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TOPOLOGICAL PROPERTIES OF LEGENDRIAN SINGULARITIES

Dedicated to Professor Haruo Suzuki on his 60th birthday.

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ABSTRACT. A wavefront set is defined to be an image of Legendrian mapping. In this note we prove that generic wavefront sets have stable local topological structures by using Mather's stratification theory.

1. **Introduction.** The notion of Legendrian singularities has been introduced by V. I. Arnol'd in order to describe wavefront sets [1,4]. In this paper we will prove that generic Legendrian singularities are singularities of MT-stable map germs.

Let N be a $(2n+1)$-dimensional smooth manifold and K be a contact structure on N (i.e. K is a non-degenerate tangent hyperplane field on N). An immersion $i: L \longrightarrow N$ is said to be *Legendrian* if $\dim L = n$ and $di_x(T_xL) \subset K_{i(x)}$ for any $x \in L$. We say that a smooth fibre bundle $\pi: E \longrightarrow M$ is *Legendrian* if its total space E is furnished with a contact structure and its fibres are Legendrian submanifolds.

DEFINITION 1.1. Let $\pi: E \longrightarrow M$ be a Legendrian bundle. For a Legendrian immersion $i: L \longrightarrow E$, $\pi \circ i: L \longrightarrow M$ is said to be *a Legendrian map.* The image of Legendrian map $\pi \circ i$ is said to be *a wavefront set* of i. It is denoted by $W(i)$.

Our theorem is the following.

THEOREM 1.2. *For a residual set of Legendrian immersions $L \longrightarrow E$, local pictures of the wavefront set $W(i)$ are given by critical value sets of MT-stable map germs. Here, we call a map germ MT-stable if it is transverse to the canonical stratification of a jet space which is introduced in* [2, 3].

Of course, the set of critical values of an MT-stable map germ has a canonical Whitney stratification and a stable local topological structure. Hence, we have a finite number of local models of generic wave front sets up to stratified equivalence.

Suppose we are given a generic initial wave front W in some space M, as an example consider a surface in Euclidian 3-space. As time progresses the wavefront propagates, for a surface in 3-space along the normal directions. At certain special times the wavefront undergoes catastrophic changes, but in between one expects its structure not to change. More precisely we expect the wavefronts to be stable at all except a discrete number

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of distances. In low dimension Zakalyukin [4] has shown that generic wavefronts are smoothly stable. In higher dimensions smooth stability is lost, but one can show that they are stable up to stratified equivalence, and it is a general result of this nature we prove here.

All map germs considered here, are differentiable of class C^∞, unless stated otherwise.

2. **Generating families.** In this section we shall describe Legendrian immersion germs in terms of generalized phase function germs and review the classification theory of Arnol'd-Zakalyukin [1,4]. For any $p \in E$, there is a local coordinate system $(x_1, \dots, x_m, y_1, \dots, y_m, z)$ around p such that

$$\pi(x_1, \dots, x_m, y_1, \dots, y_m, z) = (x_1, \dots, x_m, z)$$

and the contact structure is given by the 1-form

$$\alpha = dz - \sum_{i=1}^{m} y_i \cdot dx_i$$

(cf. [1, 20.3]). Hence, we shall only consider the standard Legendrian bundle

$$\pi\colon \mathbb{R}^{2m+1} \longrightarrow \mathbb{R}^m \times \mathbb{R}.$$

Let $F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a map germ such that d_2F is nonsingular at 0. Here,

$$d_2F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$$

is given by

$$d_2F(q,x) = \Big(\frac{\partial F}{\partial x_1}(q,x), \dots, \frac{\partial F}{\partial x_k}(q,x)\Big).$$

Then F is called a *generalized phase function germ.* The critical set $C(F) = d_2F^{-1}(0)$ is a smooth m-manifold. We can prove that the map germ

$$\Phi_F\colon \big(C(F), 0\big) \longrightarrow \mathbb{R}^{2m+1}$$

given by

$$\Phi_F(q,x) = \Big(q, \frac{\partial F}{\partial q}(q,x), F(q,x)\Big)$$

is Legendrian immersion. Then we have the following proposition.

PROPOSITION 2.1 (ARNOL'D-ZAKALYUKIN [1,4]). *Any Legendrian immersion germ is given by the above construction.*

Let $F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generalized phase function germ. We define

$$\tilde{F}\colon (\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$$

by

$$\tilde{F}(q,x,z) = F(q,x) - z.$$

We call $\tilde{F}$ *a generating family of* Φ_F.

Let $\mathcal{E}_n$ be the ring of function germs $(\mathbb{R}^n, 0) \longrightarrow \mathbb{R}$ and $\mathcal{M}_n$ be the unique maximal ideal. For each map germ $f\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, we define $f^*\colon \mathcal{E}_p \longrightarrow \mathcal{E}_n$ by $f^*(h) = h \circ f$.

DEFINITION 2.2. (1) Let $f, g\colon (\mathbb{R}^n \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that f and g are *$P - \mathcal{K}$-equivalent* if there exists a diffeomorphism germ

$$\Psi\colon (\mathbb{R}^n \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}^k, 0)$$

of the form

$$\Psi(u, x) = \big(\psi(u), \Psi_2(u, x)\big)$$

for $(u, x) \in \mathbb{R}^n \times \mathbb{R}^k$ such that

$$\Psi^*(\langle f \rangle_{\mathcal{E}_{n+k}}) = \langle g \rangle_{\mathcal{E}_{n+k}}.$$

(2) Let $f\colon (\mathbb{R}^n \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ and $g\colon (\mathbb{R}^n \times \mathbb{R}^{k'}, 0) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that f and g are *stably $P - \mathcal{K}$-equivalent* if they are $P - \mathcal{K}$-equivalent after addition of non-degenerate quadratic forms in additional variables.

The following is very important.

THEOREM 2.3 (ARNOL'D-ZAKALYUKIN [1,4]). *Let*

$$\tilde{F}\colon \big((\mathbb{R}^m \times \mathbb{R}) \times \mathbb{R}^k, 0\big) \longrightarrow (\mathbb{R}, 0)$$

and

$$\tilde{G}\colon \big((\mathbb{R}^m \times \mathbb{R}) \times \mathbb{R}^{k'}, 0\big) \longrightarrow (\mathbb{R}, 0)$$

be generating families of the Legendrian immersion germ $\Phi_F = \Phi_G$. Then $\tilde{F}$ and $\tilde{G}$ are stably $P - \mathcal{K}$-equivalent.

3. **Proof of the main result.** In this section we shall use notations and results in [2]. For our purpose, we need a stratification in the jet space $J^\ell(\mathbb{R}^k, \mathbb{R})$; by Theorem 2.3, the local choice of $\tilde{F}$ is well defined up to stable $P - \mathcal{K}$-equivalence. Hence, the stratification which we need must be contact invariant. But this means that the Looijenga's canonical stratification for smooth functions is not relevant. Instead we use Mather's stratification in [2,3]. For any $x_0 \in \mathbb{R}^k$, we let $\mathcal{E}_{x_0}$ denote the ring of smooth function germs at x_0. The unique maximal ideal in $\mathcal{E}_{x_0}$ is denoted by $\mathcal{M}_{x_0}$. We now define $\mathcal{E}_{x_0}$-module $\theta(f)_{x_0}$, for any map germ $f\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, to be the set of germs of vector fields along f. We also write $\theta(\mathbb{R}^n)_{x_0} = \theta(1_{\mathbb{R}^n})_{x_0}$. There is a $\mathcal{E}_{x_0}$-homomorphism

$$tf\colon \theta(\mathbb{R}^n)_{x_0} \longrightarrow \theta(f)_{x_0}$$

defined by

$$tf(\xi) = df \circ \xi.$$

Let $z \in J^\ell(\mathbb{R}^n, \mathbb{R}^p)$, and let $f\colon (\mathbb{R}^n, x_0) \longrightarrow (\mathbb{R}^p, y_0)$ be a map germ such that $z = j^\ell_{x_0} f$. Define

$$\chi(z) = \dim_{\mathbb{R}} \frac{\theta(f)_{x_0}}{tf(\theta(\mathbb{R}^n)_{x_0} + \big(f^*(\mathcal{M}_{x_0}) + \mathcal{M}^\ell_{x_0}\big)\theta(f)_{x_0}}.$$

Let $\mathcal{A}^\ell(\mathbb{R}^n, \mathbb{R}^p)$ be the canonical stratification of $J^\ell(\mathbb{R}^n, \mathbb{R}^p) - W^\ell(\mathbb{R}^n, \mathbb{R}^p)$ defined in ([2],IV,§ 2), where $W^\ell(\mathbb{R}^n, \mathbb{R}^p)$ is the set of $z \in J^\ell(\mathbb{R}^n, \mathbb{R}^p)$ with $\chi(z) \geq \ell$. This stratification is contact invariant. Hence, we will use this stratification.

Let $F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generalized phase function germ. We define

$$(\pi_m, F)\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^m \times \mathbb{R}, 0)$$

by

$$(\pi_m, F)(q, x) = \big(q, F(q, x)\big).$$

Then $C(F) = \Sigma(\pi_m, F)$, where $\Sigma(\pi_m, F)$ is the set of singular points of (π_m, F). By the definition,

$$\pi \circ \Phi_F = (\pi_m, F)\big|\Sigma(\pi_m, F),$$

then the wavefront set of Φ_F is the critical value set of (π_m, F). For our purpose, it is enough to prove the following theorem.

THEOREM 3.1 (THE LOCAL VERSION OF THEOREM 1.2). *Let $F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generalized phase function germ such that $j^\ell_1 F(0) \notin W^\ell(\mathbb{R}^k, \mathbb{R})$ and $j^\ell_1 F$ is transverse to $\mathcal{A}^\ell(\mathbb{R}^k, \mathbb{R})$. Then (π_m, F) is an MT-stable map germ. Here*

$$j^\ell_1 F\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow J^\ell(\mathbb{R}^k, \mathbb{R})$$

is defined by

$$j^\ell_1 F(q, x) = j^\ell F_q(x).$$

Let $J^\ell(m+k, m+1)$ be the fibre of the ℓ-jet bundle $J^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$. We now define an affine subspace $J^\ell_\pi(m+k, m+1)$ of $J^\ell(m+k, m+1)$ consisting of the ℓ-jets $j^\ell(\pi_m, f)(0)$ of function germs $f\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$. Then we have the following lemma.

LEMMA 3.2. *Let X be a contact invariant submanifold of $J^\ell(m+k, m+1)$. Then X is transverse to $J^\ell_\pi(m+k, m+1)$ in $J^\ell(m+k, m+1)$.*

PROOF. Let $f\colon (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a smooth function germ, and let

$$z = j^\ell(\pi_m, f) \in J^\ell_\pi(m+k, m+1) \cap X.$$

Then there is a natural identification (as $\mathbb{R}$-vector spaces)

$$T_z(J^\ell(m+k, m+1)) = \frac{\mathcal{M}_{m+k}\theta(\pi_m, f)}{\mathcal{M}^{\ell+1}_{m+k}\theta(\pi_m, f)}.$$

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It is easy to show that

$$T_z(J^\ell_\pi(m+k,m+1)) = \frac{\mathcal{M}_{m+k}\theta(f)}{\mathcal{M}^{\ell+1}_{m+k}\theta(f)}$$

by the above identification.

The tangent space of the contact orbit through z is given by

$$\mathcal{M}_{m+k}\left\langle \frac{\partial(\pi_m,f)}{\partial q_1},\dots,\frac{\partial(\pi_m,f)}{\partial q_m},\frac{\partial(\pi_m,f)}{\partial x_1},\dots,\frac{\partial(\pi_m,f)}{\partial x_k}\right\rangle + \langle q_1,\dots,q_m\rangle\theta(\pi_m,f) \bmod \mathcal{M}^{\ell+1}_{m+k}\theta(\pi_m,f).$$

(See [2], III, Lemma 6.7).

Since $\frac{\partial(\pi_m,f)}{\partial q_i} = e_i$, we have

$$\mathcal{M}_{m+k}\theta(\pi_m,f) \equiv \mathcal{M}_{m+k}\theta(f) + T_zX \bmod \mathcal{M}^{\ell+1}_{m+k}\theta(\pi_m,f),$$

where e_i is the canonical basis of $\mathbb{R}^m \times \mathbb{R}$ and $i = 1,\dots,m$. This completes the proof.

We have a decomposition

$$J^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) = \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times J^\ell(m+k,m+1).$$

By this decomposition, we have

$$\mathcal{A}^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) = \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times \mathcal{A}^\ell(m+k,m+1).$$

The stratum of $\mathcal{A}^\ell(m+k,m+1)$ is denoted by $S_j(m+k,m+1)$. Put

$$\mathcal{A}^\ell_\pi(m+k,m+1) = J^\ell_\pi(m+k,m+1) \cap S_j(m+k,m+1).$$

By Lemma 3.2, $\mathcal{A}^\ell_\pi(m+k,m+1)$ is a Whitney stratification of

$$J^\ell_\pi(m+k,m+1) - J^\ell_\pi(m+k,m+1) \cap W^\ell(m+k,m+1).$$

Define

$$J^\ell_\pi(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) = \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times J^\ell_\pi(m+k,m+1)$$

and

$$\mathcal{A}^\ell_\pi(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) = \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times \mathcal{A}^\ell_\pi(m+k,m+1).$$

By Lemma 3.2, we have the following simple proposition.

PROPOSITION 3.3. *Let $f\colon(\mathbb{R}^m \times \mathbb{R}^k,0) \longrightarrow (\mathbb{R},0)$ be a smooth function germ. If $j^\ell(\pi_m,f)$ is transverse to $\mathcal{A}^\ell_\pi(m+k,m+1)$ in $J^\ell_\pi(m+k,m+1)$, then $j^\ell(\pi_m,f)$ is transverse to $\mathcal{A}^\ell(m+k,m+1)$ in $J^\ell(m+k,m+1)$.*

For the proof of Theorem 3.1, we will need the following proposition.

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PROPOSITION 3.4. *Let* $F: (\mathbb{R}^m \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$ *be a generalized phase function germ such that* $j_1^\ell F(0) \notin W^\ell(\mathbb{R}^k, \mathbb{R})$. *If* $j_1^\ell F$ *is transverse to* $\mathcal{A}^\ell(\mathbb{R}^k, \mathbb{R})$ *then* $j^\ell(\pi_m, F)$ *is transverse to* $\mathcal{A}_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$ *in* $J_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$.

We now define a projection

$$\pi^\ell: J_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) \to J^\ell(\mathbb{R}^k, \mathbb{R})$$

by

$$\pi^\ell(j^\ell(\pi_m, F)(q, x)) = j^\ell F_q(x).$$

By the definition of $\mathcal{A}^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$ in ([2],IV,§2), we have the following lemmas.

LEMMA 3.5. $(\pi^\ell)^{-1}\big(S_j(k,1)\big) = S_j^\pi(m+k, m+1)$, *where* $S_j^\pi(m+k, m+1) = S_j(m+k, m+1) \cap J_\pi^\ell(m+k, m+1)$.

LEMMA 3.6. $(\pi^\ell)^{-1}(W^\ell(k,1)) = W^\ell(m+k, m+1) \cap J_\pi^\ell(m+k, m+1)$.

PROOF OF PROPOSITION 3.4. Let $\pi_1: J_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) \to J_\pi^\ell(m+k, m+1)$ and $\pi_2: J^\ell(\mathbb{R}^k, \mathbb{R}) \to J^\ell(k, 1)$ be canonical projections. It follows that $j^\ell(\pi_m, F)$ is transverse to $\mathcal{A}_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$ in $J_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R})$ if and only if $\pi_1 \circ j^\ell(\pi_m, F)$ is transverse to $\mathcal{A}_\pi^\ell(m+k, m+1)$ in $J_\pi^\ell(m+k, m+1)$. We can also prove that $j_1^\ell F$ is transverse to $\mathcal{A}^\ell(\mathbb{R}^k, \mathbb{R})$ in $J^\ell(\mathbb{R}^k, \mathbb{R})$ if and only if $\pi_2 \circ j_1^\ell F$ is transverse to $\mathcal{A}^\ell(k,1)$ in $J^\ell(k,1)$. By Lemma 3.6 and the fact that $\pi^\ell|: J_\pi^\ell(m+k, m+1) \to J^\ell(k,1)$ is a submersion, we have

$$(d\pi^\ell|)^{-1}\big(T_{\pi^\ell(z)} S_j(k,1)\big) = T_z S_j^\pi(m+k, m+1)$$

for any $S_j(k,1) \in \mathcal{A}^\ell(k,1)$.

We now consider the following commutative diagram:

$$\begin{array}{ccccc} (\mathbb{R}^m \times \mathbb{R}^k, 0) & \xrightarrow{j^\ell(\pi_m, F)} & J_\pi^\ell(\mathbb{R}^m \times \mathbb{R}^k, \mathbb{R}^m \times \mathbb{R}) & \xrightarrow{\pi_1} & J_\pi^\ell(m+k, m+1) \\ \| & & \pi^\ell \downarrow & & \pi^\ell| \downarrow \\ (\mathbb{R}^m \times \mathbb{R}^k, 0) & \xrightarrow{j_1^\ell F} & J^\ell(\mathbb{R}^k, \mathbb{R}) & \xrightarrow{\pi_2} & J^\ell(k,1) \end{array}$$

It follows that

$$(d\pi^\ell)^{-1}((d\pi_2 \circ j_1^\ell F)\big(T_0(\mathbb{R}^m \times \mathbb{R}^k)\big) = d\big(\pi_1 \circ j^\ell(\pi_m, F)\big)\big(T_0(\mathbb{R}^m \times \mathbb{R}^k)\big).$$

If $\pi_2 \circ j_1^\ell F$ is transverse to $S_j(k,1)$ in $J^\ell(k,1)$, then $\pi_1 \circ j^\ell(\pi_m, F)$ is transverse to $S_j^\pi(m+k, m+1)$ in $J_\pi^\ell(m+k, m+1)$. This completes the proof.

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