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Group Algebras with Minimal Strong Lie Derived Length

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Abstract. Let KG be a non-commutative strongly Lie solvable group algebra of a group G over a field K of positive characteristic p. In this note we state necessary and sufficient conditions so that the strong Lie derived length of KG assumes its minimal value, namely $\lceil \log_2(p+1) \rceil$.

1 Introduction

Let KG be the group algebra of a group G over a field K. As usual, we regard it as a Lie algebra under the Lie multiplication $[a,b] := ab - ba$ for all $a, b \in KG$. We put $\delta^{(0)}(KG) := \delta^{[0]}(KG) := KG$ and define by induction $\delta^{[n+1]}(KG) := [\delta^{[n]}(KG), \delta^{[n]}(KG)]$, where this symbol denotes the additive subgroup generated by all the Lie commutators $[a,b]$ with $a, b \in \delta^{[n]}(KG)$, and $\delta^{(n+1)}(KG)$ as the associative ideal generated by $[\delta^{(n)}(KG), \delta^{(n)}(KG)]$.

We say that KG is *strongly Lie solvable* if there exists an integer n such that $\delta^{(n)}(KG)= 0$; in this case, the minimal integer m such that $\delta^{(m)}(KG) = 0$ is called the *strong Lie derived length* of KG and denoted by $dl^L(KG)$. In a similar manner we define the *Lie derived length* of KG, denoted by $dl_L(KG)$. Clearly $\delta^{[n]}(KG) \subseteq \delta^{(n)}(KG)$ for all non-negative integers n. Thus a strongly Lie solvable group algebra KG is Lie solvable and $dl_L(KG) \leq dl^L(KG)$. But, as stressed in [1], the equality does not always hold. In fact, let G be a 2-group of maximal class of order 2^n with $n \geq 5$ and let K be a field of characteristic 2. Then G contains an abelian subgroup of index 2 and, by [6, Theorem 1], $dl_L(KG) \leq 3$, whereas $dl^L(KG) = n - 1$.

Let G be a non-abelian group. It is well known (see [8, Theorem V.5.1]) that KG is strongly Lie solvable if and only if K has positive characteristic p and the commutator subgroup of G is a finite p-group. I. B. S. Passi, D. S. Passman, and S. K. Sehgal stated necessary and sufficient conditions so that the group algebra KG is Lie solvable [5]. According to these results, the Lie solvability of KG occurs if and only if KG is strongly Lie solvable, under the assumption that p is odd. Instead, this is not true when $p = 2$; for instance, the group algebra $\mathbb{F}_2 S_3$, where $\mathbb{F}_2$ is the field of two elements and S_3 the symmetric group on three letters, is Lie solvable of length 3, but not strongly Lie solvable.

Very little is known about the Lie derived length of non-commutative group algebras. The most remarkable works in this area are the papers by A. Shalev [10, 11], which gave life to a range of new questions. In particular, if K is a field of positive characteristic p, then $\lceil \log_2(p+1) \rceil \leq dl_L(KG)$, where the left-hand side of

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the inequality denotes the upper integral part of $\log_2(p+1)$ (see [10, Theorem A]). Moreover, this bound is actually the correct one [10]. Indeed, if G is a nilpotent group whose commutator subgroup has order p, $dl_L(KG) = \lceil \log_2(p+1) \rceil$. By virtue of [1, (2)], this is also the value of $dl^L(KG)$. Hence the lower bound by Shalev is the best possible also for the strong Lie derived length of a group algebra. The aim of this note is to establish necessary and sufficient conditions so that this bound is achieved.

If m is a positive integer, define recursively

$$g(0,m) := 1, \quad g(t,m) := g(t-1,m) \cdot 2^{m+1} + 1 \ (t \in \mathbb{N});$$

moreover, if k is a non-negative integer, we denote by $q_{k,m}$ and $\epsilon_{k,m}$ the quotient and the remainder of the Euclidian division of $k-1$ by $m+1$, respectively. Finally, if G is a group, G' denotes its commutator subgroup and, if S is a subgroup of G, we use $C_G(S)$ for its centralizer in G.

The main result that we prove is the following.

Theorem 1 *Let KG be a non-commutative strongly Lie solvable group algebra over a field K of positive characteristic p. Let n be the positive integer such that $2^n \le p < 2^{n+1}$ and s, q (q odd) the non-negative integers such that $p - 1 = 2^s q$. The following statements are equivalent:*

(i) $dl^L(KG) = \lceil \log_2(p+1) \rceil$;

(ii) *p and G satisfy one of the following conditions:*

- (a) *$p = 2$, G' has exponent 2 and an order dividing 4 and G' is central;*
- (b) *$p \ge 3$ and G' is central of order p;*
- (c) *$5 \le p < 2^{n+2}/3$, G' is not central of order p and $|G/C_G(G')| = 2^m$ with $m \le s$ a positive integer such that $p \le 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1, m)$.*

Actually, F. Levin and G. Rosenberger characterized Lie metabelian modular group algebras and showed that this condition is equivalent to saying that the group algebra is strongly Lie metabelian [3]. Moreover, M. Sahai [7] classified group algebras over fields of odd characteristic whose strong Lie derived length is at most 3. Thus they already completed the special cases $p = 2, 3, 5, 7$. Here we shall give an independent proof also for these values of p.

Shalev observed that if G is the dihedral group of order $2p$ ($p > 2$) and K a field of characteristic p, then, by [10, Theorem C(2)], the value of $dl_L(KG)$ is $\lceil \log_2(3p/2) \rceil$ and, if $2^n < p < 2^{n+2}/3$ for some integer $n \ge 2$, one has that $dl_L(KG) = \lceil \log_2(p+1) \rceil$ (the same result was obtained in the theorem of [1] replacing $dl_L(KG)$ by $dl^L(KG)$). Thus he showed that groups G satisfying $dl_L(KG) = \lceil \log_2(p+1) \rceil$ are not necessarily nilpotent with commutator subgroup of order p. Moreover, he stressed that "their complete characterization may be a delicate task". Our main theorem gives a contribution in this direction. In fact, the groups G for which $dl_L(KG) = \lceil \log_2(p+1) \rceil$ are not only of the type described by Shalev. In particular, in the case in which G is not nilpotent, it is not necessary that the elements that do not centralize G' act by inversion on G'. Indeed, let

$$G := \langle x, y \mid x^{17} = y^8 = 1 \ y^{-1}xy = x^2 \rangle$$

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and let K be a field of characteristic 17. Then we have $dl_L(KG) = dl^L(KG) = 5$ and $|G/C_G(G')| = 8$.

The notation that we shall use is rather standard: if G is a group, $\zeta(G)$ denotes the center of G and $\gamma_i(G)$ the i-th term of its lower central series. If S, T, U are subsets of G, the symbol (S, T) means the subgroup generated by all the elements $s^{-1}t^{-1}st$, where s belongs to S and t to T, and we set $(S, T, U) := ((S, T), U)$. Moreover, if m is a positive integer, $G^m := \langle x^m \mid x \in G\rangle$ and C_m denotes the cyclic group of order m. Finally, if K is a field and $x := \sum_{g\in G} x_g g$ is an element of the group algebra KG, set $\mathrm{aug}(x) := \sum_{g\in G} x_g$.

2 Proof of Theorem 1

Let KG be the group algebra of a group G over a field K of positive characteristic p. According to a well-known result (see [8, Lemma I.2.21]), the augmentation ideal $\omega(G)$ is nilpotent if and only if G is a finite p-group. In particular, we consider a sequence of (normal) subgroups of G by setting

$$\forall n \in \mathbb{N} \quad \mathfrak{D}_n(G) := G \cap (1 + \omega(G)^n).$$

The n-th term of this series is called the n-th *dimension subgroup* of G. For the basic results about the series of the dimension subgroups we refer the reader to [4]. For our purposes, we confine ourselves to recalling that it is possible to describe the $\mathfrak{D}_m(G)$'s in the following manner:

$$\mathfrak{D}_m(G) = \begin{cases} G & \text{if } m = 1, \\ (\mathfrak{D}_{m-1}(G), G) \cdot \mathfrak{D}_{\lceil \frac{m}{p} \rceil}(G)^p & \text{if } m \geq 2. \end{cases} \tag{2.1}$$

Put $p^{d_k} := |\mathfrak{D}_k(G) : \mathfrak{D}_{k+1}(G)|$, where $k \geq 1$. Then Jennings's theory [2] provides a formula for the computation of the nilpotency index of the augmentation ideal, namely

$$t(G) = 1 + (p-1)\sum_{m\geq 1} m d_m. \tag{2.2}$$

In particular, if G is a direct product of cyclic groups of order $p^{n_1}, \dots, p^{n_k}$ respectively, the nilpotency index of the augmentation ideal is given by

$$t(G) = 1 + \sum_{i=1}^{k} (p^{n_i} - 1). \tag{2.3}$$

Before proving the main result, we present a lemma giving a fairly good estimation of the terms of the strong Lie derived series of the group algebra of a particular group.

Lemma 2 *Let K be a field of characteristic $p > 3$ and let G be a group whose commutator subgroup has order p. Suppose that $|G/C_G(G')| = 2^m$ for some integer m. For all non-negative integer n,*

$$\delta^{(n+1)}(KG) = \omega(G')^{2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1,m)} \cdot KG.$$

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Proof We proceed by induction on n. For $n = 0$, $\epsilon_{n-m,m} = 0$ and $q_{n-m,m} = -1$. Then $\delta^{(1)}(KG) = \omega(G') \cdot KG$, and the statement holds. Assume now that $n \geq 0$ and, for all non-negative integers j, set $a_j := 2^{\epsilon_{n-m+j,m}} \cdot g(q_{n-m+j,m} + 1, m)$. By induction hypothesis, we have

$$\begin{aligned}\delta^{(n+2)}(KG) &= [\delta^{(n+1)}(KG), \delta^{(n+1)}(KG)]KG\\ &= [\omega(G')^{a_0} \cdot KG, \omega(G')^{a_0} \cdot KG]KG.\end{aligned}$$

Set $C := C_G(G')$. The action of G on G' by conjugation embeds G/C into the automorphism group $\operatorname{Aut}(G')$ of G'. In particular, $\operatorname{Aut}(G') \cong C_{p-1}$. Therefore, G/C is cyclic (and $m \leq \eta$ if $p = 2^\eta q + 1$ for a suitable integer η and an odd integer q). Let z be the generator of G' and αC the generator of G/C. To obtain the statement, it is sufficient to prove that

$$(2.4) \qquad [(z-1)^{a_0}, (z-1)^{a_0}\alpha] \in \omega(G')^{a_1} \cdot KG \setminus \omega(G')^{a_1+1} \cdot KG,$$

under the assumption that $a_0 < t(G') = p$.

Suppose first that $\epsilon_{n-m+1,m} = 0$. Let $r < p$ be the integer such that $\alpha^{-1}z\alpha = z^r$. Clearly, it holds that

$$(2.5) \qquad \forall t < m \quad 1 - r^{2^t} \not\equiv 0 \pmod{p},$$

otherwise $|G/C| < 2^m$, in contradiction with our assumption. It is easily checked that

$$(2.6) \qquad \forall s \in \mathbb{N} \qquad [(z-1)^s, \alpha] = (z-1)^s(1 - (1 + z + \cdots + z^{r-1})^s)\alpha.$$

According to (2.6),

$$[(z-1)^{a_0}, (z-1)^{a_0}\alpha] = (z-1)^{2a_0}(1 - (1 + z + \cdots + z^{r-1})^{a_0})\alpha.$$

Put

$$\begin{aligned}x &:= 1 - (1 + z + \cdots + z^{r-1})^{2^{m-1} \cdot g(q_{n-m+1,m}, m)},\\ y &:= 1 + (1 + z + \cdots + z^{r-1})^{2^{m-1} \cdot g(q_{n-m+1,m}, m)}.\end{aligned}$$

Since in this case $\epsilon_{n-m,m} = m$ and $q_{n-m,m} = q_{n-m+1,m} - 1$, one has at once that $1 - (1 + z + \cdots + z^{r-1})^{a_0} = xy$. By standard computations we obtain that

$$(2.7) \qquad y = (1-z)w,$$

where $\operatorname{aug}(w) = (r-1)(p+1) \cdot g(q_{n-m+1,m}, m) \cdot 2^{m-2}$. Since $g(q_{n-m+1,m}, m) < p$, we have that p does not divide $\operatorname{aug}(w)$. Thus w is a unit of KG. In particular, by (2.7), it follows that p divides $\operatorname{aug}(y) = 1 + r^{2^{m-1} \cdot g(q_{n-m+1,m}, m)}$, hence p cannot divide $\operatorname{aug}(x)$, which means that x is a unit of KG. Therefore

$$[(z-1)^{a_0}, (z-1)^{a_0}\alpha] \in \omega(G')^{2a_0+1} \cdot KG \setminus \omega(G')^{2a_0+2} \cdot KG.$$

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But

$$2a_0 + 1 = 2^{\epsilon_{n-m,m}+1} \cdot g(q_{n-m,m} + 1, m) + 1$$
$$= 2^{m+1} \cdot g(q_{n-m,m} + 1, m) + 1 = g(q_{n-m+1,m} + 1, m) = a_1,$$

and this proves (2.4).

Finally, suppose that $\epsilon_{n-m+1,m} \neq 0$. First of all, we notice that a standard induction argument allows expressing (2.6) in the following manner:

$$\text{(2.8)} \qquad \forall s \in \mathbb{N} \quad [(z-1)^s, \alpha] = \sum_{\substack{i,j \geq 0 \\ i+j=s-1}} \alpha z (z^r - 1)^i (z^{r-1} - 1)(z-1)^j.$$

Directly by (2.8) we obtain

$$[(z-1)^s, (z-1)^s \alpha] = \sum_{i=0}^{s-1} \alpha z (1 + z + \cdots + z^{r-1})^{s+i} (1 + z + \cdots + z^{r-2})(z-1)^{2s}.$$

Set $v := \sum_{i=0}^{s-1} (1 + z + \cdots + z^{r-1})^{s+i}$. Clearly, $\operatorname{aug}(v) = r^s \cdot \sum_{i=0}^{s-1} r^i$. For the first part of the proof, p divides $\sum_{i=0}^{\beta-1} r^i$, where $\beta := 2^m \cdot g(q_{n-m,m} + 1, m)$. By combining this with (2.5) and the fact that $0 \leq \epsilon_{n-m,m} \leq m - 1$, we obtain that p does not divide $\sum_{i=0}^{a_0-1} r^i$. Then, in this case, v is a unit of KG, thus

$$[(z-1)^{a_0}, (z-1)^{a_0} \alpha] \in \omega(G')^{2a_0} \cdot KG \setminus \omega(G')^{2a_0+1} \cdot KG.$$

Since $\epsilon_{n-m,m} + 1 = \epsilon_{n-m+1,m}$ and $q_{n-m+1,m} = q_{n-m,m}$, we obtain that

$$2a_0 = 2^{\epsilon_{n-m,m}+1} \cdot g(q_{n-m,m} + 1, m) = a_1,$$

and this completes the proof. ■

Now we are in position to establish the main result.

Proof of Theorem 1 First we prove that (i) implies (ii). Assume that p is even. If $dl^L(KG) = 2$, since $\lceil \log_2(t(G') + 1) \rceil \leq dl^L(KG)$ (see [1]), it follows at once that $t(G') \leq 3$. By virtue of (2.2), $0 \leq d_1 \leq 2$. In the case in which $d_1 = 0$, by applying (2.1), we obtain that $\mathfrak{D}_j(G') = G'$ for all positive integers j, which is clearly impossible. Hence $d_1 > 0$ and the upper bound for $t(G')$ forces $d_j = 0$ for every $j \geq 2$. As a consequence, G' is elementary abelian. By (2.3) it is easily checked that either $G' \cong C_2$ or $G' \cong C_2 \times C_2$.

When the first case occurs, G is nilpotent. Then we suppose $G' \cong C_2 \times C_2$ and verify that G' is central. Assume, if possible, that G is not nilpotent. If x and y are the generators of G', then $\delta^{(2)}(KG) = \omega(G') \cdot \omega(\gamma_3(G)) \cdot KG + \omega(\gamma_3(G)) \cdot \omega(G') \cdot KG \neq 0$, since $(x-1)(y-1) \in \delta^{(2)}(KG)$, and this is a contradiction. The same argument proves that G cannot be nilpotent of class 3 and the statement (a) holds.

Let p be odd and assume, if possible, that $|G'| = p^n$ for some $n > 1$. By [1] and [9, Proposition 3.2] we obtain:

$$dl^L(KG) \geq \lceil \log_2(t(G') + 1) \rceil > \lceil \log_2(p + 1) \rceil.$$

Thus, assume that $|G'| = p$. By [10, Lemma 4.1], G has a section H/N, where $N \trianglelefteq H \leq G$, such that either H/N is nilpotent of class two with commutator subgroup of order p or $H/N = E \rtimes \langle\alpha\rangle$, where E is an elementary abelian p-group and α is an automorphism of E of prime order $d \neq p$. We claim that when the first case occurs, G is nilpotent. Now for a question of order: $H' = G'$ and $\gamma_3(H) = \langle 1\rangle$, otherwise $H' = \gamma_3(H) \leq N$ and thus H/N is abelian, in contradiction with our assumption. Since $(H', H) = (H, G, H) = (G, H, H) = \langle 1\rangle$, by the three-subgroups lemma we have $(H, H, G) = \gamma_3(G) = \langle 1\rangle$ and the claim follows.

Hence, if we assume that G is not nilpotent, there exists a section of the second type. By [10, Theorem C] one has at once that $d = 2$ and $d^L(KG) \geq \lceil \log_2(3p/2) \rceil$. Since the equality $\lceil \log_2(3p/2) \rceil = \lceil \log_2(p + 1) \rceil$ is true if and only if $p < 2^{n+2}/3$, it remains only to study the action by conjugation of G over G' when the last inequality holds.

Set $C := C_G(G')$. As a first step we verify that G/C has order a power of 2. Let G be a counterexample. Then G/C contains an element αC of prime order $r \neq 2$. Let $L := \langle\alpha, G'\rangle$. Clearly, $L' = G'$; in particular, L is also a counterexample. Thus we may replace G by L and assume that $G = G'\langle\alpha\rangle$. Since α^r centralizes both G' and α, we must have $\alpha^r \in \zeta(G)$. Moreover $\langle\alpha^r\rangle \cap G' = \langle 1\rangle$, otherwise $G' \leq \langle\alpha^r\rangle$ and thus $\alpha C = C$. But now $G/\langle\alpha^r\rangle$ is also a counterexample. We may therefore replace G by $G/\langle\alpha^r\rangle$ and assume that G is the semidirect product of G' and α, where α has order r. In this situation [10, Theorem C(1)] implies that $dl^L(KG) > \lceil \log_2(p + 1) \rceil$, contradicting our assumptions.

Obviously, if $|G/C| = 2^m$, then $m \leq s$. Now we recall that $2^n < p < 2^{n+1}$ and by invoking Lemma 2, we obtain that

$$\delta^{(\lceil \log_2(p+1) \rceil)}(KG) = \delta^{(n+1)}(KG) = \omega(G')^{2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m}+1,m)} \cdot KG,$$

and, since $t(G') = p$, it vanishes if and only if $p \leq 2^{\epsilon_{n-m,m}} \cdot g(q_{n-m,m} + 1, m)$, and the proof of the first implication is complete.

Conversely, we suppose that one of the conditions (a)–(c) holds and show that, under these assumptions, $dl^L(KG) = \lceil \log_2(p + 1) \rceil$. Now when G is nilpotent, the above equality is a direct consequence of (2.3) and [1, (2)], otherwise the result follows at once from Lemma 2. ∎

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