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MATHEMATICAL NOTES

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CENTRAL IDEMPOTENTS IN GROUP RINGS

BY
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Let R be a ring and G a group. The *group ring* RG consists of all functions $f\colon G \to R$ with finite support. Addition is pointwise and multiplication is defined for $f, h \in RG$ and $g \in G$, by

$$(fh)(g) = \sum_{x \in G} f(x)h(x^{-1}g).$$

The *support group* of f is defined to be the subgroup of G generated by the support of f. The element f is *idempotent* if $ff = f$.

We prove the following result.

THEOREM. *Suppose R and G are arbitrary and that f is a central idempotent in RG. Then f has finite support group.*

This generalizes Theorem 3.4 of Rudin and Schneider [5] and partly answers a question of theirs ([5], see also [4]). The proof depends strongly on Theorem 3.3 of [5].

A few preliminaries are needed. A group is said to be an *FC-group* if all its conjugacy classes are finite. A group is *locally normal* if every finite subset is contained in a finite normal subgroup.

LEMMA 1. *Any FC-group G is isomorphic to a subdirect product of a torsion-free abelian group A with a locally normal group B.*

Proof. By a result of B. H. Neumann [2], there is a characteristic subgroup H of G such that H is locally normal and G/H is torsion-free abelian. By a theorem of Černikov [1], G contains in its centre a torsion-free abelian subgroup K, say, such that G/K is locally normal. Clearly $H \cap K$ is trivial, whence G can be embedded as a subgroup of $G/H \times G/K$, whose projections on the direct factors are epimorphisms.

The next lemma is a special case of [5, Theorem 3.3].

LEMMA 2. *If R is a commutative ring and G is a torsion-free abelian group then every idempotent in RG has trivial support group.*

Proof of the theorem. Since f is central it is constant on each conjugacy class of

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G. Therefore, since f has finite support, the elements of the support lie in finite conjugacy classes of G. It follows easily that the support group of f is a finitely generated FC-group. By Lemma 1 we may therefore assume without loss of generality that G is a subdirect product of a group A by a group B, where A is (finitely generated) torsion-free abelian and B is finite. It is readily seen that then f is central in $R(A \times B)$, since $A \times B = GA$ and A is central in $A \times B$.

Let $\psi: R(A \times B) \to (RB)A$ be the ring isomorphism (see [5, Theorem 1.4]) defined as follows. For $h \in R(A \times B)$, $a \in A$, set

$$(\psi(h))(a) = h_a \in RB,$$

where h_a is defined for $b \in B$, by

$$h_a(b) = h(ab).$$

The centrality of $\psi(f)$ in $(RB)A$ implies that f_a is central in RB for all $a \in A$. Let R_1 denote the subring of RB generated by the set $\{f_a \mid a \in A\}$. Thus R_1 is a commutative ring and $\psi(f) \in R_1 A$. We infer from Lemma 2 that $\psi(f)$ has trivial support group. By the definition of ψ this just means that f has its support in B, and the proof is complete.

REMARK. It has been brought to the author's attention that the above theorem follows from a result of Passman [3, Theorem 2.6]. The methods used in [3] differ from those of the present note.

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