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In memory of Steve

Abstract

Let $\{\Lambda_t^\infty\}$ be an isotopy of Legendrians (possibly singular) in a unit cosphere bundle S^*M that arise as slices of a singular Legendrian $\Lambda_I^\infty \subset S^*M \times T^*I$. Let $\mathcal{C}_t = Sh(M, \Lambda_t^\infty)$ be the differential graded derived category of constructible sheaves on M with singular support at infinity contained in Λ_t^∞. We prove that if the isotopy of Legendrians embeds into an isotopy of Liouville hypersurfaces, then the family of categories $\{\mathcal{C}_t\}$ is constant in t.

Motivation and results

Let M be a smooth manifold, and let $Sh(M)$ be the co-complete differential graded (dg) derived category of weakly constructible sheaves on M with coefficient in $\mathbb{C}$. In [KS13, Tam08, GKS12] it is proved that contact isotopy of the cosphere bundle $T^\infty M = (T^*M - T^*_M M)/\mathbb{R}_+$ acts on $Sh(M)$ as equivalences of categories. In this paper, we consider a (singular) Legendrian $\Lambda^\infty \subset T^\infty M$ and the full subcategory $Sh(M, \Lambda^\infty)$ consisting of sheaves F with singular support at infinity, $SS^\infty(F) = (SS(F) - T^*_M M)/\mathbb{R}_+$, contained in Λ^∞. We define a notion of isotopy for the singular Legendrian Λ^∞ and prove that the category $Sh(M, \Lambda^\infty)$ remains invariant under such an isotopy.

Such invariances of constructible sheaf categories are possible because constructible sheaves are closely related to Lagrangians in T^*M (see [NZ09, GPS18a, NS20]) and hence enjoy the flexibility of symplectic geometry. More precisely, the full subcategory of compact objects in $Sh(M, \Lambda^\infty)$, denoted by $Sh^{\mathrm{w}}(M, \Lambda^\infty)$, is equivalent to the wrapped Fukaya category of the pair (T^*M, Λ^∞) (see [GPS18a, NS20]),

$$Sh^{\mathrm{w}}(M, \Lambda^\infty) \simeq \mathrm{Fuk}^{\mathrm{w}}(T^*M, \Lambda^\infty),$$

where the superscript 'w' stands for 'wrapped'. The traditional constructible sheaves with bounded cohomologies $Sh^{pp}(M, \Lambda^\infty)$ can be recovered as perfect modules [Nad16],

$$Sh^{pp}(M, \Lambda^\infty) = \mathrm{Fun}^{ex}(Sh^{\mathrm{w}}(M, \Lambda^\infty)^{op}, \mathrm{Perf}(\mathbb{C})).$$

There is an analogous result in the wrapped Fukaya category for Liouville sectors [GPS18b, Theorem 1.4]: given a Liouville domain W with a 'stop' $S \subset \partial W$, if the contact complement $\partial W \backslash S$ remains invariant up to contact isotopy as S moves, then the wrapped Fukaya category

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$\mathrm{Fuk}^{\mathrm{w}}(W, S)$ is invariant. Hence, combined with the comparison results of [GPS18a, NS20], we get that $Sh(M, \Lambda^\infty)$ is invariant as long as $T^\infty M \backslash \Lambda^\infty$ is invariant up to contact isotopy.

This paper gives a similar sufficient condition using 'isotopy' of Λ^∞: we replace Λ^∞ by a tube $U = U(\Lambda^\infty)$ around the Legendrian Λ^∞, and we equip U with a contact flow X that shrinks the tube U back to Λ^∞ (Definition 0.2). Although Λ^∞ may be singular, the data of (U, X) are smooth, so we can talk about isotopies of (U, X). The relation to the complement $T^\infty M \backslash \Lambda^\infty$ is that if (U_t, X_t) varies smoothly, then the complements $\{T^\infty M \backslash \Lambda_t^\infty\}_t$ are contactomorphic, where Λ_t^∞ is the limit of U_t under the shrinking flow X_t.

To state the main theorem precisely, we need some definitions.

Let (C, α) be a contact manifold with a contact 1-form α.

Definition 0.1. A singular Legendrian $\mathcal{L} \subset C$ is a Whitney stratifiable subspace such that its top-dimensional strata are smooth Legendrians and the closure of the union of the top-dimensional strata is $\mathcal{L}$.

Definition 0.2. Let $\mathcal{L} \subset C$ be a singular Legendrian. A *convex tube* (U, X) *around* $\mathcal{L}$ is an open subset U containing $\mathcal{L}$ with smooth boundary ∂U and a contact vector field X transverse to ∂U and pointing inward to ∂U, such that $\mathcal{L}_X(\alpha) = -\alpha$ and $\bigcap_{u>0} X^u(U) = \mathcal{L}$, where X^u is the time-u flow of X.

Definition 0.3. Let $I \subset \mathbb{R}$ be a closed interval and $\{(U_t, X_t, \mathcal{L}_t)\}_{t \in I}$ a family of singular Legendrians $\mathcal{L}_t$ with convex tubes (U_t, X_t). If ∂U_t and X_t vary smoothly with t, we say that $\{(U_t, X_t, \mathcal{L}_t)\}_{t \in I}$ is an *isotopy of convex tubes* over I.

Let M be a smooth manifold with Riemannian metric g. Let $S^*M \subset T^*M$ be the unit cosphere bundle, and let $\alpha = \lambda|_{S^*M}$ be a contact 1-form on S^*M where λ is the Liouville 1-form on T^*M (e.g. $\lambda = p\,dx$ on $T^*\mathbb{R}$). We identify S^*M with $T^\infty M$. We equip $S^*M \times T^*I$ with the contact form $\widetilde{\alpha} = \alpha + \tau\, dt$, where t is the coordinate of I and τ is the coordinate on the cotangent fiber. Then the composition $S^*M \times T^*I \hookrightarrow \dot{T}^*(M \times I) \to T^\infty(M \times I)$ is an open immersion and contactomorphism, with image $(x, t; [p, \tau]) \in T^\infty(M \times I)$ where $p \neq 0$.

Definition 0.4. Let $I \subset \mathbb{R}$ be a closed interval. A strong isotopy of Legendrians in S^*M over I is a Legendrian $\mathcal{L}_I \subset S^*M \times T^*I$. A strong isotopy of convex tubes is a convex tube (U_I, X_I) of $\mathcal{L}_I$ such that X_I preserves the fibers of $S^*M \times T^*I \to I$.

Our main result is the following.

Theorem 0.5. *If (U_I, X_I) is a strong isotopy of convex tubes around $\mathcal{L}_I$ in $S^*M \times T^*I$, then for any $t \in I$ we have an equivalence of categories*

$$\iota_t^* : Sh(M \times I, \mathcal{L}_I) \to Sh(M, \mathcal{L}_t),$$

where $\iota_t : M_t = M \times \{t\} \hookrightarrow M_I = M \times I$ is the inclusion of the slice over t.

Given a strong isotopy of Legendrians $\mathcal{L}_I$, we prove that to construct a tube thickening (U_I, X_I) it suffices to construct a Liouville hypersurface thickening of each slice $\mathcal{L}_t$ (see Proposition 1.14).

Although the result is expected given the analogous result in Fukaya category, and it is superseded by the recent paper [NS20], we hope that its purely sheaf-theoretic proof and the simpler cotangent bundle setting make the presentation of this result still worthwhile.

Previous work

We first recall the sheaf quantization of a contact isotopy of S^*M.

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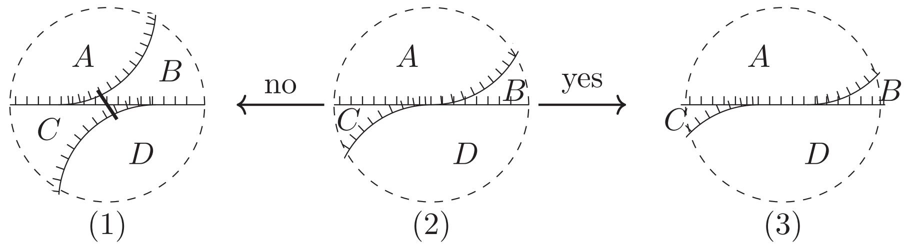

Figure 1. The deformation to the right is uniformly displaceable, and the one to the left is not, due to the appearance of a new short Reeb chord (marked by a thick line); cf. [Nad15, Example 1.5].

Theorem 0.6 [GKS12, Theorem 3.7 and Proposition 3.12]. *Let I be an open interval containing 0, and let $\varphi : I \times T^\infty M \to T^\infty M$ be a smooth map with $\varphi_t = \varphi(t, -)$. Assume φ is such that* (i) $\varphi_0 = \mathrm{id}$ *and* (ii) φ_t *are contactomorphisms for all $t \in I$. Then for each $t \in I$ we have the equivalences of categories*

$$\hat{\varphi}_t : Sh(M) \xrightarrow{\sim} Sh(M) \quad \textit{such that} \quad SS^\infty(\hat{\varphi}_t F) = \varphi_t(SS^\infty(F)).$$

Note that any isotopy of smooth Legendrians can be extended to a contact isotopy of the ambient manifold. In general, we have the following corollary.

Corollary 0.7. *If an isotopy of Legendrians $\{\Lambda_t^\infty\}_{t\in I}$ can be embedded into an isotopy $\{\varphi_t\}_{t\in I} : S^*M \to S^*M$ of the contact manifold, that is, $\Lambda_t^\infty = \varphi_t(\Lambda_0^\infty)$, then we have an equivalence of categories*

$$\hat{\varphi}_t : Sh(M, \Lambda_0^\infty) \xrightarrow{\sim} Sh(M, \Lambda_t^\infty).$$

For a deformation of singular Legendrians, there is one necessary condition for the invariance of categories, due to Nadler [Nad15].

Definition 0.8 (Displaceable Legendrian). Let (S^*M, α) be the unit cosphere bundle of a Riemannian manifold M with Reeb vector field R and time-t Reeb flow R^t. A Legendrian $\mathcal{L} \subset S^*M$ is *ϵ-displaceable* for R and for some $\epsilon > 0$ if

$$\mathcal{L} \cap R^s(\mathcal{L}) = \emptyset \quad \text{for all } 0 < |s| < \epsilon. \tag{1}$$

We say that a family of Legendrians $\{\mathcal{L}_t\}$ is *uniformly ϵ-displaceable* for R and for some $\epsilon > 0$ if each $\mathcal{L}_t$ is ϵ-displaceable.

If a family of Legendrians $\{\mathcal{L}_t\}$ can be upgraded to an isotopy of convex tubes $\{U_t, X_t, \mathcal{L}_t\}$, then $\{\mathcal{L}_t\}$ is uniformly displaceable (Proposition 1.9).

Example 0.9. Consider the example in Figure 1.

The category of constructible sheaves for the three diagrams comprises the representations of the following commutative diagrams (where each region corresponds to a vertex and an arrow between vertices goes against the direction of the hair).

$$(1) = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl] \\ C \arrow[r] & D \end{tikzcd} \qquad (2), (3) = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

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A sketch of the proof

Given a convex tube (U, X) of a Legendrian $\mathcal{L} \subset S^*M$, we may define a projection functor as the 'limit' of the flow X:

$$\Pi_{\mathcal{L}} : Sh(M, U) \to Sh(M, \mathcal{L}), \quad \Pi(F) := \lim_{T \to \infty} \hat{X}^T(F), \tag{2}$$

where $Sh(M, U)$ stands for the category of constructible sheaves with $SS^\infty(F) \subset U$ and the limit is defined using the nearby cycle functor in § 2.6.

Let $(U_I, X_I, \mathcal{L}_I)$ be a strong isotopy of convex tubes in $S^*M \times T^*I$, and let $\{(U_t, X_t, \mathcal{L}_t)\}$ be the slices. Let $F_t \in Sh(M, \mathcal{L}_t)$. We will extend F_t to a sheaf $F_I \subset Sh(M \times I, \mathcal{L}_I)$ such that $F_I|_t = F_t$.

We first show that such an extension is unique (if it exists); this is equivalent to showing that the restriction functor $Sh(M, \mathcal{L}_I) \to Sh(M, \mathcal{L}_t)$ is fully faithful (Proposition 3.1), that is,

$$\mathrm{Hom}(F_I, G_I) \xrightarrow{\sim} \mathrm{Hom}(F_t, G_t) \quad \text{for all } F_I, G_I \in Sh(M \times I, \mathcal{L}_I).$$

One needs to show that $\mathcal{H}om(F_I, G_I)(M \times (a, b))$ is independent of the size of the interval, so that one can interpolate from $(a, b) = I$ to an infinitesimally small neighborhood around t. The key technical point is to use the uniform displaceability condition to perturb G_I slicewise by positive Reeb flow for time s, $G_I \to K_s^! G_I$, to separate $SS^\infty(F_I)$ and $SS^\infty(K_s^! G_I)$.

We then show that such an extension exists locally; that is, given F_t, we can find a small neighborhood $J = (t - \delta, t + \delta)$ such that $\mathcal{L}_t \times T_J^* J \subset U_J = U_I \cap S^*M \times T^*J$ and extend F_t on M_t to F_J on M_J by defining $F_J = \Pi_{\mathcal{L}_J}(F_t \boxtimes \mathbb{C}_J)$.

Finally, we use the uniqueness of extension to patch together the local extensions, and thus we get the global extension result (cf. [GKS12, Lemma 1.13]).

Notation

We use $Sh(M)$ to denote the co-complete dg derived category of weakly constructible sheaves. With an abuse of notation, we use 'constructible sheaf' to mean a cohomologically constructible complex of sheaves. All the functors $f_*, f^*, f_!, \mathcal{H}om$ etc. are derived.

1. Convex tubes and isotopy

1.1 Basics of contact geometry

We recall the definition of co-oriented contact manifold as follows. Let C be a $(2n+1)$-dimensional manifold, and let $\xi \subset TC$ be a rank-$2n$ sub-bundle such that there exists a 1-form (contact 1-form) α (up to multiplication by a non-negative function) satisfying $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$. If we fix such an α, we have a Reeb vector field R_α given by

$$\iota_{R_\alpha} \alpha = 1, \quad \iota_{R_\alpha} d\alpha = 0.$$

We remark that different choices of α will lead to different choices of R_α.

A contact vector field X is a vector field on C that preserves the sub-bundle ξ.

Definition 1.1. Given a smooth function $H : C \to \mathbb{R}$, the *contact Hamiltonian vector field X_H* is uniquely determined by the conditions

$$\begin{cases} \langle X_H, \alpha \rangle = H, \\ \iota_{X_H} d\alpha = \langle dH, R \rangle \alpha - dH. \end{cases} \tag{3}$$

The Reeb vector field is a special case of X_H where $H = 1$.

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PROPOSITION 1.2 [Gei08, Theorem 2.3.1]. *With a fixed choice of contact form α, there is a one-to-one correspondence between the contact vector field X and smooth functions $H : C \to \mathbb{R}$. The correspondence is given by*

$$X \mapsto H = \langle \alpha, X \rangle, \quad H \mapsto X_H.$$

Unlike symplectic Hamiltonian vector fields, X_H does not preserve the level sets of H.

LEMMA 1.3. *We have that*

$$\langle X_H, dH \rangle = H \langle R, dH \rangle.$$

In particular, X_H preserves the zero set of H.

Proof. Apply ι_{X_H} to the second line of (3) and then use the first line. □

We also have the Lie derivative of α along X_H,

$$\mathcal{L}_{X_H}(\alpha) = d\iota_{X_H}\alpha + \iota_{X_H}\, d\alpha = \langle R, H \rangle \alpha.$$

PROPOSITION 1.4. *If $\mathcal{L} \subset C$ is a germ of a smooth Legendrian and H is any locally defined function vanishing on $\mathcal{L}$, then the contact flow X_H is tangential to $\mathcal{L}$.*

Proof. To show that X_H is tangential to $\mathcal{L}$ at $p \in \mathcal{L}$, we only need to show that for any tangent vector $v \in T_p\mathcal{L}$ we have $d\alpha(X_H, v) = 0$ and $\alpha(X_H)|_p = 0$, because these two conditions imply $X_H \in (T_p\mathcal{L})^{\perp_{d\alpha}} \cap \ker(\alpha) = T_p\mathcal{L}$. Indeed, $\alpha(X_H)|_p = H(p) = 0$ and

$$d\alpha(X_H, v) = \iota_{X_H}(d\alpha)(v) = [\langle R, dH \rangle \alpha - dH](v) = \langle R, dH \rangle (\alpha(v)) - H(v) = 0.$$

Hence X_H is tangential to $\mathcal{L}$. □

Example 1.5. Let M be a smooth manifold, and let T^*M the cotangent bundle with canonical Liouville 1-form λ and symplectic 2-form $\omega = d\lambda$. If we put local Darboux coordinates $(q, p) = (q_1, \dots, q_m; p_1, \dots, p_m)$ on T^*M, where $m = \dim_{\mathbb{R}} M$, then $\lambda = \sum_{i=1}^m p_i\, dq_i$ and $\omega = \sum_i dp_i \wedge dq_i$, and we will suppress the indices and summation and write simply $\lambda = p\, dq$ and $\omega = dp\, dq$. Also define $\dot{T}^*M = T^*M \backslash T^*_M M$ and $T^\infty M = \dot{T}^*M/\mathbb{R}_{>0}$. The Liouville vector field for λ is defined by $\iota_{V_\lambda}\omega = \lambda$, and here it is given by $V_\lambda = p\partial_p$. On $T(\dot{T}^*M)$, the symplectic orthogonal to the Liouville vector field defines a distribution

$$\widetilde{\xi} = \{(q, p; v_q, v_p) \in T(\dot{T}^*M) : \omega((v_q, v_p), V_\lambda) = 0\},$$

which projects to a canonical contact distribution ξ on $T^\infty M$. Let g be any Riemannian metric on M; then T^*M has an induced norm. Let $S^*M = \{(q, p) \in T^*M : |p| = 1\}$ be the unit cosphere bundle with contact form $\alpha = \lambda|_{S^*M}$; then the contact distribution can also be written as $\xi = \ker(\alpha)$.

Define the symplectization of $(C, \xi = \ker \alpha)$ by

$$S := C \times \mathbb{R}_u, \quad \lambda = e^u \alpha, \quad \omega_S = d(e^u \alpha).$$

We have the projection along $\mathbb{R}_u$ and the inclusion of zero-section

$$\pi_S : S \to C, \quad \iota_C : C \simeq C \times \{0\} \hookrightarrow S.$$

A different choice of α would give rise to the same S up to a fiber-preserving symplectomorphism that identifies the 'zero-section' $\mathrm{Im}(\iota_C)$.

A Hamiltonian function $H : C \to \mathbb{R}$ can be extended to a homogeneous degree-one function $\widetilde{H} : S \to \mathbb{R}$ by setting $\widetilde{H} = e^u H$. Then the symplectic Hamiltonian vector field $\xi_{\widetilde{H}}$, given by $\omega_S(-, \xi_{\widetilde{H}}) = d\widetilde{H}(-)$, preserves the fiber of π_S and descends to X_H.

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1.2 Convex tubes

Recall the definition of convex tubes in Definition 0.2.

Definition 1.6. A Liouville hypersurface thickening of a singular Legendrian $\mathcal{L}$ is a hypersurface $\mathcal{H} \supset \mathcal{L}$ such that $(\mathcal{H}, \alpha|_{\mathcal{H}})$ is a Liouville domain with the Liouville skeleton being $\mathcal{L}$.

First we show that a Liouville hypersurface thickening can be upgraded to a convex tube thickening of $\mathcal{L}$.

Proposition 1.7. *Let $\mathcal{L}$ be a singular Legendrian with a Liouville hypersurface thickening $\mathcal{H}$. Then $\mathcal{L}$ admits a convex tube (U, X), where the contact vector field X preserves $\mathcal{H}$ and X restricted to $\mathcal{H}$ is the downward Liouville flow of $\mathcal{H}$.*

Proof. Let $\epsilon > 0$ be small enough that for any $0 < s < \epsilon$ we have $\mathcal{H} \cap R^s\mathcal{H} = \emptyset$. Then define $U = \bigcup_{|s|\leqslant\epsilon/2} R^s\mathcal{H} \simeq \mathcal{H} \times (-\epsilon/2, +\epsilon/2)$, and let $h : U \to (-\epsilon/2, +\epsilon/2)$ be the projection. Then $X = X_h$ shrinks U to $\mathcal{L}$ and restricts to the downward Liouville flow on $\mathcal{H}$. One may smooth the corner of U and achieve transversality of X with ∂U. □

Conversely, we show that each convex tube (U, X) around $\mathcal{L}$ determines a Liouville thickening.

Proposition 1.8. *Let (U, X) be a convex tube around $\mathcal{L}$. Let $h = \alpha(X)$ and $\mathcal{H} = h^{-1}(0) \subset U$. Then $\mathcal{H}$ is a Liouville thickening of $\mathcal{L}$.*

Proof. Since $X = X_h$ preserves $\mathcal{H}$ and shrinks $\mathcal{H}$ to $\mathcal{L}$, we only need to show that $\mathcal{H}$ is transverse to R and X is the downward Liouville flow on $\mathcal{H}$.

Since $\mathcal{L}_X(\alpha) = \langle R, dh\rangle\alpha = -\alpha$, we have $R(h) = -1$. Thus R is transversal to the level sets of h, in particular $\mathcal{H}$. Hence $d\alpha$ is non-degenerate on $\mathcal{H}$, so $\mathcal{H}$ is exact symplectic. Let $\lambda = \alpha|_{\mathcal{H}}, \omega = d\lambda$. When we restrict to $T\mathcal{H}$, we have

$$\iota_{X_h}(\omega) = \iota_{X_h}(d\alpha) = \langle R, dh\rangle\alpha - dh = -\lambda,$$

and hence X_h is the downward Liouville flow on $\mathcal{H}$. □

Proposition 1.9. *Let (U, X) be a convex tube of $\mathcal{L}$. Then $\mathcal{L}$ is displaceable (see Definition 0.8). Similarly, let $I = [0, 1]$ and let (U_I, X_I) be a strong isotopy of convex tubes for $\mathcal{L}_I$. Then the family of Legendrians $\mathcal{L}_t$ are uniformly displaceable.*

Proof. Let $h = \alpha(X)$ be the Hamiltonian function generating X. Then h vanishes on $\mathcal{L}$, and by the normalization condition we have $R(h) = -1$. If there is a Reeb chord $\gamma : [0, T] \to C$ contained in U and ending on $\mathcal{L}$, then we have

$$\int_0^T \dot\gamma(dh)\, dt = \int_0^T R(dh)\, dt = \int_0^T (-1)\, dt = -T.$$

But on the other hand we also have

$$\int_0^T \dot\gamma(dh)\, dt = \int_\gamma dh = h(\gamma(T)) - h(\gamma(0)) = 0,$$

since $\gamma(T) \in \mathcal{L}$, $\gamma(0) \in \mathcal{L}$ and $h|_{\mathcal{L}} = 0$. Thus, there is no Reeb chord ending on $\mathcal{L}$ and contained in U. For any $x \in \mathcal{L}$, let $t(x) = \inf\{t \in \mathbb{R} : R^t(x) \notin U\}$; then $t(x) > 0$ and is continuous on $\mathcal{L}$. Let $\epsilon = \inf\{t(x) : x \in \mathcal{L}\}$; since $\mathcal{L}$ is compact, $\epsilon > 0$. Then $\mathcal{L}$ is displaceable.

For the uniformly displaceable statement, note that $I = [0, 1]$ is compact and $\epsilon(t)$ for (U_t, X_t) is continuous in t; hence $\epsilon = \inf\{\epsilon(t)\} > 0$. □

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1.3 The construction of strong isotopies of convex tubes

Consider the unit cosphere bundle (S^*M, α) and a closed interval $I \subset \mathbb{R}$. Let a point in S^*M be denoted by $(x,p) \in T^*M$ with $|p| = 1$. Let a point in T^*I be denoted by $(t, \tau) \in I \times \mathbb{R}$. Let $S^*M \times T^*I$ be equipped with the contact 1-form

$$\alpha_I = \alpha + \tau\, dt.$$

Let $\pi_t : S^*M \times T^*I \to I$.

Proposition 1.10. *The Reeb flow R_I on $S^*M \times T^*I$ for α_I is the pullback of the Reeb flow R on S^*M.*

Proof. Let R denote the pullback to $S^*M \times T^*I$. We may verify that $\iota_R(\alpha_I) = 1$ and $\iota_R(d\alpha_I) = 0$. □

Let $\mathcal{L}_I$ be a strong isotopy of Legendrians. Let

$$\mathcal{L}_t = \{(x,p) \in S^*M \mid \exists (x,p,t,\tau) \in \mathcal{L}_I\}.$$

Lemma 1.11. *We have that $\mathcal{L}_t$ is a singular Legendrian in S^*M.*

Proof. Take any $p \in \mathcal{L}_t$ that is the image of a point $\widetilde{p}$ in the smooth loci $\mathcal{L}_I^{sm}$; any tangent vector $v \in T_p\mathcal{L}_t$ can be lifted to $\widetilde{v} \in T_{\widetilde{p}}\mathcal{L}_I$. Concretely, $\widetilde{v} = v + c\partial_\tau$. Since $0 = (\alpha + \tau\, dt)(\widetilde{v}) = \alpha(v)$, we see that $T_p\mathcal{L}_t$ is in $\ker(\alpha)$. Hence a dense open part of $\mathcal{L}_t$ is Legendrian, and thus $\mathcal{L}_t$ is a singular Legendrian. □

Let $(U_I, X_I, \mathcal{L}_I)$ be a strong isotopy of convex tubes. First we define the restriction to $S^*M \times T_t^*I$. Since X_I preserves the t coordinate, for each t we have the vector field

$$\hat{X}_t := X_I|_t \in \mathrm{Vect}(S^*M \times T_t^*I).$$

Also define the restriction

$$\hat{U}_t = U_I \cap S^*M \times T_t^*I, \quad \hat{\mathcal{L}}_t = \mathcal{L}_I \cap S^*M \times T_t^*I.$$

Next, we define (U_t, X_t). Define the projection map $\hat{\pi}_t : S^*M \times T_t^*I \to S^*M$ and write

$$U_t = \hat{\pi}_t(\hat{U}_t), \quad \mathcal{L}_t = \hat{\pi}_t(\hat{\mathcal{L}}_t).$$

Let $h_I = \alpha_I(X_I)$; since X_I has no ∂_t component, $\partial_\tau h_I = 0$ and hence h_I is independent of τ. For each $t \in I$ we define

$$h_t(x,p) := h_I(x,p,t) \quad \text{for all } (x,p) \in U_t,$$

and we let X_t be the contact vector field generated by h_t.

Proposition 1.12. *With the above setup, we have*

$$\hat{X}_t = X_t + (-\tau - \partial_t h_t)\partial_\tau.$$

Proof. We split a tangent vector v on $S^*M \times T^*I$ into two components as $v = v_1 + v_2$, where v_1 and v_2 are along the S^*M and T^*I factors, respectively. Similarly, we decompose X_I as $X_I = X_{I,1} + X_{I,2}$, where $X_{I,2} = a\,\partial_\tau$.

By the definition of X_I, we have

$$\iota_{X_I}(\alpha + \tau\, dt) = h_I$$

and

$$\iota_{X_I} d(\alpha + \tau\, dt) = \langle R_I, h_I \rangle (\alpha + \tau\, dt) - dh_I,$$

which we will refer to as the first and second equations below.

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Since $\tau\, dt(X_{I,2}) = 0$, the first equation becomes

$$\alpha(X_{I,1}) = h_t(x,p).$$

For the second equation, if we restrict to the tangent space on S^*M, we have

$$\iota_{X_{I,1}}\, d\alpha = \langle R, h_t\rangle \alpha - dh_t.$$

Thus $X_{I,1} = X_t$ is the contact vector field on S^*M generated by $h_t(x,p)$.

Finally, if we restrict the second equation to the tangent space of T^*I, we get

$$\iota_{X_{I,2}}(d\tau \wedge dt) = \langle R, h_I\rangle(\tau\, dt) - \partial_t h_t\, dt.$$

If we plug in $X_{I,2} = a\partial_t$ and $\langle R, h_I\rangle = -1$, we get the desired result. □

Proposition 1.13. *For any $t \in I$, the (U_t, X_t) defined above is a convex tube for $\mathcal{L}_t$. Furthermore, the family $\{(U_t, X_t)\}_t$ varies smoothly with t and hence is an isotopy of convex tubes for $\{\mathcal{L}_t\}$.*

Proof. From Proposition 1.12 we know that the flow of $\hat{X}_t$ preserves the fibers of $S^*M \times T^*_t I \to S^*M$ and the induced flow on S^*M is generated by X_t. Since the flow of $\hat{X}_t$ shrinks $\hat{U}_t$ to $\hat{\mathcal{L}}_t$, i.e. $\hat{\mathcal{L}}_t = \bigcap_{u>0} \hat{X}_t^u \hat{U}_t$, the sequence of open sets $\hat{X}_t^u \hat{U}_t$ is monotonically decreasing in u; furthermore, $\hat{\pi}_t(\hat{X}_t^u \hat{U}_t) = X_t^u(U_t)$, so we have

$$\mathcal{L}_t = \bigcap_{u>0} X_t^u(U_t). \qquad \square$$

The final proposition allows us to upgrade from an isotopy of Liouville hypersurfaces to a strong isotopy of convex tubes.

Proposition 1.14. *If $\mathcal{L}_I$ is a Legendrian in $S^*M \times T^*I$ and if $\{\mathcal{H}_t\}$ is a smooth family of Liouville hypersurfaces in S^*M such that $\mathcal{L}_t$ is the skeleton of $\mathcal{H}_t$, then we have a strong isotopy of convex tubes (U_I, X_I) around $\mathcal{L}_I$.*

Proof. First we use $\mathcal{H}_t$ to get a family of convex tubes (U_t, X_t) and the associated Hamiltonian functions h_t, where $h_t|_{\mathcal{H}_t} = 0$ and $R(h_t) = -1$. The family of functions h_t determines the lifted function $h_I(x,p,t,\tau) = h_t(x,p)$ which is defined when $(x,p) \in U_t$. In turn, h_I determines the contact vector field X_I, which restricts to the fiber $S^*M \times T^*_t I$ as given by Proposition 1.12. Thus, we only need to specify the subset $\hat{U}_t \subset U_t \times T^*_t I$ such that its boundary $\partial\hat{U}_t$ is transverse to the vector field $\hat{X}_t$ and it is compressed by the flow of $\hat{X}_t$ to $\hat{\mathcal{L}}_t = \mathcal{L}_I \cap S^*M \times T^*_t I$.

Let

$$C = 1 + \sup\{|\partial_t h_t(x,p)| : (x,p) \in \bar{U}_t,\ t \in I\}$$

and let

$$\hat{U}_t = U_t \times (-C, C) \subset S^*M \times T^*I.$$

Then the flow $\hat{X}_t$ is transverse to the boundary ∂U_t. We only need to show that $\bigcap_{u>0} \hat{X}_t^u(\hat{U}_t) = \hat{\mathcal{L}}_t$. Since $\hat{U}_t \to U_t$ with fiber $(-C, C)$, $\hat{X}_t$ restricted to the fiber gives the equation for τ as

$$(d/du)\tau(u) = -\tau - \partial_t h_t(x(u), p(u)).$$

This is a contracting flow with unit contraction rate in the sense that for any initial condition (τ_1, τ_2) at $u = 0$ we have $\tau_1(u) - \tau_2(u) = (\tau_1 - \tau_2)e^{-u}$ for $u > 0$.

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Under the projection map $\hat\pi_t : S^*M \times T^*_t I \to S^*M$, we have the surjection

$$\hat\pi_t : \hat{\mathcal{L}}'_t := \bigcap_{u>0} \hat X^u_t \hat U_t \to \bigcap_{u>0} X^u_t U_t = \mathcal{L}_t,$$

and by the contracting property of the flow $\hat X_t$, the fiber can consist of only one point; thus $\hat\pi_t : \hat{\mathcal{L}}'_t \to \mathcal{L}_t$ is a bijection.

Let $U_I = \bigcup_{t\in I} \hat U_t \subset S^*M \times T^*I$, and put the slices $\hat{\mathcal{L}}'_t$ together into $\mathcal{L}'_I = \bigcap_{u>0} X^u_I U_I$. Recall $\hat\pi : S^*M \times T^*I \to S^*M \times I$; then $\hat\pi(\mathcal{L}_I) = \hat\pi(\mathcal{L}'_I)$ and $\mathcal{L}'_I$ is homeomorphic to its image. Since a smooth family of smooth Legendrians in $S^*M \times I$ has a unique lift to $S^*M \times T^*I$, $\mathcal{L}_I$ and $\mathcal{L}'_I$ agree over the smooth loci of $\hat\pi(\mathcal{L}_I)$. Since $\mathcal{L}_I$ is the closure of its smooth part, we have $\mathcal{L}_I = \mathcal{L}'_I$, finishing the proof of the proposition. □

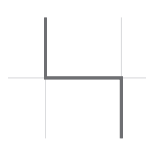
2. Non-characteristic isotopy of sheaves

2.1 Constructible sheaves

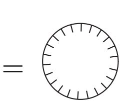
We give a quick working definition of constructible sheaves used in this paper, and refer to [KS13] for a proper treatment. A constructible sheaf F on M is a sheaf valued in a chain complex of $\mathbb{C}$-vector spaces, such that its cohomology is locally constant with finite rank with respect to some Whitney stratification[1] $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha\in A}$ on M, where the $\mathcal{S}_\alpha$ are disjoint locally closed smooth submanifolds with a nice adjacency condition and $M = \bigsqcup_{\alpha\in A} \mathcal{S}_\alpha$. The singular support $SS(F)$ of F is a closed conical Lagrangian in T^*M, contained in $\bigcup_{\alpha\in A} T^*_{\mathcal{S}_\alpha} M$, such that $SS(F) \cap T^*_M M$ equals the support of F and $(p, q) \in SS(F)\backslash T^*_M M$ if there exists a locally defined function f with $f(q) = 0$ and $df(q) = p$ such that the restriction map $F(B_\epsilon(q) \cap \{f < \delta\}) \to F(B_\epsilon(q) \cap \{f < -\delta\})$ fails to be a quasi-isomorphism for $0 < \delta \ll \epsilon \ll 1$. We denote by $SS^\infty(F) = SS(F) \cap S^*M$ the singular support of F at infinity.

If $\Lambda \subset T^*M$ is a conical Lagrangian containing the zero-section (as assumed throughout this paper), we write $Sh(M, \Lambda^\infty)$ for the dg derived category of constructible sheaves with object F satisfying $SS^\infty(F) \subset \Lambda^\infty$.

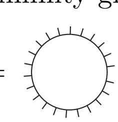
Example 2.1. On $\mathbb{R}$, let $\mathbb{C}_{[0,1]}$ (respectively $\mathbb{C}_{(0,1)}$) denote the constant sheaf with stalk $\mathbb{C}$ on $[0, 1]$ (respectively on $(0, 1)$) and zero stalk elsewhere. Then the singular supports of $\mathbb{C}_{[0,1]}$ and $\mathbb{C}_{(0,1)}$ in $T^*\mathbb{R}$ are

$SS(\mathbb{C}_{[0,1]}) =$  , $SS(\mathbb{C}_{(0,1)}) =$  .

Example 2.2. Let $j : U = B(0, 1) \hookrightarrow \mathbb{R}^2$ be the inclusion of an open unit ball in $\mathbb{R}^2$. Then $j_*\mathbb{C}_U$ is supported on the closed set $\overline{U}$, with singular support at infinity as

$SS^\infty(j_*\mathbb{C}_U) = \{(x, \eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = -d|x|\} =$ 

And $j_!\mathbb{C}_U$ is supported on the open set U, with singular support at infinity given by

$SS^\infty(j_!\mathbb{C}_U) = \{(x, \eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = d|x|\} =$ 

Here the Legendrians are represented by co-oriented hypersurfaces in $\mathbb{R}^2$ with hairs indicating the co-orientation.

[1] More precisely, a μ-stratification; see [KS13, § 8.3].

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2.2 Operation on sheaves

In this subsection, we deviate from our running convention and use $Sh(X)$ to denote the co-complete dg derived category of sheaves on X without any constructibility condition. Let $f : Y \to X$ be a map of real analytic manifolds. Then we have the following pairs of adjoint functors:

$$- \otimes F \colon Sh(X) \leftrightarrow Sh(X) \colon \mathcal{H}om(F, -),$$
$$f^* \colon Sh(X) \leftrightarrow Sh(Y) \colon f_*,$$
$$f_! \colon Sh(Y) \leftrightarrow Sh(X) \colon f^!.$$

Given an open subset U of X and its closed complement Z,

$$\text{open inclusion:} \quad U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z, \quad \text{closed inclusion,}$$

we have $j^* = j^!$ and $i_* = i_!$. Furthermore, there are exact triangles

$$i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{[1]}, \quad j_! j^! \to \mathrm{id} \to i_* i^* \xrightarrow{[1]}.$$

These are sheaf-theoretic incarnations of excisions: upon applying to the constant sheaf on X and taking global sections, we get

$$H^*(Z, i^! \mathbb{C}) \to H^*(X, \mathbb{C}) \to H^*(U, \mathbb{C}) \xrightarrow{[1]}, \quad H^*_c(U, \mathbb{C}) \to H^*_c(X, \mathbb{C}) \to H^*_c(Z, \mathbb{C}) \xrightarrow{[1]}.$$

Let X_i, $i = 1, 2$, be spaces, and let $K \in Sh(X_1 \times X_2)$. We define the pair of adjoint functors

$$K_! : Sh(X_1) \leftrightarrow Sh(X_2) : K^!, \tag{4}$$
$$K_! : F \mapsto \pi_{2!}(K \otimes \pi_1^* F), \quad K^! : G \mapsto \pi_{1*}(\mathcal{H}om(K, \pi_2^! G)). \tag{5}$$

In [KS13], $K_! = \Phi_K$ and $K^! = \Psi_K$ with X_1 and X_2 switched. The notation here is suggestive of their being adjoint functors.

2.3 Isotopy of constructible sheaves

Let $I = (a, b) \subset \mathbb{R}$. For any $t \in I$, let

$$j_t : M_t := M \times \{t\} \hookrightarrow M_I := M \times I$$

be the inclusion of the t-slice M_t into the total space M_I, and let $\pi_I : M_I \to I$ be the projection. Let $\mathbb{C}_{M_t}$ be the constant sheaf on M_t with stalk $\mathbb{C}$. We then have

$$SS(\mathbb{C}_{M_t}) = \{(x, t; 0, \tau) \in T^* M_I\}, \quad SS^\infty(\mathbb{C}_{M_t}) = \{(x, t; 0, \pm 1) \in S^* M_I \simeq T^\infty M\}.$$

Definition 2.3. Let M be a smooth manifold and I a closed interval of $\mathbb{R}$.

(i) An *isotopy of (constructible) sheaves* is a constructible sheaf $F_I \in Sh(M \times I)$ such that $SS^\infty(F_I)$ is a strong isotopy of Legendrians in $S^* M \times T^* I$ (Definition 0.4). Equivalently, for any $t \in I$ we have

$$SS^\infty(F_I) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset.$$

If F_I is an isotopy of sheaves, then for any $t \in I$ we denote the *restriction of F_I at t* by

$$F_t := F_I|_{M_t} \in Sh(M).$$

(ii) Two isotopies of sheaves $F_I, G_I \in Sh(M \times I)$ are said to be *non-characteristic* if

$$SS^\infty(F_I)|_t \cap SS^\infty(G_I)|_t = \emptyset \quad \text{for all } t \in I.$$

Some easy-to-check properties are the following.

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Proposition 2.4. *Let M be a compact real analytic manifold.*

(1) *If F_I is an isotopy of sheaves and $\Lambda_I^\infty = SS^\infty(F_I)$, then*
$$SS^\infty(F_t) \subset \Lambda_t^\infty.$$

(2) *If F_I is an isotopy of sheaves and $\pi_I : M_I \to I$, then $(\pi_I)_* F_I$ is a local system on I.*

2.4 Invariance of morphisms under non-characteristic isotopies

We use the same notation for $M_I = M \times I$, M_t, $\mathbb{C}_{M_t}$ etc. as in the previous subsection.

Lemma 2.5. *Let $F \in Sh(M)$ and let $\varphi : M \to \mathbb{R}$ be a C^1 function such that $d\varphi(x) \neq 0$ for $x \in \varphi^{-1}([0,1])$.*

(1) *For $s \in (0,1)$ let $U_s = \{x : \varphi(x) < s\}$, and let $U_1 = \bigcup_s U_s$. If*
$$SS^\infty(\mathbb{C}_{U_s}) \cap SS^\infty(F) = \emptyset \quad \textit{for all } 0 < s < 1,$$
then
$$\mathrm{Hom}(\mathbb{C}_{U_1}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{U_s}, F) \quad \textit{for all } 0 < s < 1.$$

(2) *For $s \in (0,1)$ let $Z_s = \{x : \varphi(x) \leqslant s\}$, and let $Z_0 = \bigcap_s Z_s$. If*
$$SS^\infty(\mathbb{C}_{Z_s}) \cap SS^\infty(F) = \emptyset \quad \textit{for all } 0 < s < 1,$$
then
$$\mathrm{Hom}(\mathbb{C}_{Z_s}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{Z_0}, F) \quad \textit{for all } 0 < s < 1.$$

Proof. Assertion (1) is a special case of [GKS12, Proposition 1.8]. Assertion (2) follows from (1) and the fact that
$$0 \to \mathbb{C}_{M \setminus \mathbb{Z}_s} \to \mathbb{C}_M \to \mathbb{C}_{Z_s} \to 0. \qquad \square$$

The following lemma is also often used.

Lemma 2.6 (Petrowsky theorem for sheaves [KS13]). *Let $F, G \in Sh(M)$ be (cohomologically) constructible complexes of sheaves. If $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then the natural morphism*
$$\mathcal{H}om(F, \mathbb{C}_M) \otimes G \to \mathcal{H}om(F, G)$$
is an isomorphism.

Corollary 2.7. *If F_I is an isotopy of sheaves, then*
$$\mathcal{H}om(\mathbb{C}_{M_t}, F_I) \simeq \mathbb{C}_{M_t}[-1] \otimes F_I.$$

Proposition 2.8. *Let G_I and F_I be non-characteristic isotopies of sheaves; then $\mathcal{H}om(F_I, G_I)$ is an isotopy of sheaves. In particular,*
$$\mathrm{Hom}(F_t, G_t) \simeq \mathrm{Hom}(F_s, G_s) \quad \textit{for all } t, s \in I.$$

Proof. As G_I and F_I being non-characteristic implies $SS^\infty(G_I) \cap SS^\infty(F_I) = \emptyset$, we can bound the singular support of the hom-sheaf as [KS13]
$$SS(\mathcal{H}om(F_I, G_I)) \subset SS(G_I) + SS(F_I)^a.$$
Again, using that G_I and F_I are non-characteristic, we obtain
$$SS^\infty(\mathcal{H}om(F_I, G_I)) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset \quad \text{for all } t, s \in I.$$

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Hence $\mathcal{H}om(F_I, G_I)$ is an isotopy of sheaves. For the second statement, we have

$$\begin{aligned}\mathrm{Hom}(F_t, G_t) &= \mathrm{Hom}({j_t}^* F_I, {j_t}^* G_I) \simeq \mathrm{Hom}(F_I, j_{t*} j_t^* G_I) \simeq \mathrm{Hom}(F_I, \mathbb{C}_{M_t} \otimes G_I)\\ &\simeq \mathrm{Hom}(F_I, \mathcal{H}om(\mathbb{C}_{M_t}, G_I)[1]) \simeq \mathrm{Hom}(\mathbb{C}_{M_t}, \mathcal{H}om(F_I, G_I))[1]\\ &\simeq \mathrm{Hom}(\mathbb{C}_t, \pi_{I*}\mathcal{H}om(F_I, G_I))[1] \simeq [\pi_{I*}\mathcal{H}om(F_I, G_I)]_t. \end{aligned} \tag{6}$$

The result then follows since $\pi_{I*}(\mathcal{H}om(F_I, G_I))$ is a local system. ☐

2.5 Invariance of morphisms under Reeb perturbations

Sometimes we want to vary G and F while preserving $\mathrm{Hom}(F, G)$, but $SS^\infty(G) \cap SS^\infty(F) \neq \emptyset$, e.g. $F = G$. Here we borrow an idea from the infinitesimally wrapped Fukaya category [NZ09], namely that to compute $\mathrm{Hom}_{Fuk}(L_1, L_2)$ one needs to perform a perturbation to separate L_1 and L_2 at infinity; one can perturb $L_2 \rightsquigarrow R^t L_2$ or $L_1 \rightsquigarrow R^{-t} L_1$, where R^t is the unit-speed geodesic flow on T^*M (smoothed near the zero-section) for positive small times t, small enough that no new intersections are created between L_1 and L_2 at infinity.

Fix a Riemannian metric g on M and identify S^*M with $T^\infty M$, so that the Reeb flow R^t is the unit-speed geodesic flow. Let $r_{\mathrm{inj}}(M, g)$ be the injective radius of (M, g). Let $\hat{R}^t$ be the GKS quantization of R^t. The rest of this subsection will be devoted to proving the following proposition.

Proposition 2.9. *Let $\Lambda^\infty \subset T^\infty M$ be a Legendrian, and let $0 < \epsilon < r_{\mathrm{inj}}(M, g)$ be small enough that*

$$\Lambda^\infty \cap R^t \Lambda^\infty = \emptyset \quad \textit{for all } 0 < |t| < \epsilon.$$

(1) *For any $F \in Sh(M, \Lambda)$ and $0 \leqslant t < \epsilon$, we have a canonical morphism*

$$F \to \hat{R}^t F.$$

(2) *For any $F, G \in Sh(M, \Lambda)$ and $0 \leqslant t < \epsilon$, we have canonical quasi-isomorphisms*

$$\mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(F, \hat{R}^t G), \quad \mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(\hat{R}^{-t} F, G).$$

Proof. For any $0 \leqslant t < \epsilon$, define

$$K_t = \mathbb{C}_{\{(x,y) | d_g(x,y) \leqslant t\}} \in Sh(M \times M).$$

Then, from [GKS12], we have

$$\hat{R}^t F = \pi_{1*}\mathcal{H}om(K_t, \pi_2^! F) = K_t^! F$$

and

$$\hat{R}^{-t} F = \pi_{2!}(K_t \otimes \pi_1^* F) = (K_t)_! F,$$

where π_1 and π_2 are the projections from $M \times M$ to the first and second factors, and $\mathcal{H}om$ is the (dg derived) sheaf-hom. From the canonical restriction morphism $K_t \to K_0 = \mathbb{C}_\Delta$, where $\Delta \subset M \times M$ is the diagonal subset, we have

$$F = \pi_{1*}\mathcal{H}om(K_0, \pi_2^! F) \to \pi_{1*}\mathcal{H}om(K_t, \pi_2^! F) = \hat{R}^t F.$$

For statement (2) of the proposition, we first prove the following lemma.

Lemma 2.10. *We have that*

$$SS^\infty(K_t) \cap SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G)) = \emptyset \quad \textit{for all } 0 < t < \epsilon. \tag{7}$$

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Proof. We identify the contact infinity $T^\infty M$ with the unit cosphere bundle S^*M. Assume that the intersection is non-empty and contains the point (x_1, x_2, p_1, p_2). Since $(x_1, x_2; p_1, p_2) \in SS^\infty(K_t)$, we have

$$d_g(x_1, x_2) = t.$$

Since $t < \epsilon < r_{\mathrm{inj}}(M, g)$, there is a unique length-t geodesic γ connecting x_1 and x_2, and p_i is the unit tangent vector along γ at x_i pointing to the interior of the geodesic,

$$p_i = -\partial_{x_i} d_g(x_1, x_2).$$

Hence the geodesic flow on S^*M relates (x_i, p_i) via

$$R^t(x_1, p_1) = (x_2, -p_2), \quad R^t(x_2, p_2) = (x_1, -p_1). \tag{8}$$

On the other hand, since $(x_1, x_2; p_1, p_2) \in SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G))$, we have

$$(x_1, -p_1) \in SS^\infty(F), \quad (x_2, p_2) \in SS^\infty(G). \tag{9}$$

Hence, combining (8) and (9), we have

$$(x_1, -p_1) \in R^t(SS^\infty(G)) \cap SS^\infty(F) \subset R^t\Lambda^\infty \cap \Lambda^\infty.$$

This contradicts the displaceability of Λ^∞ for $t < \epsilon$. □

Now we come back to the proof of Proposition 2.9. We have

$$\begin{aligned}
\mathrm{Hom}(F, G) &\simeq \Gamma(M, \mathcal{H}om(F, G)) \\
&\simeq \Gamma(M \times M, \mathcal{H}om(\mathbb{C}_\Delta, \mathcal{H}om(\pi_1^* F, \pi_2^! G))) \\
&\xrightarrow{\sim} \Gamma(M \times M, \mathcal{H}om(K_t, \mathcal{H}om(\pi_1^* F, \pi_2^! G))) \\
&\simeq \Gamma(M \times M, \mathcal{H}om(\pi_1^* F, \mathcal{H}om(K_t, \pi_2^! G))) \\
&\simeq \Gamma(M, \mathcal{H}om(F, \pi_{1*}\mathcal{H}om(K_t, \pi_2^! G))) \\
&\simeq \mathrm{Hom}(F, \hat{R}^t G),
\end{aligned}$$

where in the third step we used the canonical morphism $K_t \to \mathbb{C}_\Delta$ when replacing $\mathbb{C}_\Delta$ by K_t, and used Lemmas 2.10 and 2.5(2) to show that it is a quasi-isomorphism. □

We will use the following purely sheaf-theoretic statement later to study the family of GKS quantization.

Proposition 2.11. *Let $I = (0, 1)$, and let $K_I \in Sh(M \times M \times I)$ be an isotopy of sheaves such that $K_t = \mathbb{C}_{\Delta_t}$ for some closed subsets $\{\Delta_t\}_{0<t<1}$ satisfying*

$$\Delta_t \subset \Delta_s \ \forall\, 0 < t < s < 1 \quad \textit{and} \quad \bigcap_{t \in I} \Delta_t = \Delta_M = \{(x, x) : x \in M\}.$$

Let $F, G \in Sh(M, \Lambda)$, and let $\mathcal{H}om(\pi_1^* F, \pi_2^! G) \in Sh(M \times M)$ be the hom-sheaf. Assume that

$$SS^\infty(K_t) \cap SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G)) = \emptyset \quad \textit{for all } t \in I;$$

then

$$\mathrm{Hom}(F, G) \simeq \mathrm{Hom}(F, K_t^! G) \simeq \mathrm{Hom}(K_{t!} F, G) \quad \textit{for all } t \in I,$$

where $K_t^!, K_{t!}$ are defined in (5).

The proof is exactly the same as that of Proposition 2.9(2), where the condition provided in Lemma 2.10 is put into the hypothesis, so we do not repeat it here.

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2.6 Limit of contact isotopy

Let $I=(0,1)$ and define the inclusions

$$(0,1) \xhookrightarrow{j_I} \mathbb{R} \xhookleftarrow{j_0} \{0\}.$$

Proposition 2.12 [TWZ19, Lemma 7.1]. *Let $F_I \in Sh(M_I)$ be an isotopy of constructible sheaves, and let $\Lambda_I^\infty = SS^\infty(F_I)$. Suppose the family $(\Lambda_t^\infty, t) \subset T^\infty M \times (0,1)$ has a closure in $T^\infty M \times [0,1)$ whose intersection with $T^\infty M \times \{0\}$ is a Legendrian Λ_0^∞. Then the sheaf*

$$F_0 := (j_0)^*(j_I)_* F_I \tag{10}$$

has $SS^\infty(F_0) \subset \Lambda_0^\infty$.

Proof. These are corollaries of results in [KS13]. By [KS13, Theorem 6.3.1], a point $(x,p;0,-1) \in \dot{T}^*M \times T^*\mathbb{R}$ belongs to $SS((j_I)_*F_I)$ only if (x,p) is the limit of a sequence of points $(x_n,p_n) \in \Lambda_{t_n}$ where $t_n \to 0$, i.e. $(x,p) \in \Lambda_0$. By [KS13, Proposition 5.4.5], $SS(F_0) \subset SS((j_I)_*F_I)|_0 = \Lambda_0$; hence $SS^\infty(F_0) \subset \Lambda_0^\infty$. □

Let (U,X) be a convex tube for a Legendrian $\mathcal{L} \subset S^*M$. Let X be extended from a neighborhood of $\bar{U}$ to all of S^*M. Let

$$\hat{X}^{[0,\infty)} : Sh(M) \to Sh(M \times [0,\infty))$$

be the sheaf quantization of the flow X, and let

$$j_{[0,\infty)}: \ [0,\infty) \hookrightarrow \ [0,\infty] \ \hookleftarrow \{\infty\}: \ j_\infty.$$

Then we define the functor $\Pi_X := (\mathrm{id}_M \times j_\infty)^* \circ (\mathrm{id}_M \times j_{[0,\infty)})_* \circ \hat{X}^{[0,\infty)} : Sh(M) \to Sh(M)$.

Let $Sh(M,U)$ denote the subcategory of $Sh(M)$ consisting of sheaves F with $SS^\infty(F) \subset U$.

Proposition 2.13. *When restricted to $Sh(M,U)$, we have that*

$$\Pi_{U,X} = \Pi_X|_{Sh(M,U)} : Sh(M,U) \to Sh(M,\mathcal{L}).$$

Proof. This follows from the definition of a convex tube and Proposition 2.12. □

3. Existence and uniqueness of the extension

In this section we prove our main result, Theorem 0.5. In this section, we will sometimes identify $\Lambda_t^\infty \subset T^\infty M$ with $\mathcal{L}_t \subset S^*M$ and identify Reeb flow with geodesic flow.

3.1 Uniqueness of extension

Recall from Proposition 1.9 that existence of strong isotopy of convex tubes implies uniform displaceability of the family $\{\mathcal{L}_t\}$.

Proposition 3.1. *Let Λ_t^∞ be a family of Legendrians in $T^\infty M$ that are uniformly displaceable with parameter ϵ. Then the restriction functor ι_t^* is fully faithful for all t.*

Proof. For $0 \leqslant s < \epsilon$ we define a family of kernels in $Sh((M_1 \times I_1) \times (M_2 \times I_2))$ as

$$K_s := \mathbb{C}_{d(x_1,x_2) \leqslant s} \boxtimes \mathbb{C}_{t_1=t_2}. \tag{11}$$

One can check that K_s generates the slicewise geodesic flow, i.e. if $F_I \in Sh(M_I)$ and

$$K_s^! F_I := \pi_{1*}\mathcal{H}om(K_s, \pi_2^! F_I),$$

then

$$SS^\infty((K_s^! F_I)|_{M_t}) = R^s SS^\infty(F_I|_{M_t}),$$

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where π_i is the projection from $(M_1 \times I_1) \times (M_2 \times I_2)$ to $M_i \times I_i$ and R^s is the Reeb (geodesic) flow for time s.

We first prove the following claim: for any $F_I, G_I \in Sh(M_I, \Lambda_I^\infty)$,

$$\mathrm{Hom}(\mathbb{C}_{M\times(a,b)}, \mathcal{H}om(F_I, G_I)) \text{ is independent of } (a,b) \subset I.$$

It suffices to prove the case for the right endpoint b. To use the estimate of the singular support of the hom-sheaf, we would like to perturb G_I by the fiberwise Reeb flow.

Lemma 3.2. *For any $0 < s < \epsilon$,*

$$\mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, G_I)) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I)).$$

Furthermore, $\mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I))$ *is independent of t. The same is true if we replace $\{t\}$ by any subinterval, e.g. $[a,b]$ and (a,b) of I.*

Proof. Unwinding the definition of $K_s^!$, we have

$$\begin{aligned}
&\mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I))\\
&\quad = \mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, \pi_{1*}\mathcal{H}om(K_s, \pi_2^! G_I)))\\
&\quad = \mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \pi_{1*}\mathcal{H}om(\pi_1^* F_I, \mathcal{H}om(K_s, \pi_2^! G_I)))\\
&\quad = \mathrm{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))).
\end{aligned}$$

We claim that

$$SS^\infty(\pi_1^*\mathbb{C}_{M\times\{t\}}) \cap SS^\infty\mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)) = \emptyset \quad \text{for all } 0 < s < \epsilon. \tag{12}$$

The verification is straightforward, though a bit tedious, and we leave it to the reader.

From this claim and the fact that

$$\begin{aligned}
&\mathrm{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)))\\
&\quad \simeq \mathrm{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)),
\end{aligned}$$

we may apply Lemma 2.5(2) on the shrinking closed set to get

$$\mathrm{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)) \simeq \mathrm{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_0, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))$$

for all $0 < s < \epsilon$. This proves the first statement of the lemma.

The statement about independence of t follows from (12) and Proposition 2.8.

The subinterval case can be proved similarly, and we omit the details. ☐

Now we finish proving the proposition. By Lemma 3.2,

$$\mathrm{Hom}(\mathbb{C}_{M\times(a,b)}, \mathcal{H}om(F_I, G_I))$$

is independent of (a,b), so we may shrink from $(0,1)$ to an arbitrary small neighborhood of t. Then we have

$$\begin{aligned}
\mathrm{Hom}(F_I, G_I) &\simeq [\pi_{I*}(\mathcal{H}om(F_I, G_I))]_t \simeq [\pi_{I*}(\mathcal{H}om(F_I, K_s^! G_I))]_t\\
&\simeq \mathrm{Hom}(\iota_t^* F_I, \iota_t^* K_s^! G_I) \simeq \mathrm{Hom}(F_t, R^s G_t) \simeq \mathrm{Hom}(F_t, G_t),
\end{aligned}$$

where $0 < s < \epsilon$ and we have used a small Reeb perturbation to make $F_I, K_s^! G_I$ a non-characteristic isotopy of sheaves and then applied (6) from the proof of Proposition 2.8. ☐

Proposition 3.3. *Let $\{\Lambda_t^\infty\}$ be a family of Legendrians in $T^\infty M$ that are uniformly displaceable with parameter ϵ. For a given t, let $F_t \in Sh(M, \Lambda_t^\infty)$. Suppose we have F_I' and F_I'' in $Sh(M_I, \Lambda_I^\infty)$*

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and isomorphisms

$$f : F'_I|_t \xrightarrow{\sim} F_t, \quad g : F''_I|_t \xrightarrow{\sim} F_t.$$

Then there exists a canonical isomorphism

$$\Phi : F'_I \to F''_I$$

such that $\Phi|_t = g^{-1} \circ f : F'_I|_t \to F''_I|_t$.

Proof. The proof follows from Proposition 3.1 by standard arguments. □

3.2 Existence of local extension

Proposition 3.4. *Let $I = [0,1]$. Let $\mathcal{L}_I$ be a strong isotopy of Legendrians in $S^*M \times T^*I$ with the slice over t denoted by $\mathcal{L}_t$. Let (U_I, X_I) be a strong isotopy of convex tubes for $\mathcal{L}_I$. Then for any $t \in I$ and $F_t \in Sh(M, \mathcal{L}_t)$, there exists an interval $J \supset t$ and $F_J \in Sh(M_J, \mathcal{L}_J)$ such that $F_J|_t = F_t$, where $M_J = M \times J$ and $\mathcal{L}_J = \mathcal{L}_I \cap S^*M \times T^*J$.*

Proof. For any interval $J \subset I$, let $U_J = U_I \cap S^*M \times T^*J$. Then for small enough J containing t, we have $\mathcal{L}_t \times T^*_J J \subset U_J$. Let X_J denote the restriction of X_I to X_J; then if we define (see Proposition 2.13 for definition of $\Pi_{U,X}$)

$$F_J := \Pi_{(U_J, X_J)}(F_t \boxtimes \mathbb{C}_J),$$

we have $F_J|_t = F_t$ and $SS^\infty(F_J) \in \mathcal{L}_J$. □

3.3 Proof of Theorem 0.5

By the local extension result (Proposition 3.4) and uniqueness of extension result, for any $t \in I = [0,1]$ and $F_t \in Sh(M, \mathcal{L}_t)$ we can extend F_t to $F_I \in Sh(M_I, \mathcal{L}_I)$ such that $F_I|_t = F_t$. Hence the functor ι_t^* is fully faithful (Proposition 3.1) and essentially surjective; thus it is an equivalence.

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Peng Zhou pzhou.math@gmail.com

University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720, USA

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