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ON GROUPS WITH CHAIN CONDITIONS

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Our aim in this note is to generalize results of Baer in [**3**; **5**]. In § 1 an arbitrary formation $\mathfrak{n}$ is considered, the key result being Proposition 1.5. This is applied in § 2 to characterize various finiteness conditions, for example the classes of groups with maximum [minimum] condition on subgroups, subnormal subgroups, and normal subgroups respectively, or the class of (not necessarily soluble) polyminimax groups (see Theorems 2.1 and 2.6). These results may also be regarded as generalizations of the well-known theorem of Malcev-Baer that a radical group satisfies the maximum condition [is a polyminimax group] if all its abelian subgroups satisfy the maximum condition [are minimax groups].

Notation.

$X \circ Y$ = set of all commutators $x \circ y = x^{-1}y^{-1}xy$ with x in X, y in Y.

$G' = G \circ G$ = commutator subgroup of the group G.

$\mathfrak{c}X$ = centralizer of the subset X of G.

$\mathfrak{z}G$ = centre of G.

Factor = epimorphic image of a subgroup.

A group theoretical property $\mathfrak{e}$ is a non-empty isomorphism closed class of groups.

A group is an $\mathfrak{e}$-group if it has the property $\mathfrak{e}$.

If N is a normal subgroup of G, then $N\,\mathfrak{e}\,G$ if and only if $G/\mathfrak{c}N$ is an $\mathfrak{e}$-group.

Almost-$\mathfrak{e}$-group = group with an $\mathfrak{e}$-subgroup of finite index.

Locally-$\mathfrak{e}$-group = group whose finitely generated subgroups are $\mathfrak{e}$-groups.

Radical group = group whose non-trivial epimorphic images possess non-trivial locally nilpotent normal subgroups.

Poly-$\mathfrak{e}$-group = group with a finite (normal) chain from 1 to G with $\mathfrak{e}$-factors.

Soluble group = group G with $G^{(i)} = 1$ for almost all i.

Minimax group = group which possesses a normal subgroup with maximum condition such that its quotient group satisfies the minimum condition.

$\mathfrak{R}G$ = radical of the group G = product of all radical normal subgroups of G.

A set of normal subgroups of a group is independent if their product is direct.

Accessible subgroup of a group G = subgroup of G that can be connected with G by a well-ordered ascending normal chain.

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1. Formations of groups. In this section we consider a group theoretical property $\mathfrak{n}$ meeting certain requirements. Most of the discussion may be read either with the word "characteristic subgroup" or with the word "normal subgroup". Therefore we will always use the word "invariant subgroup" which may be read "normal subgroup" or "characteristic subgroup" throughout. The following requirements are imposed upon $\mathfrak{n}$:

(H) Epimorphic images (modulo invariant subgroups) of $\mathfrak{n}$-groups are $\mathfrak{n}$-groups.

(R) If X and Y are invariant subgroups of the group G and G/X and G/Y are $\mathfrak{n}$-groups, then also $G/(X \cap Y)$ is an $\mathfrak{n}$-group.

A group theoretical property $\mathfrak{n}$ meeting these two requirements is called a *formation*. Since $G/(X \cap Y)$ is isomorphic to a subgroup of the direct product $G/X \times G/Y$, requirement (R) is always satisfied if $\mathfrak{n}$ is inherited by subgroups and direct products.

If X is a normal subgroup of the group G, then $G/\mathfrak{c}X$ is essentially the group of automorphisms induced in X by G. We write $X \,\mathfrak{n}\, G$ if and only if $G/\mathfrak{c}X$ is an $\mathfrak{n}$-group.

If X and Y are normal subgroups of the group G such that $X \subseteq Y$, then $\mathfrak{c}Y \subseteq \mathfrak{c}X$. Thus by (H), $Y \,\mathfrak{n}\, G$ implies $X \,\mathfrak{n}\, G$ and $(Y/X)\,\mathfrak{n}\,(G/X)$. Since $\mathfrak{c}(XY) = \mathfrak{c}X \cap \mathfrak{c}Y$ for any two normal subgroups X and Y of G, by (R), $X \,\mathfrak{n}\, G$ and $Y \,\mathfrak{n}\, G$ imply $XY \,\mathfrak{n}\, G$. Furthermore, it should be noted that the centralizer of an invariant subgroup of any group is an invariant subgroup of this group.

LEMMA 1.1. *If* 1 *is the only abelian invariant subgroup of the group* G *and if* X *and* Y *are invariant subgroups of* G *such that* $X \subseteq Y$, *then*

$$Y \,\mathfrak{n}\, G \text{ if and only if } X \,\mathfrak{n}\, G \text{ and } (Y/X)\,\mathfrak{n}\,(G/X).$$

Proof. The necessity of the second condition is a consequence of the above remarks. Assume that $X \,\mathfrak{n}\, G$ and $(Y/X)\,\mathfrak{n}\,(G/X)$. Let T be the uniquely determined invariant subgroup of G such that $X \subseteq T$ and $T/X = \mathfrak{c}(Y/X)$. The hypotheses imply that $G/\mathfrak{c}X$ and

$$(G/X)/\mathfrak{c}(Y/X) = (G/X)/(T/X) \simeq G/T$$

are $\mathfrak{n}$-groups. By (R) also $G/(T \cap \mathfrak{c}X)$ is an $\mathfrak{n}$-group. If t is an element in $T \cap \mathfrak{c}X$, then t stabilizes the normal subgroup X of Y. This implies that $t \circ Y \subseteq \mathfrak{z}X$; see for instance [**10**, p. 88, proof of *Satz* 19]. Since $\mathfrak{z}X$ is a characteristic subgroup of X, it is an invariant subgroup of G. Since 1 is the only abelian invariant subgroup of G, $\mathfrak{z}X = 1$. This shows that $t \circ Y = 1$, and t is contained in $\mathfrak{c}Y$. Thus $T \cap \mathfrak{c}X = \mathfrak{c}Y$, so that $G/(T \cap \mathfrak{c}X) = G/\mathfrak{c}Y$ is an $\mathfrak{n}$-group and $Y \,\mathfrak{n}\, G$.

For the group theoretical property $\mathfrak{n}$ and any group theoretical property $\mathfrak{f}$ and any ordinal β we define inductively the following properties:

(*o*) $\mathfrak{n}^0(\mathfrak{f}) = \mathfrak{n}$;

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(β) The group G is an $\mathfrak{n}^\beta(\mathfrak{f})$-group if and only if every epimorphic image $H \neq 1$ of G (modulo an invariant subgroup) possesses an invariant subgroup $N \neq 1$ which is an $\mathfrak{f}$-group or $N\, \mathfrak{n}^{\beta'}(\mathfrak{f})\, H$ for some $\beta' < \beta$.

($*$) The group G is an $\mathfrak{n}^*(\mathfrak{f})$-group if and only if it is an $\mathfrak{n}^\beta(\mathfrak{f})$-group for some β.

Thus a group G is an $\mathfrak{n}^*(\mathfrak{f})$-group if and only if every epimorphic image $H \neq 1$ of G (modulo an invariant subgroup) possesses an invariant subgroup $N \neq 1$ which is an $\mathfrak{f}$-group or $N\, \mathfrak{n}^*(\mathfrak{f})\, H$.

It follows from the definition that epimorphic images (modulo invariant subgroups) of $\mathfrak{n}^\beta(\mathfrak{f})$-groups and $\mathfrak{n}^*(\mathfrak{f})$-groups are likewise $\mathfrak{n}^\beta(\mathfrak{f})$-groups and $\mathfrak{n}^*(\mathfrak{f})$-groups, respectively, so that these properties always satisfy (H). Also, a group is an $\mathfrak{n}^\beta(\mathfrak{f})$-group for any $\beta > 0$ if its epimorphic images $H \neq 1$ (modulo invariant subgroups) possess invariant subgroups, not 1. The above definitions are generalizations of group classes considered by Baer in [**3**].

LEMMA 1.2. *If the group theoretical property* $\mathfrak{f}$ *is factor inherited, then the group* G *is an* $\mathfrak{n}^\beta(\mathfrak{f})$*-group for some ordinal* β [*an* $\mathfrak{n}^*(\mathfrak{f})$*-group*] *if and only if every invariant subgroup* $K \neq 1$ *of the epimorphic image* $H \neq 1$ (*modulo an invariant subgroup*) *contains an invariant subgroup* $N \neq 1$ *such that* $1 \subset N \subseteq K$ *and* N *is an* $\mathfrak{f}$*-group or* $N\, \mathfrak{n}^{\beta'}(\mathfrak{f})\, H$ *for some* $\beta' < \beta$ [$N\, \mathfrak{n}^*(\mathfrak{f})\, H$].

Proof. This may be proved in the same way as [**4**, p. 17, Lemma 3.2]. The property (R) is not needed in this proof.

LEMMA 1.3. *If the group theoretical property* $\mathfrak{f}$ *is inherited by normal subgroups, then the properties* $\mathfrak{n}^\beta(\mathfrak{f})$ *for any* β *and* $\mathfrak{n}^*(\mathfrak{f})$ *satisfy* (R).

Proof. Note first that a group G is an $\mathfrak{n}^\beta(\mathfrak{f})$-group if and only if it possesses an ascending invariant series (G_i) leading from 1 to G with G_i invariant in G and such that G_{i+1}/G_i is an $\mathfrak{f}$-group of $G_{i+1}/G_i\, \mathfrak{n}^{\beta'}(\mathfrak{f})\, G/G_i$ for some $\beta' < \beta$.

Let X and Y be invariant subgroups of the group G with $\mathfrak{n}^\beta(\mathfrak{f})$-quotient groups G/X and G/Y. If $\beta = 0$, then (R) implies that $G/(X \cap Y)$ is an $\mathfrak{n}^\beta(\mathfrak{f})$-group. Thus it may be assumed that β is a positive ordinal. By the remark above there exist invariant series

$$X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_\gamma = G \quad \text{for some ordinal } \gamma,$$
$$Y = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_\delta = G \quad \text{for some ordinal } \delta,$$

such that

$$X_{i+1}/X_i \text{ is an } \mathfrak{f}\text{-group or } (X_{i+1}/X_i)\, \mathfrak{n}^{\beta'}(\mathfrak{f})\, (G/X_i) \quad \text{for some } \beta' < \beta,$$
$$Y_{i+1}/Y_i \text{ is an } \mathfrak{f}\text{-group or } (Y_{i+1}/Y_i)\, \mathfrak{n}^{\beta''}(\mathfrak{f})\, (G/Y_i) \quad \text{for some } \beta'' < \beta.$$

It has to be shown that there exists an invariant series of G leading from $X \cap Y$ to G whose factors are $\mathfrak{f}$-groups or G induces $\mathfrak{n}^{\beta'''}(\mathfrak{f})$-groups of automorphisms in them for $\beta''' < \beta$.

Clearly $X_i \cap Y$ is an invariant series of G leading from $X_0 \cap Y = X \cap Y$ to $X \cap Y = G \cap Y = Y$. Now $(X_{i+1} \cap Y)/(X_i \cap Y)$ is isomorphic to a normal subgroup of X_{i+1}/X_i, and if X_{i+1}/X_i is an $\mathfrak{f}$-group, then so is

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$(X_{i+1} \cap Y)/(X_i \cap Y)$. Assume now that $(X_{i+1}/X_i)\ \mathfrak{n}^{\beta'}(\mathfrak{f})\ (G/X_i)$ for some $\beta' < \beta$, and let C/X_i be the centralizer of X_{i+1}/X_i in G/X_i. Then $(G/X_i)/(C/X_i)$ is an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group. If x is an element in C, then $x \circ X_{i+1} \subseteq X_i$. Therefore $x \circ (X_{i+1} \cap Y) \subseteq (X_i \cap Y)$ and $C(X_i \cap Y)/(X_i \cap Y)$ is in the centralizer of $(X_{i+1} \cap Y)/(X_i \cap Y)$ in $G/(X_i \cap Y)$. Thus the group of automorphisms induced by $G/(X_i \cap Y)$ in $(X_{i+1} \cap Y)/(X_i \cap Y)$ is an epimorphic image of the $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group G/C, and thus itself an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group.

COROLLARY 1.4. *If* $\mathfrak{f}$ *is a factor inherited group theoretical property, then the following properties are formations:*

(a) *The class of groups whose non-trivial epimorphic images possess non-trivial normal* $\mathfrak{f}$*-subgroups;*

(b) *The class of groups whose non-trivial epimorphic images possess non-trivial subnormal* $\mathfrak{f}$*-subgroups.*

Proof. That the class in (a) is a formation follows from Lemma 1.3 if $\mathfrak{n}$ is the universal property of being a group. That the class in (b) is also a formation follows from this by using transfinite induction and, for instance, [**6**, p. 413, *Satz* 5.7].

PROPOSITION 1.5. *Let* $\mathfrak{f}$ *be a factor inherited group theoretical property such that abelian groups are* $\mathfrak{f}$*-groups, and assume that the group* G *satisfies the following two requirements:*

(a) 1 *is the only invariant* $\mathfrak{f}$*-subgroup of* G,

(b) G *is an* $\mathfrak{n}^*(\mathfrak{f})$*-group.*

Then the following properties hold:

(I) *For every invariant subgroup* $X \neq 1$ *of* G *there exists an invariant subgroup* Y *of* G *such that* $1 \subset Y \subseteq X$ *and* $Y\ \mathfrak{n}^*(\mathfrak{f})\ G$.

(II) *If* G *is not an* $\mathfrak{n}$*-group, then there exists an infinite independent set of invariant subgroups* X *of* G *such that* $X\ \mathfrak{n}^*(\mathfrak{f})\ G$.

Proof. (I) is a consequence of Lemma 1.2 and the absence of non-trivial invariant $\mathfrak{f}$-subgroups of G.

Since G is an $\mathfrak{n}^*(\mathfrak{f})$-group, it is also an $\mathfrak{n}^{\beta}(\mathfrak{f})$-group for a least ordinal β. If G is not an $\mathfrak{n}$-group, then $0 < \beta$. By induction we construct invariant subgroups N_i of G with the following properties:

(+) $P_0 = N_0 = 1$, if $P_n = \prod_{i=0}^{n} N_i$, then $P_n \cap N_{n+1} = 1$ and $P_n\ \mathfrak{n}^{\beta'}(\mathfrak{f})\ G$ for some $\beta' < \beta$.

Assume now that $0 < n$ and that invariant subgroups $N_1, \ldots, N_n$ have already been constructed. By (+), $G/\mathfrak{c}P_n$ is an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group for some $\beta' < \beta$. By the minimality of β, G is not an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group and thus $\mathfrak{c}P_n \neq 1$. By Lemma 1.3 there exists an invariant subgroup $Y = N_{n+1}$ of G such that $1 \subset N_{n+1} \subseteq \mathfrak{c}P_n$ and $N_{n+1}\ \mathfrak{n}^{\beta'}(\mathfrak{f})\ G$. Here we note that N_{n+1} cannot be an $\mathfrak{f}$-group because of (a). There exist no non-trivial abelian invariant subgroups of G by (a) and thus

$$P_n \cap N_{n+1} \subseteq P_n \cap \mathfrak{c}P_n = \mathfrak{z}P_n = 1,$$

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and $P_n \cap N_{n+1} = 1$. Since $P_n\ \mathfrak{n}^{\beta'}(\mathfrak{f})\ G$ and $N_{n+1}\ \mathfrak{n}^{\beta'}(\mathfrak{f})\ G$, the groups $G/\mathfrak{c}P_n$ and $G/\mathfrak{c}N_{n+1}$ are both $\mathfrak{n}^{\beta'}(\mathfrak{f})$-groups. Lemma 1.4 and (R) imply that

$$G/(\mathfrak{c}P_n \cap \mathfrak{c}N_{n+1})$$

is an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group. However, $\mathfrak{c}P_n \cap \mathfrak{c}N_{n+1} = \mathfrak{c}P_{n+1}$, where $P_{n+1} = P_nN_{n+1}$, so that $G/\mathfrak{c}P_{n+1}$ is an $\mathfrak{n}^{\beta'}(\mathfrak{f})$-group. Thus (+) is proved. It is well known that (+) implies that the N_i form an infinite independent set of invariant subgroups of G.

COROLLARY 1.6. *Let all abelian groups be* $\mathfrak{f}$*-groups, let the group theoretical property* $\mathfrak{f}$ *be factor inherited and assume that the group* G *has the following properties:*

(a) 1 *is the only invariant* $\mathfrak{f}$*-subgroup of* G*;*

(b) G *is an* $\mathfrak{n}^*(\mathfrak{f})$*-group;*

(c) *There exists no infinite independent set of invariant subgroups* X *of* G*. Then* G *is an* $\mathfrak{n}$*-group.*

This follows immediately from Proposition 1.5.

2. Application to finiteness conditions. In this section we take for the group theoretical property $\mathfrak{n}$ certain finiteness conditions. Also in the definition of the properties $\mathfrak{n}^{\beta}(\mathfrak{f})$ and $\mathfrak{n}^*(\mathfrak{f})$ we take for the property $\mathfrak{f}$ the class $\mathfrak{r}$ of radical groups or the class $\mathfrak{a}$ of abelian groups. It follows then that radical groups are $\mathfrak{n}^{\beta}(\mathfrak{r})$-groups for every $\beta > 0$ and therefore also $\mathfrak{n}^*(\mathfrak{r})$-groups. Also, hyperabelian groups are $\mathfrak{n}^{\beta}(\mathfrak{a})$-groups for every $\beta > 0$ and therefore also $\mathfrak{n}^*(\mathfrak{a})$-groups.

It is well known that in every group the product $\mathfrak{R}G$ of all radical normal subgroups of G is a radical characteristic subgroup of G, the radical of G, whose quotient group $G^* = G/\mathfrak{R}G$ possesses no non-trivial radical accessible subgroups and therefore also no non-trivial radical normal subgroups; see for instance [**8**, p. 44, 3.2.2]. It follows from Corollary 1.6 that every $\mathfrak{n}^*(\mathfrak{r})$-group, whose epimorphic images modulo characteristic subgroups possess no infinite independent sets of invariant subgroups, is an extension of its radical by an $\mathfrak{n}$-group.

A group G is a minimax group if G possesses a normal subgroup with maximum condition whose quotient group satisfies the minimum condition (see [**7**]). Abelian minimax groups may be specialized in the following way. If $\mathfrak{p}$ is a set of primes, then the abelian minimax group A is a $\mathfrak{p}$-minimax group if $\mathfrak{p}$ contains every prime p such that A contains an infinite p-quotient group. It is easy to see that subgroups and quotient groups of $\mathfrak{p}$-minimax groups are $\mathfrak{p}$-minimax groups. Abelian extensions of $\mathfrak{p}$-minimax groups by $\mathfrak{p}$-minimax groups are $\mathfrak{p}$-minimax groups. An abelian $\mathfrak{p}$-minimax group satisfies the maximum condition if and only if $\mathfrak{p}$ is the empty set of primes.

THEOREM 2.1. (a) *If the abelian subgroups of the* $\mathfrak{n}^*(\mathfrak{r})$*-group* G *are* $\mathfrak{p}$*-minimax*

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groups, then G is an extension of a characteristic soluble poly-$\mathfrak{p}$-minimax subgroup by an $\mathfrak{n}$-group.

(b) *If the abelian accessible subgroups of the $\mathfrak{n}^*(\mathfrak{a})$-group G are $\mathfrak{p}$-minimax groups and no epimorphic image of G modulo a characteristic subgroup contains an infinite independent set of invariant subgroups, then G is also an extension of a characteristic soluble poly-$\mathfrak{p}$-minimax group by an $\mathfrak{n}$-group.*

Proof. If the abelian subgroups of the group G are $\mathfrak{p}$-minimax groups for the set of primes $\mathfrak{p}$, application of [**7**, p. 30, *Satz* 5.1] yields the following.

(1) $\mathfrak{R}G$ is soluble and abelian factors of $\mathfrak{R}G$ are $\mathfrak{p}$-minimax groups; 1 is the only radical normal subgroup of $G^* = G/\mathfrak{R}G$. Abelian subgroups of G^* are $\mathfrak{p}$-minimax groups.

The last statement implies, in particular, that G^* cannot contain infinite elementary abelian subgroups; see [**7**, p. 3, *Lemma* 1.2]. Thus G^* cannot contain an infinite independent set of invariant subgroups, and since G^* is an $\mathfrak{n}^*(\mathfrak{r})$-group, application of Corollary 1.6 yields:

(2) $G^* = G/\mathfrak{R}G$ is an $\mathfrak{n}$-group.

This proves the first part of the theorem.

If the abelian accessible subgroups of G are $\mathfrak{p}$-minimax groups, application of [**7**, p. 35, *Hauptsatz* 7.1] yields:

(3) The product $\mathfrak{R}^*G$ of all hyperabelian normal subgroups of G is a characteristic soluble poly-$\mathfrak{p}$-minimax group of G.

This implies that 1 is the only abelian normal subgroup of $G/\mathfrak{R}^*G$. Application of the hypotheses of (b) and Corollary 1.6 now yield that $G/\mathfrak{R}^*G$ is an $\mathfrak{n}$-group. Thus the theorem is proved.

Theorem 2.1 may be used to give characterizations of various classes of groups as may be seen by the following corollary. The group theoretical property $\mathfrak{n}$ is *extension inherited* if every group G is an $\mathfrak{n}$-group whenever the characteristic subgroup N of G and its quotient group G/N are $\mathfrak{n}$-groups.

COROLLARY 2.2. *Let $\mathfrak{n}$ be an extension inherited formation which is inherited by abelian subgroups and such that for a set of primes $\mathfrak{p}$ an abelian group is an $\mathfrak{n}$-group if and only if it is a $\mathfrak{p}$-minimax group. Then the following holds:*

(a) *A group G is an $\mathfrak{n}$-group if and only if it is an $\mathfrak{n}^*(\mathfrak{r})$-group and its abelian subgroups are $\mathfrak{n}$-groups;*

(b) *If no epimorphic image of G modulo a characteristic subgroup possesses an infinite independent set of invariant subgroups, then G is an $\mathfrak{n}$-group if and only if it is an $\mathfrak{n}^*(\mathfrak{a})$-group and its abelian accessible subgroups are $\mathfrak{n}$-groups.*

Proof. If G is an $\mathfrak{n}$-group, then by (H) it is also an $\mathfrak{n}^*(\mathfrak{r})$-group and its abelian subgroups are $\mathfrak{n}$-groups by hypotheses. Conversely, if G is an $\mathfrak{n}^*(\mathfrak{r})$-group whose abelian subgroups are $\mathfrak{n}$-groups, then its abelian subgroups are $\mathfrak{p}$-minimax groups for some set of primes $\mathfrak{p}$, and by Theorem 2.1, G is an extension of a soluble poly-$\mathfrak{p}$-minimax characteristic subgroup by an $\mathfrak{n}$-group.

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Since abelian $\mathfrak{p}$-minimax groups are $\mathfrak{n}$-groups and $\mathfrak{n}$ is extension inherited, G is an $\mathfrak{n}$-group. Thus (a) is proved. The proof of (b) is similar.

Remark 2.3. Among the group theoretical properties $\mathfrak{n}$ that satisfy the hypotheses of Corollary 2.2 are the following:

(a) The class of (not necessarily soluble) polyminimax groups;

(b) The class of (almost) soluble poly-$\mathfrak{p}$-minimax groups for any set of primes $\mathfrak{p}$;

(c) The class of groups in which every ascending [descending] chain of subgroups U_i has only finitely many finite indices $|U_{i+1}: U_i|$.

Since all these classes are factor and direct product inherited, they are clearly formations. If $\mathfrak{p}$ is the set of all primes, then an abelian group is a $\mathfrak{p}$-minimax group if and only if it is one of the classes under (c); see [**7**, p. 3, *Lemma* 1.2]. It is then evident that in all cases, Corollary 2.2 is applicable.

Theorem 2.1 contains as a special case that a radical group whose abelian subgroups are $\mathfrak{p}$-minimax groups for a set of primes p is a poly-$\mathfrak{p}$-minimax group, since this property is extension inherited; see [**7**]. Thus it contains the theorem of Baer used essentially in its proof. Theorem 2.1 may therefore be regarded as another generalization of the theorem of Malcev and Baer that a radical group satisfies the maximum condition [is a polyminimax group] if all its abelian subgroups satisfy the maximum condition [are minimax groups].

Let $\mathfrak{m}$ be one of the following six classes of groups: the classes of groups with minimum [maximum] condition on subgroups, subnormal subgroups, and normal subgroups, respectively. It is well known that a group G is an $\mathfrak{m}$-group if its normal subgroup N and the quotient group G/N are $\mathfrak{m}$-groups; see [**9**, p. 9, Lemma 1.31]. Since the classes of groups with minimum [maximum] condition on subgroups are also subgroup inherited, they are formations. That also the other four classes are formations follows from the following well-known lemma which may for example be proved by a generalization of [**5**, p. 167, the proof of *Hilfssatz* 2.1].

LEMMA 2.4. *If X and Y are normal subgroups of the group G, then for any of the above classes of groups $\mathfrak{m}$, $G/(X \cap Y)$ is an $\mathfrak{m}$-group if and only if G/X and G/Y are $\mathfrak{m}$-groups.*

Proof. By the above remark we may assume that $\mathfrak{m}$ is the class of groups with maximum [minimum] condition on subnormal or on normal subgroups, respectively. Let $\mathfrak{S}$ be the set of all subnormal subgroups or, in the second case, the set of all normal subgroups of G. Then G/J is an $\mathfrak{m}$-group if and only if the subgroups in $\mathfrak{S}$ containing J satisfy the maximum [minimum] condition. In particular, this remark implies the necessity of our condition.

Assume now that G/X and G/Y are $\mathfrak{m}$-groups. Let $\mathfrak{M}$ be a non-empty set of subgroups in $\mathfrak{S}$ containing $X \cap Y$. Then the set of all AX with A in $\mathfrak{M}$ is a non-empty set of subgroups in $\mathfrak{S}$ containing X. Since G/X is an $\mathfrak{m}$-group,

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there exists a maximal [minimal] element M in $\mathfrak{M}$. Then the set $\mathfrak{N}$ of all subgroups N in $\mathfrak{M}$ such that $M = XN$ is non-empty.

Also the set of all $\mathfrak{S}$-subgroups $Y(X \cap B)$ with B in $\mathfrak{N}$ is non-empty. Since G/Y is an $\mathfrak{m}$-group, there exists a maximal [minimal] element in $\mathfrak{N}$. This is an $\mathfrak{S}$-subgroup W such that

$$N = Y(X \cap W) \text{ with } W \text{ in } \mathfrak{N}.$$

Since W is in $\mathfrak{M}$, $X \cap Y \subseteq W$. Let V be an element of $\mathfrak{M}$ such that $W \subseteq V$ [$V \subseteq W$]. Then

$$M = XW \subseteq XV \qquad [XV \subseteq XW = M]$$

and the maximality [minimality] of M implies $XW = XV$, so that V is in $\mathfrak{N}$. Furthermore,

$$N = Y(X \cap W) \subseteq Y(X \cap V) \quad [Y(X \cap V) \subseteq Y(X \cap W) = N]$$

and the maximality [minimality] of N implies that $Y(X \cap W) = Y(X \cap V)$. Application of Dedekind's Modular Law to the relations

$$X \cap W \subseteq X \cap V \subseteq Y(X \cap W),\ W \subseteq V \subseteq XW,$$
$$[Y(X \cap W) \subseteq X \cap V \subseteq X \cap W,\ XW \subseteq V \subseteq W]$$

yields

$$X \cap V = (X \cap W)(Y \cap X \cap V) = X \cap W,$$

since $X \cap Y \subseteq X \cap W$ [$X \cap W \subseteq X \cap Y$]. Furthermore,

$$V = W(X \cap V) = W(X \cap W) = W,$$

so that W is maximal [minimal] in $\mathfrak{M}$. Thus $G/(X \cap Y)$ is an $\mathfrak{m}$-group.

Remark 2.5. The proof of Lemma 2.4 may still be exploited to yield similar results for other finiteness conditions, for example for the maximum condition on accessible subgroups or the maximum condition for accessible subgroups of at most a certain defect. Also the following is proved in the same way:

If X and Y are characteristic subgroups of the group G, then $G/(X \cap Y)$ satisfies the maximum [minimum] condition on characteristic subgroups if and only if G/X and G/Y satisfy the maximum [minimum] condition on characteristic subgroups.

THEOREM 2.6. *Let $\mathfrak{n}$ be an extension inherited formation such that Abelian groups with minimum [maximum] condition are $\mathfrak{n}$-groups. Then the following properties of the $\mathfrak{n}^*(\mathfrak{r})$-group G are equivalent:*

(I) *G is an $\mathfrak{n}$-group;*

(II) *If G is not an $\mathfrak{n}$-group, then the abelian subgroups of G satisfy the minimum [maximum] condition.*

(III) *If G is not an $\mathfrak{n}$-group, then the normal subgroups of any characteristic subgroup of G satisfy the minimum condition [maximum condition or are finitely generated].*

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Proof. The equivalence of (I) and (II) is a special case of Theorem 2.1. If (III) is satisfied, then it follows from [**1**, p. 348, *Satz* A; **2**, p. 9, Theorem] that the radical $\mathfrak{R}G$ of G satisfies the minimum [maximum] condition. Also 1 is the only abelian invariant subgroup of $G^* = G/\mathfrak{R}G$, and by (III), G^* cannot possess an infinite independent set of invariant subgroups. Application of Corollary 1.6 implies that G^* is an $\mathfrak{n}$-group. Since $\mathfrak{n}$ is extension inherited, G is an $\mathfrak{n}$-group, and the equivalence of (I) and (III) is shown.

Remark 2.7. Among the group theoretical properties that may be characterized by Theorem 2.6 are the following:

(a) The classes of groups with minimum [maximum] condition on subgroups, subnormal subgroups, and normal subgroups respectively (see Lemma 2.5).†

(b) The classes of almost abelian groups with minimum condition, the class of (almost) polycyclic groups, and the class of finite groups.

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†It is easy to see that the characterization of the class of groups with maximum condition on subgroups in [**5**, see introduction or *Hauptsatz*, p. 175] is contained in Theorem 2.6.

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