta \cup W).$$

Then $\pi_* \mathscr{T}_{\widetilde{Z}}(-\log \widetilde{\Delta}_W)$ is reflexive.

Proof. Let $U \subset Z$ be an open subset and let $\xi \in H^0(U \setminus (Z, \Delta)_{\mathrm{sing}}, \mathscr{T}_Z(-\log \Delta))$ be a vector field. Since $\mathscr{T}_Z(-\log \Delta) = \Omega^1_Z(\log \Delta)^*$ is reflexive, ξ extends to a logarithmic vector field on U, i.e., to an element $\xi \in H^0(U, \mathscr{T}_Z(-\log \Delta))$. Lifting the local $\mathbb{C}$-action Φ_ξ that corresponds to ξ with the help of Proposition 4.6, we obtain a local $\mathbb{C}$-action on $\pi^{-1}(U)$ that stabilizes $\widetilde{\Delta}_W$. The corresponding vector field $\widetilde{\xi} \in H^0(\pi^{-1}(U), \mathscr{T}_{\widetilde{Z}}(-\log \widetilde{\Delta}_W))$ is an extension of ξ considered as an element of $H^0(\pi^{-1}(U \setminus (Z, \Delta)_{\mathrm{sing}}), \mathscr{T}_{\widetilde{Z}}(-\log \widetilde{\Delta}_W))$. □

Part II. Extension theorems for log canonical pairs

5. Proof of Theorem 1.1 for n-forms

In this section, we consider the extension problem for logarithmic n-forms. The proof of the case $p = n$ of Theorem 1.1 immediately follows from the following, slightly stronger result. The *discrepancy* of an exceptional divisor has been introduced in Definition 3.10 above.

Proposition 5.1. *Let (Z, Δ) be an n-dimensional log canonical logarithmic pair. Let $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a log resolution and $E_{\mathrm{lc}} \subset \widetilde{Z}$ the union of all π-exceptional prime divisors $E \not\subseteq \widetilde{\Delta}$ with discrepancy $a(E, Z, \Delta) = -1$, endowed with the structure of a reduced subscheme of $\widetilde{Z}$. Then the sheaf $\pi_* \Omega^n_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_{\mathrm{lc}}))$ is reflexive.*

Proof. After shrinking Z if necessary, it suffices to show that the pull-back of any n-form $\sigma \in H^0(Z, \Omega^{[n]}_Z(\log \Delta))$ extends to an element of $H^0(\widetilde{Z}, \Omega^n_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_{\mathrm{lc}})))$. Using the argument in the proof of Corollary 3.3 and the discrepancy calculation in the proof of [KM98, Proposition 5.20], we see that it is sufficient to prove the claim under the additional assumption that $K_Z + \Delta$ is Cartier.

First, we renumber the exceptional prime divisors $E_1, \ldots, E_m$ of π in such a way that:

(5.1.1) $\pi(E_j) \subset \Delta$ if and only if $j = 1, \ldots, k$;

(5.1.2) $a(E_j, Z, \Delta) \geqslant 0$ for $j = k + 1, \ldots, l$;

(5.1.3) $E_{\mathrm{lc}} = \bigcup_{j=l+1}^{m} E_j$.

Using the assumption that $a(E_j, Z, \Delta) \geqslant -1$ for all j, we obtain that

$$K_{\widetilde{Z}} + \pi_*^{-1}(\Delta) - \sum_{j=1}^{k} a(E_j, Z, \Delta)E_j = K_{\widetilde{Z}} + \widetilde{\Delta} - \sum_{j=1}^{k} c_j E_j \tag{5.1.4}$$

for some $c_j \geqslant 0$. From (5.1.4) and the definition of discrepancy, we conclude that

$$\pi^*(K_Z + \Delta) = K_{\widetilde{Z}} + \widetilde{\Delta} + E_{\mathrm{lc}} - \sum_{j=1}^{l} b_j E_j, \tag{5.1.5}$$

for some $b_j \geqslant 0$; note that the b_j are integral because $K_Z + \Delta$ is Cartier. Equation (5.1.5) then implies that any n-form $\sigma \in H^0(Z, \Omega_Z^{[n]}(\log \Delta)) = H^0(Z, \mathscr{O}_Z(K_Z + \Delta))$ extends to an element of $H^0(\widetilde{Z}, \Omega^n_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_{\mathrm{lc}})))$. □

Remark 5.2. It follows from the proof of Proposition 5.1 that the assumption 'log canonical' is indeed necessary for the case $p = n$ of the Theorem 1.1.

6. Proof of Theorem 1.1 for $(n-1)$-forms

In this section, we consider the case $p = n - 1$ of Theorem 1.1. We recall the statement as follows.

Proposition 6.1. *Let (Z, Δ) be a log canonical logarithmic pair of dimension n. Let $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ be a log resolution and set*

$$\widetilde{\Delta}_{\mathrm{lc}} := \text{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \text{non-klt locus of } (Z, \Delta)).$$

Then $\pi_* \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive.

Proof. After shrinking Z, it suffices to show that the pull-back $\pi^*\sigma$ of any $\sigma \in H^0(Z, \Omega_Z^{[n-1]}(\log \Delta))$ extends to an element of $H^0(\widetilde{Z}, \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}))$. By Corollary 3.3, we may assume that $K_Z + \Delta$ is Cartier and, possibly after a further shrinking of Z, that $K_Z + \Delta$ is trivial. Finally, due to Lemma 2.13 we may assume that $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ is a functorial log resolution.

Since $\Omega_Z^{[1]}(\log \Delta)^* \cong \mathscr{T}_Z(-\log \Delta)$, there exists a unique logarithmic vector field $\eta \in H^0(Z, \mathscr{T}_Z(-\log \Delta))$ that corresponds to σ via the perfect pairing

$$\Omega_Z^{[1]}(\log \Delta) \times \Omega_Z^{[n-1]}(\log \Delta) \to \mathscr{O}_Z(K_Z + \Delta) \cong \mathscr{O}_Z.$$

Since the non-klt locus is invariant under the local $\mathbb{C}$-action Φ_η of η, we can lift η to a vector field $\widetilde{\eta} \in H^0(\widetilde{Z}, \mathscr{T}_{\widetilde{Z}}(-\log \widetilde{\Delta}_{\mathrm{lc}}))$ using Corollary 4.7. The assumption that (Z, Δ) is log canonical implies, via a discrepancy computation similar to (5.1.5) in the proof of Proposition 5.1, that $\mathscr{O}_{\widetilde{Z}}(K_{\widetilde{Z}} + \widetilde{\Delta}_{\mathrm{lc}}) \cong \mathscr{O}_{\widetilde{Z}}(D)$ for some effective divisor D on $\widetilde{Z}$. Hence, the logarithmic vector field $\widetilde{\eta}$ corresponds to an element $\widetilde{\sigma} \in H^0(\widetilde{Z}, \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}) \otimes \mathscr{O}_{\widetilde{Z}}(-D))$ via the pairing

$$\Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}) \times \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}) \to \mathscr{O}_{\widetilde{Z}}(K_{\widetilde{Z}} + \widetilde{\Delta}_{\mathrm{lc}}) \cong \mathscr{O}_{\widetilde{Z}}(D).$$

This yields the desired extension of σ. □

Remark 6.2. If $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ is a log resolution of a log canonical surface pair (Z, Δ), not only $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ but also $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta})$ is reflexive, i.e., 1-forms extend over the exceptional set of π without acquiring further logarithmic poles, see [Wah85, Lemma 1.3].

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We conclude this section with an example showing that the assumption 'log canonical' in Theorem 1.1 is necessary also for the cases $p=1$ and $n-1$, cf. Remark 5.2.

Example 6.3. Let Z be the affine cone over a smooth curve C of degree four in $\mathbb{P}^2$. Let $\widetilde{Z}$ be the total space of the line bundle $\mathscr{O}_C(-1)$. Then the contraction of the zero section E of $\widetilde{Z}$ yields a log resolution $\pi : (\widetilde{Z}, \emptyset) \to (Z, \emptyset)$. An elementary intersection number computation shows that the discrepancy of E with respect to Z is equal to -2. If $Z = \{f=0\}$ for some quartic form f in three variables z_0, z_1, z_2, the (rational) differential form

$$\tau = \frac{dz_1 \wedge dz_2}{\partial f / \partial z_0}$$

yields a global generator for $\Omega_Z^{[2]}$, cf. [Rei87, Example 1.8]. Let $\bar{\tau} := \pi^*(\tau) \in H^0(\widetilde{Z},\, \Omega^2_{\widetilde{Z}}(2E))$ be the associated rational 2-form on $\widetilde{Z}$, and observe that $\bar{\tau}$, seen as a section in $\Omega^2_{\widetilde{Z}}(2E)$, does not vanish along E. Finally, let ξ be the vector field induced by the canonical $\mathbb{C}^*$-action on $\widetilde{Z}$. Contracting $\bar{\tau}$ by ξ, we obtain a regular 1-form $\sigma = \imath_\xi \bar{\tau}$ on $\widetilde{Z} \setminus E$ that does not extend to an element of $H^0(\widetilde{Z},\, \Omega^1_{\widetilde{Z}}(\log E))$. To see this, let U be an open subset of C such that $\mathscr{O}_C(-1)|_U$ is trivial, and such that there exists a local coordinate z on U. If the bundle projection is denoted by $p : \widetilde{Z} \to C$, consider $\widetilde{U} := p^{-1}(U) \cong U \times \mathbb{C}$. If w is a linear fiber coordinate on $\widetilde{U}$, we have $\widetilde{U} \cap E = \{w = 0\}$. In these coordinates, $\bar{\tau}|_{\widetilde{U}} = (g(z,w)/w^2)\, dz \wedge dw$ for some nowhere vanishing $g \in \mathscr{O}_{\widetilde{U}}(\widetilde{U})$, and $\xi|_{\widetilde{U}} = w(\partial/\partial w)$. Hence, in the chosen coordinates we have $\sigma|_{\widetilde{U}} = -(g(z,w)/w)\, dz \notin H^0(\widetilde{U}, \Omega^1_{\widetilde{Z}}(\log E))$.

7. Proof of Theorem 1.1 for 1-forms

The aim of the present section is to prove the Theorem 1.1 for 1-forms. This is an immediate consequence of the following stronger proposition.

Proposition 7.1. *Let (Z, Δ) be a reduced log canonical pair. Let $\pi : \widetilde{Z} \to Z$ be a birational morphism such that $\widetilde{Z}$ is smooth, the π-exceptional set $\operatorname{Exc}(\pi) \subset \widetilde{Z}$ is of pure codimension one and $\operatorname{supp}(\pi^{-1}(\Delta) \cup \operatorname{Exc}(\pi))$ is a divisor with simple normal crossings. Let*

$$\widetilde{\Delta}_{\mathrm{lc}} := \textit{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \textit{non-klt locus of } (Z,\Delta)). \tag{7.1.1}$$

Then the sheaf $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive.

Remark 7.1.2. Observe that the morphism π in Proposition 7.1 need not be a log resolution in the sense of Definition 2.6, as we do not assume that π is isomorphic over the set where (Z, Δ) is snc. The setup of Proposition 7.1 has the advantage that it behaves well under hyperplane sections. This makes it easier to proceed by induction.

We will prove Proposition 7.1 in the remainder of the present section. Since the proof is somewhat involved, we chose to present it as a sequence of clearly marked and relatively independent steps.

7.1 Proof of Proposition 7.1: setup of notation

For notational convenience, we call a birational morphism *admissible* if it satisfies the assumptions made in Proposition 7.1.

Notation 7.2 (Admissible morphism). Throughout this section, if (X, D) is a logarithmic pair, we call a birational morphism $\eta : \widetilde{X} \to X$ *admissible* if $\widetilde{X}$ is smooth, the η-exceptional set $\operatorname{Exc}(\eta)$

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is of pure codimension one and

$$\operatorname{supp}(\eta^{-1}(D) \cup \operatorname{Exc}(\eta))$$

has simple normal crossings.

Notation 7.3. In the setup of Proposition 7.1, we denote the irreducible components of $\operatorname{Exc}(\pi)$ by $E_i \subset \widetilde{Z}$. Further, let $T \subset X$ denote the set of fundamental points of π^{-1}. For $x \in T$, let $F_x := \pi^{-1}(x)$ be the associated fiber and $F_{x,i} := F_x \cap E_i$ the obvious decomposition.

7.2 Proof of Proposition 7.1: technical preparations

To prove Proposition 7.1, we argue using repeated hyperplane sections of Z. We show that the induced resolutions of general hyperplanes are again admissible.

Lemma 7.4. *In the setup of Proposition 7.1, assume that* $\dim Z > 1$ *and let* $H \subset Z$ *be a general hyperplane section.*

(7.4.1) *If* $\Delta_H := \operatorname{supp}(H \cap \Delta)$, *then the pair* (H, Δ_H) *is again log canonical.*

(7.4.2) *If* $\widetilde{H} := \pi^{-1}(H)$, *then the restricted morphism* $\pi|_{\widetilde{H}} : \widetilde{H} \to H$ *is admissible.*

(7.4.3) *If* $\widetilde{\Delta}_{\widetilde{H},\mathrm{lc}}$ *is the largest reduced divisor contained in*

$$\pi^{-1}(\Delta_H \cup \textit{non-klt locus of } (H, \Delta_H)),$$

then $\widetilde{\Delta}_{\widetilde{H},\mathrm{lc}} \subset \widetilde{\Delta}_{\mathrm{lc}} \cap \widetilde{H}$.

Remark 7.4.4. The inclusion $\widetilde{\Delta}_{\widetilde{H},\mathrm{lc}} \subset \widetilde{\Delta}_{\mathrm{lc}} \cap \widetilde{H}$ of (7.4.3) might be strict.

Proof. Seidenberg's theorem asserts that H is normal, cf. [BS95, Theorem 1.7.1]. Recall from [KM98, Lemma 5.17] that discrepancies do not decrease when taking general hyperplane sections. It follows that the pair (H, Δ_H) is log canonical since (Z, Δ) is. This shows (7.4.1). Assertion (7.4.3) follows from [KM98, Lemma 5.17(1)].

Since $\widetilde{H}$ is general in its linear system, Bertini's theorem guarantees that $\widetilde{H}$ is smooth. Zariski's main theorem [Har77, V Theorem 5.2] now asserts that a point $z \in \widetilde{Z}$ is in $\operatorname{Exc}(\pi)$ if and only if the fiber that contains z is positive dimensional; the same holds for $\pi|_{\widetilde{H}}$. By construction, we then have that

$$\operatorname{Exc}(\pi|_{\widetilde{H}}) = \operatorname{Exc}(\pi) \cap \widetilde{H}, \tag{7.4.5}$$

$$\operatorname{supp}(\pi|_{\widetilde{H}}^{-1}(\Delta_H) \cup \operatorname{Exc}(\pi|_{\widetilde{H}})) = \operatorname{supp}(\pi^{-1}(\Delta) \cup \operatorname{Exc}(\pi)) \cap \widetilde{H}. \tag{7.4.6}$$

The left-hand side of (7.4.5) is thus of pure codimension one in $\widetilde{H}$, and another application of Bertini's theorem implies that the left-hand side of (7.4.6) is a divisor in $\widetilde{H}$ with simple normal crossings. The admissibility asserted in (7.4.2) is thus shown. □

The following elementary corollary of Mumford's contractibility criterion [Mum61, p. 6] helps in the discussion of linear systems of divisors supported on fibers over isolated points.

Proposition 7.5. *Let* $\phi : \widetilde{Y} \to Y$ *be a projective birational morphism between quasi-projective, normal varieties of dimension* $\dim Y > 1$ *and assume that* $\widetilde{Y}$ *is smooth. Let* $y \in Y$ *be a point whose preimage* $\phi^{-1}(y)$ *has codimension one*[1] *and let* $F_0, \ldots, F_k \subset \operatorname{supp}(\phi^{-1}(y))$ *be the reduced*

[1] We do not assume that $\phi^{-1}(y)$ has *pure* codimension one.

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divisorial components. If all the F_i are smooth and if $\sum k_i F_i$ is a non-trivial, effective linear combination, then there exists a number j, $0 \leqslant j \leqslant k$, such that $k_j \neq 0$ and such that

$$h^0\left(F_j,\ \mathscr{O}_{\widetilde{Y}}\left(\sum k_i F_i\right)\bigg|_{F_j}\right) = 0. \tag{7.5.1}$$

Proof. If j is any number with $k_j = 0$, then the trivial sheaf $\mathscr{O}_{F_j}$ injects into $\mathscr{O}_{\widetilde{Y}}(\sum k_i F_i)|_{F_j}$ and (7.5.1) cannot hold. To prove Proposition 7.5, it therefore suffices to find a number j such that (7.5.1) holds; the assertion $k_j \neq 0$ is then automatic.

In order to do this, consider general hyperplanes $\widetilde{H}_1, \ldots, \widetilde{H}_{\dim Y-2} \subset \widetilde{Y}$, and let $\widetilde{H} = \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_{\dim Y-2}$ be their intersection. Then $\widetilde{H}$ is a smooth surface and the intersections $C_i := \widetilde{H} \cap F_i$ are smooth curves. The Stein factorization of $\phi|_{\widetilde{H}}$,

$$\widetilde{H} \xrightarrow{\ \alpha\ } \widetilde{H}' \xrightarrow{\ \beta\ } Y, \qquad \beta \circ \alpha = \phi|_{\widetilde{H}}$$

gives $\alpha : \widetilde{H} \to \widetilde{H}'$, a birational morphism that maps to a normal surface and contracts precisely the curves $C_i \subset \widetilde{H}$. Using Mumford's criterion that the intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative definite, we see that there exists a $j \in \{1, \ldots, k\}$ such that

$$\deg_{C_j} \mathscr{O}_{\widetilde{Y}}\left(\sum k_i F_i\right)\bigg|_{C_j} = C_j \cdot \left(\sum k_i F_i|_{\widetilde{H}}\right) < 0,$$

where the intersection product in the middle term is that of curves on the smooth surface $\widetilde{H}$, cf. [KMM87, Lemma 5-1-7]. In particular, any section $\sigma \in H^0(F_j,\ \mathscr{O}_{F_j}(\sum k_i F_i|_{F_j}))$ vanishes on C_j and on all of its deformations. Since the $\widetilde{H}_i$ are general, those deformations dominate F_j, and the section σ must vanish on all of F_j. This shows (7.5.1) and completes the proof.

7.3 Proof of Proposition 7.1: extendability over isolated points

Before proving Proposition 7.1 in full generality in § 7.4 below, we consider the case where reflexivity of $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is already known away from a finite set. This result will be used as the anchor for the inductive argument used in § 7.4. The argument relies on a vanishing result of Steenbrink [Ste85].

Proposition 7.6. *In the setup of Proposition 7.1, let $\Sigma \subset T$ be a finite set of points. Assume that $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive away from Σ. Then $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive.*

Proof. If $n := \dim Z = 2$, the result is shown in Proposition 6.1 above. We will thus assume for the remainder of the proof that $n \geqslant 3$. Since the assertion is local on Z, we can shrink Z and assume without loss of generality that the following holds:

(7.6.1) the set Σ contains only a single point, $\Sigma = \{z\}$; and

(7.6.2) either $\Delta = \emptyset$ or every irreducible component of Δ contains z.

By Lemma 2.13, we are free to blow up $\widetilde{Z}$ further, if necessary. Thus, we can also assume that the following holds:

(7.6.3) the reduced fiber $F_z := (\pi^{-1}(z))_{\mathrm{red}}$ is a simple normal crossings divisor on $\tilde{Z}$; and

(7.6.4) the divisor $\widetilde{\Delta}'_{\mathrm{lc}} := (\widetilde{\Delta}_{\mathrm{lc}} + F_z)_{\mathrm{red}}$ is a simple normal crossings divisors on $\widetilde{Z}$.

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To prove Proposition 7.6, after shrinking Z more, if necessary, we need to show that the natural restriction map

$$H^0(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})) \to H^0(\widetilde{Z} \setminus F_z, \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})) \tag{7.6.5}$$

is surjective. The proof proceeds in two steps. First, we show surjectivity of (7.6.5) when we replace $\widetilde{\Delta}_{\mathrm{lc}}$ by the slightly larger divisor $\widetilde{\Delta}'_{\mathrm{lc}}$. Surjectivity of (7.6.5) is then shown in a second step.

Step 1: extension with logarithmic poles along $\widetilde{\Delta}'_{\mathrm{lc}}$. Since $n \geqslant 3$, a vanishing result of Steenbrink [Ste85, Theorem 2.b] asserts that

$$R^{n-1}\pi_*(\mathscr{J}_{\widetilde{\Delta}'_{\mathrm{lc}}} \otimes \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) = 0. \tag{7.6.6}$$

Theorem A.1 states that for any locally free sheaf $\mathscr{F}$ on $\widetilde{Z}$ and any number $0 \leqslant j \leqslant n$, there exists an isomorphism

$$((R^j\pi_*\mathscr{F})_z)\widehat{\ } \cong H^{n-j}_{\pi^{-1}(z)}(\widetilde{Z},\ \mathscr{F}^* \otimes \omega_{\widetilde{Z}})^*,$$

where $\widehat{\ }$ denotes completion with respect to the maximal ideal $\mathfrak{m}_z$ of the point $z \in Z$. Setting $\mathscr{F} := \mathscr{J}_{\widetilde{\Delta}'_{\mathrm{lc}}} \otimes \Omega^{n-1}_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})$ and using $\mathscr{F}^* \otimes \omega_{\widetilde{Z}} \cong \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})$, we see that the vanishing (7.6.6) implies that the following cohomology group support vanishes:

$$H^1_{F_z}(\widetilde{Z}, \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) = \{0\}.$$

The standard sequence for cohomology with supports [Har77, III ex. 2.3e],

$$\cdots \to H^0(\widetilde{Z}, \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) \to H^0(\widetilde{Z} \setminus F_z,\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) \to H^1_{F_z}(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) \to \cdots,$$

then shows surjectivity of the restriction map (7.6.5) for the larger boundary divisor $\widetilde{\Delta}'_{\mathrm{lc}}$.

Step 2: extension as a form with logarithmic poles along $\widetilde{\Delta}_{\mathrm{lc}}$. To prove surjectivity of (7.6.5), we will show that the natural inclusion

$$H^0(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})) \to H^0(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}'_{\mathrm{lc}})) \tag{7.6.7}$$

is surjective. The results of Step 1 will then finish the proof of Proposition 7.6.

If $z \in \Delta$, or if z is contained in the non-klt locus, then the divisors $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ agree after some additional shrinking of Z, and (7.6.7) is the identity map. So, we may assume that $z \notin \Delta$, and that the pair (Z, Δ) is log terminal (i.e., plt) in a neighborhood of z. Assumption (7.6.2) then asserts that $\Delta = \emptyset$. It follows that $\widetilde{\Delta}_{\mathrm{lc}} = \emptyset$ and that $\widetilde{\Delta}'_{\mathrm{lc}} = F_z$. In this setup, recall the well-known result that Z has only rational singularities at z, cf. [KM98, Theorem 5.22]. For rational singularities, surjectivity of (7.6.7) has been shown by Namikawa [Nam01, Lemma 2]. □

7.4 Proof of Proposition 7.1: end of proof

To finish the proof of Proposition 7.1, after possibly shrinking Z, let $\sigma \in H^0(\widetilde{Z} \setminus E, \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}))$ be any form defined outside the π-exceptional set $E := \mathrm{Exc}(\pi)$ and let $\widetilde{\sigma} \in H^0(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})(*E))$ be its extension to $\widetilde{Z}$ as a logarithmic form, possibly with poles along E.

We need to show that indeed $\widetilde{\sigma}$ does not have any poles as a logarithmic form. More precisely, if $E' \subset E$ is any irreducible component, then we show that $\widetilde{\sigma}$ does not have any poles along E',

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i.e.,

$$\widetilde{\sigma} \in H^0(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})(*(E-E'))). \tag{7.6.8}$$

To prove this, we proceed by induction on pairs $(\dim Z, \operatorname{codim} \pi(E'))$, which we order lexicographically as indicated in Table 1.

Table 1. Lexicographical ordering of dimensions and codimensions.

No.	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\dim Z$	2	3	3	4	4	4	5	5	5	5	$\cdots$
$\operatorname{codim} \pi(E')$	2	2	3	2	3	4	2	3	4	5	$\cdots$

For convenience of notation, we renumber the irreducible components E_i of E, if necessary, and assume that $E' = E_0$ and that there exists a number k such that

$$\{E_0, \dots, E_k\} = \{E_i \subset E \text{ an irreducible component} | \pi(E_i) = \pi(E_0)\}.$$

Further, let $k_i \in \mathbb{N}$ be the pole orders of $\widetilde{\sigma}$ along the E_i, i.e., the minimal numbers such that

$$\widetilde{\sigma} \in H^0\left(\widetilde{Z},\ \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}) \otimes \mathscr{O}_{\widetilde{Z}}\left(\sum k_i E_i\right)\right).$$

To prove (7.6.8), it is then equivalent to show that $k_0 = 0$.

Start of induction. In the case $\dim Z = \operatorname{codim} \pi(E_0) = 2$, the set T of fundamental points is necessarily isolated, and Proposition 7.6 applies.[2]

Inductive step. Our induction hypothesis is that the extension statement as in (7.6.8) holds for all log canonical pairs (X, D), all admissible morphisms $\pi_X : \widetilde{X} \to X$, all logarithmic forms on $\widetilde{X}$ defined outside the π_X-exceptional set and all π_X-exceptional divisors $E'_X \subset \widetilde{X}$, where either

$$\dim X < \dim Z \quad \text{or} \quad (\dim X = \dim Z \text{ and } \operatorname{codim} \pi_X(E'_X) < \operatorname{codim} \pi(E_0)).$$

If $\dim Z = \operatorname{codim} \pi(E_0)$, then the induction hypothesis asserts that the set of points where $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is not already known to be reflexive is at most finite. But then Proposition 7.6 again implies that $\pi_* \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})$ is reflexive everywhere, and the claim holds. We will therefore assume without loss of generality for the remainder of this proof that $\dim Z > \operatorname{codim} \pi(E_0)$ or, equivalently, that $\dim \pi(E_0) > 0$.

Now choose general hyperplanes $H_1, \dots, H_{\dim \pi(E_0)} \subset Z$ and consider their intersection $H := H_1 \cap \cdots \cap H_{\dim \pi(E_0)}$ and its preimage $\widetilde{H} := \pi^{-1}(H)$. Setting $\Delta_H := \operatorname{supp}(\Delta \cap H)$ and $\widetilde{H} := \pi^{-1}(H)$, a repeated application of Lemma 7.4 then guarantees that the pair (H, Δ_H) is log canonical, and the restricted morphism $\pi|_{\widetilde{H}}$ is admissible. If $\widetilde{\Delta}_{H,\mathrm{lc}} \subset \widetilde{H}$ is the divisor discussed in Lemma 7.4, the induction hypothesis applies to forms on $\widetilde{H}$ with logarithmic poles along $\widetilde{\Delta}_{H,\mathrm{lc}} \subset \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}$.

The variety H then intersects $\pi(E_0)$ in finitely many points, which are general in $\pi(E_0)$. Let $z \in H \cap \pi(E_0)$ be one of them, and let $F_z := \pi^{-1}(z)$ be the fiber over z. Shrinking Z, if necessary, we may assume without loss of generality that z is the only point of intersection, $\{z\} = H \cap \pi(E_0)$.

[2] Alternatively, Proposition 6.1 would also apply.

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The fiber $F_z \subset \widetilde{H}$ will generally be reducible, and need not be of pure dimension. However, if we set

$$F_{z,i} := F_z \cap E_i$$

then an elementary computation of dimensions and codimensions shows that the first $(k+1)$ intersections, $F_{z,0}, \ldots, F_{z,k} \subset F_z$, are precisely those irreducible components of F_z that have codimension one in $\widetilde{H}$. For $0 \leqslant i \leqslant k$, we also obtain that

$$F_{z,i} := E_i \cap \widetilde{H}.$$

In particular, since $\pi|_{\widetilde{H}}$ is admissible by Lemma 7.4 and the E_i are all smooth by assumption, Bertini's theorem applies to show that the $(F_{z,i})_{0 \leqslant i \leqslant k}$ are smooth as well. Note that all prerequisites of Proposition 7.5 are thus satisfied. We will apply that proposition later near the end of the proof.

Now consider the standard restriction sequence for logarithmic forms, cf. [KK08a, Lemma 2.13 and references therein],

$$0 \longrightarrow N^*_{\widetilde{H}/\widetilde{Z}} \longrightarrow \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})|_{\widetilde{H}} \xrightarrow{\varrho} \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}) \longrightarrow 0,$$

its twist with $\mathscr{F} := \mathscr{O}_{\widetilde{H}}(\sum k_i E_i|_{\widetilde{H}})$ and its restriction to $F_{z,j}$, for $0 \leqslant j \leqslant k$. We have the following.

$$\begin{array}{ccccc}
N^*_{\widetilde{H}/\widetilde{Z}} \otimes \mathscr{F} & \xrightarrow{\alpha} & \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})|_{\widetilde{H}} \otimes \mathscr{F} & \xrightarrow{\beta} & \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}) \otimes \mathscr{F} \\
\downarrow & & \downarrow{\scriptstyle r_{1,j}} & & \downarrow{\scriptstyle r_{2,j}} \\
N^*_{\widetilde{H}/\widetilde{Z}} \otimes \mathscr{F}|_{F_{z,j}} & \xrightarrow[\alpha_j]{} & \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}}) \otimes \mathscr{F}|_{F_{z,j}} & \xrightarrow[\beta_j]{} & \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}) \otimes \mathscr{F}|_{F_{z,j}}
\end{array}$$

The induction hypothesis now asserts that $\widetilde{\sigma}|_{\widetilde{H}}$ is a regular logarithmic form on $\widetilde{H}$. More precisely, using the notation $\varrho : \Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})|_{\widetilde{H}} \to \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}})$ from above, we have

$$\begin{aligned}
\varrho(\widetilde{\sigma}|_{\widetilde{H}}) &\in H^0(\widetilde{H},\ \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{H,\mathrm{lc}})) && \text{by the induction hypothesis} && (7.6.9)\\
&\subseteq H^0(\widetilde{H},\ \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}})) && \text{because } \widetilde{\Delta}_{H,\mathrm{lc}} \subseteq \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}} \text{ by (7.4)} && (7.6.10)\\
&\subseteq H^0(\widetilde{H},\ \Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}) \otimes \mathscr{F}) && \text{because } \mathscr{O}_{\widetilde{H}} \subseteq \mathscr{F}. && (7.6.11)
\end{aligned}$$

If j is any number with $k_j > 0$, we can say more. The choice of the k_j guarantees that $\widetilde{\sigma}|_{\widetilde{H}}$ is a section in $\Omega^1_{\widetilde{Z}}(\log \widetilde{\Delta}_{\mathrm{lc}})|_{\widetilde{H}} \otimes \mathscr{F}$ that does not vanish along $\widetilde{H} \cap E_j$. On the other hand, (7.6.9)–(7.6.11) assert that $\beta(\widetilde{\sigma}|_{\widetilde{H}})$, i.e., $\varrho(\widetilde{\sigma}|_{\widetilde{H}})$, viewed as a section of $\Omega^1_{\widetilde{H}}(\log \widetilde{\Delta}_{\mathrm{lc}}|_{\widetilde{H}}) \otimes \mathscr{F}$, must necessarily vanish along $\widetilde{H} \cap E_j$. In other words, we obtain that

$$r_{1,j}(\widetilde{\sigma}|_{\widetilde{H}}) \neq 0 \quad \text{and} \quad (\beta_j \circ r_{1,j})(\widetilde{\sigma}|_{\widetilde{H}}) = (r_{2,j} \circ \beta)(\widetilde{\sigma}|_{\widetilde{H}}) = 0.$$

In other words, $r_{1,j}(\widetilde{\sigma}|_{\widetilde{H}})$ is a non-trivial section in the kernel of β_j. Consequently, $h^0(F_{z,j}, N^*_{\widetilde{H}/\widetilde{Z}} \otimes \mathscr{O}_{\widetilde{H}}(\sum k_i \cdot E_i)) \neq 0$ for all j with $k_j > 0$. Note, however, that the restriction of the conormal bundle $N^*_{\widetilde{H}/\widetilde{Z}}$ to F_z, and hence to $F_{z,j}$, is trivial because it is a pull-back from H, that is, $N^*_{\widetilde{H}/\widetilde{Z}} = (\pi|_{\widetilde{H}})^*(N^*_{H/Z})$.

Summing up, we obtain that

$$h^0\Big(F_{z,j}, \mathscr{O}_{F_{z,j}}\Big(\sum k_i E_i|_{F_{z,j}}\Big)\Big) \neq 0 \quad \text{for all } j \text{ with } k_j > 0. \qquad (7.6.12)$$

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Now, if there *was* a number $0 \leqslant j \leqslant k$ with $k_j > 0$, then inequality (7.6.12) would clearly contradict Proposition 7.5. It follows that all $(k_j)_{0 \leqslant j \leqslant k}$ must be zero. In particular, $k_0 = 0$, as claimed. This completes the proof of Proposition 7.1 and thus the proof of Theorem 1.1 for 1-forms. □

Part III. Bogomolov–Sommese vanishing on singular spaces

8. Pull-back properties for sheaves of differentials, proof of Theorem 1.4

In this section we apply the Theorem 1.1 to sheaves of reflexive differentials on singular pairs, i.e., sheaves of differentials that are defined away from the singular set. In good situations, we show that the pull-back of a sheaf of reflexive differentials to a log resolution can still be interpreted as a sheaf of differentials, and that the Kodaira–Iitaka dimension of the sheaves does not change in the process. The Bogomolov–Sommese vanishing theorem, Theorem 1.4, follows as an immediate corollary.

Theorem 8.1 (Extension for sheaves of differentials). *Let (Z, Δ) be a logarithmic pair and $\pi : (\widetilde{Z}, \widetilde{\Delta}) \to (Z, \Delta)$ a log resolution. Let T be a reflexive tensor operation and suppose that there exists a reflexive sheaf $\mathscr{A}$ with inclusion $\iota : \mathscr{A} \to \mathsf{T}\Omega^1_Z(\log \Delta)$. Further, assume that one of the following two additional assumptions holds:*

(8.1.1) *the pair (Z, Δ) is finitely dominated by analytic snc pairs; or*

(8.1.2) *the pair (Z, Δ) is log canonical, the sheaf $\mathscr{A}$ is $\mathbb{Q}$-Cartier and $\mathsf{T} = \bigwedge^{[p]}$, where $p \in \{\dim Z, \dim Z - 1, 1\}$.*

Then there exists a factorization

$$\pi^{[*]}\mathscr{A} \hookrightarrow \mathscr{C} \hookrightarrow \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta} + E_\Delta)),$$

where $E_\Delta \subset \widetilde{Z}$ is the union of those π-exceptional divisors that are not contained in $\widetilde{\Delta}$, $\mathscr{C}$ is invertible and $\kappa(\mathscr{C}) = \kappa(\mathscr{A})$.

Warning 8.2. Since $\pi^{[*]}\mathscr{A}$ is a subsheaf of $\mathscr{C}$, it might be tempting to believe that the equality $\kappa(\mathscr{C}) = \kappa(\mathscr{A})$ is immediate. Note, however, that the reflexive tensor products used in Definition 2.3 of the Kodaira–Iitaka dimension generally *do not* commute with pull-back. The Kodaira–Iitaka dimension $\kappa(\pi^{[*]}\mathscr{A})$ could therefore be strictly smaller than $\kappa(\mathscr{A})$.

Before proving Theorem 8.1 in § 8.2 below, we remark that the following, slightly stronger variant of the Bogomolov–Sommese vanishing Theorem 1.4 for log canonical threefolds and surfaces follows as an immediate corollary to Theorem 8.1.

Theorem 8.3 (Bogomolov–Sommese vanishing for log canonical pairs). *Let (Z, Δ) be a log canonical logarithmic pair. If $p \in \{\dim Z, \dim Z - 1, 1\}$ and if $\mathscr{A} \subset \Omega^{[p]}_Z(\log \Delta)$ is any $\mathbb{Q}$-Cartier reflexive subsheaf of rank one, then $\kappa(\mathscr{A}) \leqslant p$.*

Proof of Theorems 8.3 and 1.4. We argue by contradiction and assume that there exist a number $p \in \{\dim Z, \dim Z - 1, 1\}$ and a $\mathbb{Q}$-Cartier reflexive subsheaf $\mathscr{A} \subset \Omega^{[p]}_Z(\log \Delta)$ of rank one, with Kodaira–Iitaka dimension $\kappa(\mathscr{A}) > p$.

Let $\pi:(\widetilde{Z},\widetilde{\Delta})\to(Z,\Delta)$ be any log resolution. Theorem 8.1 then asserts the existence of an invertible sheaf $\mathscr{C}\subset\Omega^p_{\widetilde{Z}}(\log\widetilde{\Delta}+E_\Delta)$ with $\kappa(\mathscr{C})=\kappa(\mathscr{A})$. This contradicts the classical Bogomolov–Sommese vanishing theorem for snc pairs [EV92, Corollary 6.9]. □

8.1 Preparations for the proof of Theorem 8.1

As a preparation for the proof of Theorem 8.1, we show that the pull-back of a sheaf of reflexive differentials can be interpreted as a sheaf of differentials if the extension theorem holds.

Proposition 8.4. *Let (Z,Δ) be a logarithmic pair, T a reflexive tensor operation and assume that the extension theorem holds for T-forms on (Z,Δ), in the sense of Definition 2.8. If $\pi:(\widetilde{Z},\widetilde{\Delta})\to(Z,\Delta)$ is any log resolution and $E_\Delta\subset\widetilde{Z}$ the union of those π-exceptional components that are not contained in $\widetilde{\Delta}$, then there exists an embedding*

$$\pi^{[*]}\mathsf{T}\Omega^1_Z(\log\Delta)\hookrightarrow\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)). \tag{8.4.1}$$

Proof. As π induces an isomorphism $\widetilde{Z}\setminus\mathrm{Exc}(\pi)\simeq Z\setminus\pi(\mathrm{Exc}(\pi))$, the assumption that the extension theorem holds for T-forms on (Z,Δ) immediately implies that

$$\mathsf{T}\Omega^1_Z(\log\Delta)\simeq\pi_*\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)),$$

because both sides are reflexive and agree in codimension one and Z is S_2 since it is normal. Consequently, we obtain a morphism

$$\pi^*\mathsf{T}\Omega^1_Z(\log\Delta)\simeq\pi^*\pi_*\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))\to\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)),$$

which is an isomorphism, in particular an embedding, on $\widetilde{Z}\setminus\mathrm{Exc}(\pi)$. This remains true after taking the double dual of these sheaves. Therefore, the kernel of the map $\pi^{[*]}\mathsf{T}\Omega^1_Z(\log\Delta)\to\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$ is a torsion sheaf. Since $\pi^{[*]}\mathsf{T}\Omega^1_Z(\log\Delta)$ is torsion free, this implies the statement. □

It is well understood that tensor operations commute with pull-back. However, this is generally not true for reflexive tensor operations, cf. [HK04]. Thus, if we are in the setup of Proposition 8.4 and if $\mathscr{A}\subset\mathsf{T}\Omega^1_Z(\log\Delta)$ is any sheaf, it is generally not at all clear if the embedding (8.4.1) induces a map between reflexive tensor products,

$$\pi^{[*]}\mathscr{A}^{[m]}\xrightarrow{\exists ?}\mathrm{Sym}^m\,\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)).$$

If the sheaf $\mathscr{A}$ is invertible, we can obviously say more.

Lemma 8.5. *In the setup of Proposition 8.4, let $\mathscr{A}\subset\mathsf{T}\Omega^1_Z(\log\Delta)$ be an invertible subsheaf. If $m\in\mathbb{N}$ is arbitrary, then the embedding (8.4.1) induces a map*

$$\pi^{[*]}\mathscr{A}^{[m]}\hookrightarrow\mathrm{Sym}^m\,\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)). \tag{8.5.1}$$

Proof. Since $\mathscr{A}$ is invertible, all tensor operations on $\mathscr{A}$ are automatically reflexive. In particular, we have that $\mathscr{A}^{[m]}=\mathscr{A}^{\otimes m}$ and $\pi^{[*]}\mathscr{A}^{[m]}\cong\pi^*(\mathscr{A}^{\otimes m})\cong(\pi^*\mathscr{A})^{\otimes m}$. The existence of (8.5.1) then follows from Proposition 8.4. □

8.2 Proof of Theorem 8.1

We maintain the notation and the assumptions of Theorem 8.1. By Theorem 1.1 or Remark 3.5, respectively, the extension theorem holds for the pair (Z,Δ). Proposition 8.4 then gives an

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embedding $\psi^{[*]}\mathscr{A} \hookrightarrow \mathrm{Sym}^n\,\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$. Let $\mathscr{C} \subset \mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$ be the saturation of the image, which is automatically reflexive by [OSS80, Lemma 1.1.16, p. 158]. By [OSS80, Lemma 1.1.15, p. 154], $\mathscr{C}$ is then invertible, as desired. Further, observe that, for any $m \in \mathbb{N}$, the subsheaf $\mathscr{C}^{\otimes m} \subset \mathrm{Sym}^m\,\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$ is likewise saturated.

8.2.1 *Proof of Theorem 8.1 if (Z,Δ) is finitely dominated by analytic snc pairs.* If assumption (8.1.1) of Theorem 8.1 holds and $m \in \mathbb{N}$ is arbitrary, then again by Remark 3.5 and Proposition 8.4 there exists an embedding

$$\bar{\iota}^{[m]} : \psi^{[*]}\mathscr{A}^{[m]} \hookrightarrow \mathrm{Sym}^m\,\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta)).$$

It is easy to see that $\bar{\iota}^{[m]}$ factors through $\mathscr{C}^{\otimes m}$ as it does so on the open set where ψ is isomorphic, and because $\mathscr{C}^{\otimes m}$ is saturated in the locally free sheaf $\mathrm{Sym}^m\,\mathsf{T}\Omega^1_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$. It follows that $\kappa(\mathscr{C}) = \kappa(\mathscr{A})$. This completes the proof in the case when assumption (8.1.1) holds. □

8.2.2 *Proof of Theorem 8.1 if (Z,Δ) is log canonical.* It remains to consider the case when assumption (8.1.2) of Theorem 8.1 holds. Let $m \in \mathbb{N}$ and $\sigma \in H^0(Z,\, \mathscr{A}^{[m]})$ a section. Then $\pi^*(\sigma)$ can be seen as a section in $\mathscr{C}^{\otimes m}$, with poles along the exceptional set $E := \mathrm{Exc}(\pi)$, i.e., $\pi^*(\sigma) \in H^0(\widetilde{Z}, \mathscr{C}^{\otimes m}(*E))$. To show that $\kappa(\mathscr{C}) = \kappa(\mathscr{A})$, it suffices to prove that $\pi^*(\sigma)$ does not have any poles as a section in $\mathscr{C}^{\otimes m}$, i.e., that

$$\pi^*(\sigma) \in H^0(\widetilde{Z}, \mathscr{C}^{\otimes m}) \subset H^0(\widetilde{Z}, \mathscr{C}^{\otimes m}(*E)). \tag{8.5.2}$$

Since $\mathscr{C}^{\otimes m}$ is saturated in $\mathrm{Sym}^m\,\Omega^p_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))$, to show (8.5.2) it suffices in turn to show that $\pi^*(\sigma)$ does not have any poles as a section in the sheaf of symmetric differentials, i.e., that

$$\pi^*(\sigma) \in H^0(\widetilde{Z},\ \mathrm{Sym}^m\,\Omega^p_{\widetilde{Z}}(\log(\widetilde{\Delta}+E_\Delta))). \tag{8.5.3}$$

Since that question is local in Z in the analytic topology, we can shrink Z, use that $\mathscr{A}$ is $\mathbb{Q}$-Cartier and assume without loss of generality that there exists a number r such that $\mathscr{A}^{[r]} \cong \mathscr{O}_Z$. Similar to the construction in the proof of the finite covering trick, Proposition 3.1, we obtain a commutative diagram.

$$\begin{array}{ccc}
(\widetilde{X}, \widetilde{D}) & \xrightarrow{\ \widetilde{\gamma},\ \text{finite}\ } & (\widetilde{Z}, \widetilde{\Delta}) \\
{\scriptstyle \widetilde{\pi}}\ {\scriptstyle\text{contracts } \widetilde{E}} \downarrow & & \downarrow {\scriptstyle \pi}\ {\scriptstyle\text{log resolution}}\ {\scriptstyle\text{contracts } E} \\
(X, D) & \xrightarrow[\ \gamma,\ \text{finite}\ ]{} & (Z, \Delta)
\end{array}$$

Here γ is the index-one-cover associated with $\mathscr{A}$, $\widetilde{X}$ is the normalization of the fiber product $X \times_Z \widetilde{Z}$ and $\widetilde{D} \subset \widetilde{X}$ is the reduced preimage of $\widetilde{\Delta}$. As before, let

$$\widetilde{E} := \mathrm{Exc}(\widetilde{\pi}) = \mathrm{supp}(\widetilde{\gamma}^{-1}(E)) = \mathrm{supp}((\gamma \circ \widetilde{\pi})^{-1}(Z,\Delta)_{\mathrm{sing}})$$

be the exceptional set of the morphism $\widetilde{\pi}$. Since γ is étale away from the singularities of Z, the morphism $\widetilde{\gamma}$ is étale outside of $E \subset \widetilde{\Delta} \cup E_\Delta$. In particular, the pull-back morphism of differentials gives an isomorphism

$$\widetilde{\gamma}^{[*]}(\mathrm{Sym}^m\,\Omega^p_{\widetilde{Z}}(\log\widetilde{\Delta}+E_\Delta)) \cong \mathrm{Sym}^{[m]}\,\Omega^{[p]}_{\widetilde{X}}(\log\widetilde{D}+\widetilde{E}_D),$$

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where again $\widetilde{E}_D \subset \widetilde{X}$ is the union of the $\widetilde{\pi}$-exceptional divisors not already contained in $\widetilde{D}$. In order to prove (8.5.3), it then suffices to show that

$$\widetilde{\gamma}^{[*]}(\pi^*(\sigma)) = \widetilde{\pi}^{[*]}\gamma^{[*]}(\sigma) \in H^0(\widetilde{X},\ \mathrm{Sym}^{[m]}\,\Omega^{[p]}_{\widetilde{X}}(\log(\widetilde{D}+\widetilde{E}_D))), \tag{8.5.4}$$

cf. case (2.12.(i)) of Corollary 2.12. Since the pair (X, D) is again log canonical by [KM98, Proposition 5.20], Theorem 1.1 applies to show that the extension theorem holds for (X, D). In particular, Lemma 8.5 applies to the invertible sheaf $\widetilde{\mathscr{A}} := \gamma^{[*]}(\mathscr{A}) \subset \Omega^{[p]}_X(\log D)$. Inclusion (8.5.4) follows if one applies the embedding

$$\widetilde{\pi}^{[*]}(\widetilde{\mathscr{A}}^{\,[m]}) \hookrightarrow \mathrm{Sym}^m\, \mathsf{T}\Omega^1_{\widetilde{X}}(\log(\widetilde{D}+\widetilde{E}_D))$$

to the section $\widetilde{\sigma} := \gamma^{[*]}(\sigma) \in H^0(X,\ \widetilde{\mathscr{A}})$. This completes the proof of Theorem 8.1 in the case when assumption (8.1.2) holds. □

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Part IV. Appendix

Appendix A. Duality for cohomology with support

The proof of Proposition 7.6 relies on the following version of Hartshorne's formal duality theorem. Since this is not exactly the version contained in the main reference [Har70], we recall the relevant facts and include a full proof for the reader's convenience.

Theorem A.1 (Formal duality [Har70, Theorem 3.3]). *Let $\pi : \widetilde{Z} \to Z$ be a projective birational morphism of quasi-projective varieties, where $\widetilde{Z}$ is non-singular and Z is normal. Let $z \in Z$ and $F := \pi^{-1}(z)$ the fiber over z. Then, for any locally free sheaf $\mathscr{F}$ on $\widetilde{Z}$ and any number $0 \leqslant j \leqslant n$, there exists a canonical isomorphism*

$$(R^j\pi_*\mathscr{F}_z)\widehat{\ } \cong H^{n-j}_F(\widetilde{Z},\ \mathscr{F}^* \otimes \omega_{\widetilde{Z}})^*,$$

where $\widehat{\ }$ denotes completion with respect to the maximal ideal $\mathfrak{m}_z$ of the point $z \in Z$.

We recall a few facts before giving the proof.

Fact A.2 (Excision for local cohomology [Har77, III Example 2.3f]). Let Z be an algebraic variety, Y a subvariety and $U \subseteq Z$ an open subset that contains Y. If i is any number and $\mathscr{F}$ any sheaf, then there exists a canonical isomorphism $H^i_Y(Z,\ \mathscr{F}) \cong H^i_Y(U,\ \mathscr{F}|_U)$. □

Fact A.3 (Serre duality on $\widetilde{Z}$ [Har77, III Theorem 7.6]). Let $\widetilde{Z}$ be a non-singular projective variety of dimension n. Then there exists a canonical isomorphism

$$H^j(\widetilde{Z}, \mathscr{G}) \cong (\mathrm{Ext}^{n-j}_{\widetilde{Z}}(\mathscr{G}, \omega_{\widetilde{Z}}))^*$$

for all $j \geqslant 0$ and for every coherent sheaf $\mathscr{G}$ on $\widetilde{Z}$. □

Fact A.4 (Approximation of cohomology with support [Har67, Theorem 2.8]). In the notation of Theorem A.1 above, if $\mathscr{I}$ is any sheaf of ideals defining the subset $F \subseteq \widetilde{Z}$, the local cohomology

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groups with support on F and values in a coherent algebraic sheaf $\mathscr{G}$ can be computed as follows:

$$H^j_F(\widetilde{Z}, \mathscr{G}) = \varinjlim_m \operatorname{Ext}^j_{\widetilde{Z}}(\mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m, \mathscr{G}). \qquad \square$$

Fact A.5 (Theorem on formal functions [Har77, ch. III.11]). In the notation of Theorem A.1 above, if $\mathscr{J}$ is the $\mathscr{O}_{\widetilde{Z}}$-ideal generated by the image of the maximal ideal $\mathfrak{m}_z$ under the natural map $\pi^{-1}\mathscr{O}_Z \to \mathscr{O}_{\widetilde{Z}}$, and if $\mathscr{G}$ is any coherent sheaf on $\widetilde{Z}$, then we have

$$(R^j\pi_*\mathscr{G}_z)\widehat{\ } \cong \varprojlim_m H^j(F_m, \mathscr{G}_m),$$

where $F_m = (F, \mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m)$ is the mth infinitesimal neighborhood of the fiber F and where $\mathscr{G}_m = \mathscr{G} \otimes \mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m$. $\square$

Fact A.6 [Har77, ch. III.6, Proposition 6.7]. Let $\widetilde{Z}$ be an algebraic variety. For coherent sheaves $\mathscr{M}$ and $\mathscr{N}$ on $\widetilde{Z}$, we have

$$\operatorname{Ext}^j_{\widetilde{Z}}(\mathscr{F} \otimes \mathscr{M}, \mathscr{N}) \cong \operatorname{Ext}^j_{\widetilde{Z}}(\mathscr{M}, \mathscr{F}^* \otimes \mathscr{N})$$

for every locally free sheaf $\mathscr{F}$ on $\widetilde{Z}$. $\square$

Proof of Theorem A.1. Using the excision theorem for local cohomology, Fact A.2, we may compactify Z and $\widetilde{Z}$ and assume without loss of generality that both Z and $\widetilde{Z}$ are projective. By Fact A.5, we have

$$(R^j\pi_*\mathscr{F}_z)\widehat{\ } = \varprojlim H^j(F_m, \mathscr{F}_m). \tag{A1}$$

The cohomology group on the right-hand side of (A1) is computed as follows.

$$\begin{aligned} H^j(F_m, \mathscr{F}_m) &= H^j(\widetilde{Z}, \mathscr{F}_m) \\ &\cong (\operatorname{Ext}^{n-j}_{\widetilde{Z}}(\mathscr{F}_m, \omega_{\widetilde{Z}}))^* && \text{by Fact A.3} \\ &\cong (\operatorname{Ext}^{n-j}_{\widetilde{Z}}(\mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m, \mathscr{F}^* \otimes \omega_{\widetilde{Z}}))^* && \text{by Fact A.6.} \end{aligned}$$

Substituting this into (A1), we obtain

$$\begin{aligned} (R^j\pi_*\mathscr{F}_z)\widehat{\ } &\cong \varprojlim (\operatorname{Ext}^{n-j}_{\widetilde{Z}}(\mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m, \mathscr{F}^* \otimes \omega_{\widetilde{Z}}))^* \\ &= (\varinjlim \operatorname{Ext}^{n-j}_{\widetilde{Z}}(\mathscr{O}_{\widetilde{Z}}/\mathscr{J}^m, \mathscr{F}^* \otimes \omega_{\widetilde{Z}}))^* \\ &= (H^{n-j}_F(\widetilde{Z}, \mathscr{F}^* \otimes \omega_{\widetilde{Z}}))^* \quad \text{by Fact A.4,} \end{aligned}$$

as claimed. $\square$

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