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STRUCTURE THEOREMS FOR GROUPS WITH DIHEDRAL 3-NORMALISERS

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0. Introduction

In this paper we prove five structure theorems for groups with dihedral 3-normalisers. The interest in these theorems lies not so much in the results themselves as in what can be proved from them. The original versions of the results are contained in our doctoral thesis (**1**) where they are used to prove the following theorem, of which this paper, together with (**2**), (**3**) and other papers in preparation, will constitute a published proof:

Theorem. *Let G be a finite group with cyclic Sylow 3-subgroups. Let d be an element of G of order 3 and suppose that $N_G(\langle d\rangle) = \langle H, \tau\rangle$, where $(h\tau)^2 = 1$ for all $h \in H$, H is abelian and 4 does not divide $|H|$. Suppose further that any simple group of order prime to 3 involved in G is isomorphic to $Sz(r)$ for some r. Then one of the following holds*:

(a) *3 divides $|S(G)|$, in which case G is soluble and $G = O_{2',2}(O_{3'}(G))N_G(\langle d\rangle)$*;

(b) *$G/O_{2',2}(G) \cong SL(2, 2^n)$ for some integer $n \geq 2$*;

(c) *$G/O_{2'}(G) \cong PSL(2, q)$ where q is a prime power, $q > 5$ and $q \equiv \pm 5$, ± 7 or ± 11 (mod 24)*;

(d) *$G/O_{2',2}(G) \cong PSL(2, q)$ where q is a prime power, $q > 5$, $q \equiv \pm 5$ or ± 7 (mod 24) and a Sylow 2-subgroup of $O_{2',2}(G)$ has order 2*;

(e) *$G/O_{2',2}(G) \cong PSL(2, 7) \cong GL(3, 2)$ and a Sylow 2-subgroup of $O_{2',2}(G)$ is elementary abelian of order 8*;

(f) *$G/O_{2',2}(G) \cong S_5$*;

(g) *G has a subgroup K of index 2 such that $G = KC_G(d)$ and $K/O_{2'}(G) \cong PSL(2, q)$ where q is a prime power, $q > 5$ and $q \equiv \pm 5$ or ± 7 (mod 24).*

The five theorems proved in this paper are the following.

Theorem 1. *Let G be a finite soluble group with a normal subgroup K of order prime to 3 such that $G/K \cong D_6$. Let d be an element of G of order 3 and suppose that $N_G(\langle d\rangle) = \langle H, \tau\rangle$, where $(h\tau)^2 = 1$ for all $h \in H$, H is abelian and H has a cyclic Sylow 2-subgroup. Then*

$$K = O_{2',2}(K)C_K(d).$$

Theorem 2. *Assume the same hypotheses as Theorem 1. Let T be a $\langle d, \tau\rangle$-invariant Sylow 2-subgroup of K. Then $[T', \langle d\rangle] \leq O_2(K)$.*

Theorem 3. *Let G be a finite group with cyclic Sylow 3-subgroups. Let d be an element of order 3 and suppose that $N_G(\langle d\rangle) = \langle H, \tau\rangle$ where $(h\tau)^2 = 1$ for all $h \in H$ and H is abelian. Suppose further that G has a subgroup $V \cong V_4$ such that $\langle V, d\rangle \cong A_4$ and a soluble normal subgroup K of order prime to 2 and 3 such that all involutions of G/K are conjugate in G/K. Then $C_K(V) = 1$.*

(*Note*: this theorem is designed specifically to deal with a group G with a normal subgroup K such that $G/K \cong PSL(2, q)$, q odd.)

Theorem 4. *Let G be a finite group with a normal soluble subgroup K of order prime to 2 and 3 such that $G/K \cong SL(2, 2^n)$, $n \geqslant 2$. Let d be an element of G of order 3 and suppose that $N_G(\langle d\rangle) = \langle H, \tau\rangle$ where $(h\tau)^2 = 1$ for all $h \in H$ and H is abelian. Then either*

(a) *$K = 1$, or*

(b) *$n \leqslant 3$ and, if T is a Sylow 2-subgroup of G, $C_K(T) = 1$.*

Theorem 5. *Let G be a finite group with a normal 2-subgroup T such that $G/T \cong PSL(2, p^n)$, $p > 3$, $p^n > 5$. Let d be an element of G of order 3 and suppose that $C_T(d)$ is cyclic. Then either*

(a) *$T \cong Z_{2^m}$ for some $m \geqslant 0$, or*

(b) *$T \cong Z_{2^m} \times Z_{2^m} \times Z_{2^m}$ for some $m \geqslant 1$, $G/T \cong PSL(2, 7)$ and G/T acts as $GL(3, 2)$ on each elementary abelian section of T of order 8.*

Notation. Throughout this paper we use the notation of Gorenstein's book (**4**).

1. Proof of Theorem 1

Let G be a minimal counterexample to Theorem 1 and let K, H, d, τ be as in the statement of the theorem.

Lemma 1.1. *$O_{2'}(K) = 1$ and $\Phi(O_2(K)) = 1$.*

Proof. Immediate, since G is a minimal counterexample.

Lemma 1.2. *$K = O_{2,2'}(K)$.*

Proof. If $O_{2,2'}(K) < K$ then $O_{2,2'}(K)\langle d, \tau\rangle < G$ whence $O_{2,2'}(K) = O_2(K)C_{O_{2,2'}(K)}(d)$. Let bars denote images under the natural map $G \to G/O_2(K)$. Then d centralises $O_{2'}(\bar{K}) = \overline{O_{2,2'}(K)}$. Thus $[O_{2',2}(\bar{K}), \langle \bar{d}\rangle] \leqslant [O_{2',2}(\bar{K}), C_{\bar{G}}(O_{2'}(\bar{K}))] \leqslant O_{2',2}(\bar{K}) \cap C_{\bar{G}}(O_{2'}(\bar{K})) \leqslant O_{2'}(\bar{K})$ by the Hall-Higman centraliser lemma. By (**4**, Theorem 6.2.2) $O_{2',2}(\bar{K}) \leqslant C_G(\bar{d})$, which is abelian. But $O_2(\bar{K}) = 1$. Therefore $O_{2',2}(\bar{K}) = O_{2'}(\bar{K})$ which implies that $K = O_{2'}(\bar{K})$, i.e. $K = O_{2,2'}(K)$ contrary to assumption.

Lemma 1.3. *$K = O_2(K)P$ for some prime p and Sylow p-subgroup P of K.*

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Proof. For any odd prime p dividing $|K|$ we may, by (**4**, Theorem 6.2.2), choose a Sylow p-subgroup P of K such that $O_2(K)P$ is $\langle d, \tau\rangle$-invariant. If $O_2(K)P < K$ then Theorem 1 applies to $O_2(K)P\langle d, \tau\rangle$, giving $P \leqslant C_K(d)$ whence $K = O_2(K)C_K(d)$, which is a contradiction.

Lemma 1.4. *$P\langle d, \tau\rangle$ acts irreducibly on $M = O_2(K)$ with P acting faithfully.*

Proof. P certainly acts faithfully on M by Lemma 1.1 and the Hall-Higman centraliser lemma. If $P\langle d, \tau\rangle$ does not act irreducibly on M let N be a composition factor of M, regarding M as a $GF(2)P\langle d, \tau\rangle$ – module. Then $N \neq M$. Let G_1 be the semidirect product $N.P\langle d, \tau\rangle$. By the minimality of G, $N.P = O_{2',2}(N.P)C_{N.P}(d)$. Thus $[P, \langle d\rangle] \leqslant O_{2'}(N.P) \leqslant C(N)$. This holds for all choices of N so that $[P, \langle d\rangle] \leqslant C(M)$. But P acts faithfully on M. Therefore $[P, \langle d\rangle] = 1$, forcing $K = O_2(K)C_K(d)$, which is a contradiction.

Lemma 1.5. $C_M(d) \cong Z_2$.

Proof. $C_T(d)$ is cyclic by hypothesis and M is elementary. Therefore the lemma is true or $C_M(d) = 1$. If the latter holds then (**7**, Corollary 3.2) gives us the usual contradiction that $K = O_2(K)C_K(d)$.

Lemma 1.6. *M is homogeneous as a $GF(2)P$-module.*

Proof. We apply Clifford's Theorem (**4**, Theorem 3.4.1) to the action of $P\langle d, \tau\rangle$ on M taking P as the normal subgroup. Clearly M has 6, 3, 2 or 1 homogeneous (or Wedderburn) components.

If there are six components we may write $M = V \oplus Vd \oplus Vd^2 \oplus W \oplus Wd \oplus Wd^2$ where $W = V\tau$. Then, for any $v \in V$, $v + vd + vd^2$ is centralised by d but not inverted by τ, contrary to the structure of $N_G(\langle d\rangle)$.

If there are three components we may write $M = V \oplus Vd \oplus Vd^2$ where V is a homogeneous component. Since $C_M(d) \cong Z_2$ it follows that $V \cong Z_2$. But then P centralises V whence P centralises M, contrary to Lemma 1.4.

If there are two components then we may write $M = V \oplus V\tau$ where both V and $V\tau$ are $\langle d\rangle$-invariant. Since τ has to invert $C_M(d)$, $C_V(d) = C_{V\tau}(d) = 1$. So $C_M(d) = 1$, contrary to Lemma 1.5. Thus there is one component, which proves the lemma.

Lemma 1.7. *$Z(P)$ acts trivially on M.*

Proof. Let N be an irreducible P-submodule of M. By Lemma 1.6 P acts faithfully and irreducibly on M. Thus $Z(P)$ is cyclic. But $Z(P)$ admits $\langle d, \tau\rangle$. So $Z(P)$ is centralised by d. But $C_K(d)$ is abelian and $C_M(d) \neq 1$. Therefore $C_M(Z(P)) \neq 1$. By the irreducibility of M, $Z(P)$ centralises M.

Since Lemmas 1.4 and 1.7 contradict each other we have proved Theorem 1.

2. Proof of Theorem 2

Let G be a minimal counterexample to Theorem 2 and let K, H, T, d, τ be as in the statement of the theorem. Note that T exists for the following reasons. We can certainly choose a d-invariant Sylow 2-subgroup of K. Its normaliser will contain a Sylow 2-subgroup of G and by a suitable conjugation in $N_G(\langle d\rangle)$ we obtain a $\langle d, \tau\rangle$-invariant Sylow 2-subgroup of K.

Lemma 2.1. $O_2(K) = 1$.

Proof. Otherwise we could apply the theorem to $G/O_2(K)$ and deduce a contradiction.

Lemma 2.2. $K = O_{2',2}(K)$.

Proof. By Theorem 1, $K = O_{2',2}(K)C_K(d)$. Since $C_K(d)$ is abelian $T \leq O_{2',2}(K)$. If $O_{2',2}(K) < K$ then, by the minimality of G, $[T', \langle d\rangle] \leq O_2(O_{2',2}(K)) \leq O_2(K)$.

Lemma 2.3. *$M = O_{2'}(K)$ is an elementary abelian p-group for some prime p ($p \neq 2$ or 3) on which $T\langle d, \tau\rangle$ acts irreducibly and T acts faithfully.*

Proof. By Lemma 2.1 T acts faithfully on M.

Suppose there exists a subgroup L with $1 \neq L < O_{2'}(K)$ and $L \lhd G$. By the minimality of G, $[T', \langle d\rangle] \leq O_2(LT)$ and also $[T', \langle d\rangle]L/L \leq O_2(K/L)$. Thus $[T', \langle d\rangle]$ centralises L and $O_{2'}(K)/L$ whence $[T', \langle d\rangle]$ centralises $O_{2'}(K)$. This means that $[T', \langle d\rangle] = 1$, which is a contradiction. We conclude that no such subgroups L exist, from which the lemma follows immediately.

Lemma 2.4. *Either M has 3 homogeneous components as $GF(p)T$-module or $C_M(d) = 1$.*

Proof. We apply Clifford's Theorem to the action of $T\langle d, \tau\rangle$ on M with T as the normal subgroup. Clearly M has 6, 3, 2 or 1 homogeneous components.

If M has six components we apply the argument of the six component case of Lemma 1.6 to obtain a contradiction.

If M has two components the argument of the two component case of Lemma 1.6 shows that $C_M(d) = 1$.

If M has one component then T acts faithfully and irreducibly on each irreducible T-submodule of M. Thus $Z(T)$ is cyclic, whence $Z(T) \leq C(d)$. Since $C_M(Z(T))$ is $T\langle d, \tau\rangle$-invariant and T acts faithfully on M, $C_M(Z(T)) = 1$. But $C(d)$ is abelian and $Z(T) \leq C(d)$. We conclude that $C_M(d) = 1$.

Hence either M has three components or $C_M(d) = 1$.

Lemma 2.5. *M has 3 homogeneous components as $GF(p)T$-module. Furthermore we may choose a component V such that $V = V\tau$ and $M = V \oplus Vd \oplus Vd^2$.*

Proof. If M does not have 3 components then by Lemma 2.4 $C_M(d)=1$. Let T_1 be an abelian characteristic subgroup of T. Then, by (**7**, Corollary 3.2) applied to $MT_1\langle d\rangle$, $T_1 \leq C(d)$ forcing T_1 to be cyclic.

We conclude that T has no noncyclic abelian characteristic subgroups. By a theorem of P. Hall (**4**, Theorem 5.4.9) T' is cyclic, whence $[T', \langle d\rangle]=1$ which contradicts the hypothesis that G is a counterexample. So M has 3 components. The existence of V follows immediately.

Lemma 2.6. $C_T(d)=1$.

Proof. Since $v+vd+vd^2 \in C_M(d)$ for any $v \in V$ and $C_G(d)$ is abelian $(v+vd+vd^2)x = v+vd+vd^2$ for all $x \in C_T(d)$. But V, Vd and Vd^2 are all T-modules so that $v=vx$, $vd=vdx$, $vd^2=vd^2x$. Thus x acts trivially on M, whence $x=1$. This proves that $C_T(d)=1$.

Lemma 2.7. $[\tau, T] \leq C_T(V)$.

Proof. For any $v \in V$, $v+vd+vd^2 \in C_M(d)$ and so $(v+vd+vd^2)\tau = -v-vd-vd^2$. Since $V=V\tau$ it follows that $v\tau=-v$ so that τ is in the centre of the representation of $T\langle\tau\rangle$ on V. Thus $[\tau, T] \leq C_T(V)$.

Lemma 2.8. $C_T(V)\cap(C_T(V))^d=1$.

Proof. By Lemma 2.6 $C_T(d)=1$. It follows from (**4**, Lemma 10.1.1 (ii)) that, if $t \in C_T(V)\cap(C_T(V))^d$, $t=(t^d t^{d^2})^{-1} \in (C_T(V))^{d^2}$ so that $C_T(V)\cap(C_T(V))^d = C_T(V)\cap(C_T(V))^d\cap(C_T(V))^{d^2}$, which is trivial since it is the kernel of the action of T on M.

Lemma 2.9. $T=\langle C_T(V), (C_T(V))^d\rangle$.

Proof. Let $t \in T$ and let $u=\tau t^{-1}\tau$. By Lemma 2.7 $[\tau, u^{d^{-1}}]^d[\tau, u]^{-1} \in \langle C_T(V), (C_T(V))^d\rangle$. An easy calculation shows that this element is in fact $t^{d^{-1}}t^{-1}$. Because $C_T(d)=1$ (**4**, Lemma 10.1.1) tells us that every element of T can be written in the form $t^{d^{-1}}t^{-1}$, whence the lemma.

Lemma 2.10. $T'=1$.

Proof. $C_T(V) \lhd T$ and so, by Lemmas 2.8 and 2.9, $T=C_T(V)\times(C_T(V))^d$ and hence also $T=(C_T(V))^d\times(C_T(V))^{d^2}=(C_T(V))^{d^2}\times C_T(V)$. Thus $T=\langle(C_T(V))^d, (C_T(V))^{d^2}\rangle \leq C(C_T(V))$. Therefore $C_T(V)$ is abelian and hence so also is $(C_T(V))^d$. Therefore T is abelian.

We have now proved Theorem 2 because Lemma 2.10 contradicts the supposition that G is a counterexample.

3. Proof of Theorem 3

Suppose Theorem 3 is false. Let G be a minimal counterexample and let d, τ, H, K and V be as in the statement of the theorem. Note first that since K has odd order all involutions of G are conjugate. Let M be a non-trivial minimal normal subgroup of G contained in K. M is then an elementary abelian p-group for some prime $p \neq 2$ or 3. Let $M^* = M \otimes_{GF(p)} \mathcal{K}$ where $\mathcal{K}$ is a field of characteristic p containing a primitive 6-th root of unity.

G acts on M^* and M^* is completely reducible as $\langle d, \tau\rangle$-module. Since τ inverts $C_G(d)$ it follows that M^* has a basis such that $\langle d, \tau\rangle$ is represented by block diagonal matrices, each block being given by

(a) $d \to [1]$ $\quad\tau \to [-1]$, or

(b) $d \to \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}$ $\quad\tau \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

where η is a primitive cube root of unity.

If we let a and b be the dimensions of M^* and $C_{M^*}(d)$ respectively, we see that $\dim(C_{M^*}(\tau)) = \frac{1}{2}(a-b)$.

Consider now the action of $\langle V, d\rangle$ on M^*. We may write $M^* = C_{M^*}(V) \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_k$ where $M_1, M_2, \ldots, M_k$ are irreducible $\langle V, d\rangle$-modules on which V acts faithfully. Applying Clifford's Theorem to M_1 with V as the normal subgroup we see that M_1 has 1 or 3 homogeneous components. If there is only one then V acts faithfully on every irreducible V-submodule of M_1. Since V is not cyclic this is impossible. Therefore M_1 has 3 components. The irreducibility of M_1 now implies that $M_1 = N \oplus Nd \oplus Nd^2$ for some irreducible V-submodule N. $V/C_V(N)$ acts faithfully and irreducibly on N and therefore $C_V(N) \cong Z_2$. Now $(C_V(N))^d = C_V(Nd)$ and $(C_V(N))^{d^2} = C_V(Nd^2)$. It follows that if $t \in V^\#$ then $\dim(C_{M_1}(t)) = \frac{1}{3}\dim M_1$. Also, since $C_{M_1}(d) = \{n + nd + nd^2 : n \in N\}$, $\dim(C_{M_1}(d)) = \frac{1}{3}\dim M_1$. Similarly for $M_2, \ldots, M_k$.

If we let $c = \dim(C_{M^*}(V))$ then, since t and τ are conjugate, $\dim(C_{M^*}(\tau)) = \dim(C_{M^*}(t)) = c + \frac{1}{3}(a-c)$. Also $\dim(C_{M^*}(d)) \geq \frac{1}{3}(a-c)$. Comparing these with the earlier calculations we deduce that $c = 0$. Therefore $C_{M^*}(V) = 0$ and so $C_M(V) = 1$.

G is a minimal counterexample. We can therefore apply the theorem to G/M, obtaining $C_{K/M}(VM/M) = 1$. We conclude that $C_K(V) = 1$, which is a contradiction.

4. Proof of Theorem 4

We shall require the following lemma concerning $SL(2, 2^n)$ for $n \geq 2$.

Lemma 4.1. *Let $\langle a\rangle$ be a cyclic subgroup of $SL(2, 2^n)$ of order $2^n - 1$, and let $\langle b\rangle$ be a cyclic subgroup of $SL(2, 2^n)$ of order $2^n + 1$. Then $\{1, \tau, a^k, b^l : 1 \leq k \leq 2^{n-1} - 1, 1 \leq l \leq 2^{n-1}\}$ is a set of representatives for the conjugacy classes of $SL(2, 2^n)$ and the character table of $SL(2, 2^n)$ is as follows:*

	1	a^k	τ	b^l
A	1	1	1	1
B	2^n	1	0	-1
C_i	$2^n - 1$	0	-1	$-(\omega^{il} + \omega^{-il})$
D_j	$2^n + 1$	$\theta^{jk} + \theta^{-jk}$	1	0

where ω is a primitive (2^n+1)th root of unity, θ is a primitive (2^n-1)th root of unity, $1 \leq i \leq 2^{n-1}$, and $1 \leq j \leq 2^{n-1}-1$. Furthermore, the Brauer character of any irreducible p-modular representation of $SL(2, 2^n)$, for $p \neq 2$ or 3, is the restriction of an ordinary irreducible character to the p-regular elements.

Proof. The character table is well known; the calculations concerning the Brauer character are to be found in (**8**). [These are similar in nature to the calculations in (**9**, Proposition 3.1).]

Now let G be a counterexample to Theorem 4 and let K, T, d, τ, n be as in the statement of the theorem.

Lemma 4.2. *For some prime $p \neq 2$ or 3, G/K has an irreducible p-modular representation M_1 with the property that $m\tau = -m$ for all $m \in C_{M_1}(d)$.*

Proof. Since $h^\tau = h^{-1}$ for all $h \in C_K(d)$, to obtain M_1 we take a suitable elementary abelian homomorphic image of K, tensor it with a large field of characteristic p and then take an irreducible G/K-submodule.

Lemma 4.3. *If $r = \dim M_1$ and $s = \dim C_{M_1}(d)$ and χ is the Brauer character of M_1 then*

$$\chi(1) = r, \quad \chi(d) = \tfrac{1}{2}(3s - r), \quad \chi(\tau) = -s.$$

Proof. M_1 is completely reducible as $\langle d, \tau\rangle$-module and, using also Lemma 4.2, M_1 therefore has a basis with respect to which $\langle d, \tau\rangle$ is represented by block diagonal matrices, each block being given by

(a) $d \to [1]$ $\qquad \tau \to [-1]$, or

(b) $d \to \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix} \qquad \tau \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

where η is a primitive cube root of unity. Since $\eta + \eta^{-1} = -1$, $\chi(1) = r$, $\chi(d) = s - \frac{1}{2}(r - s) = \frac{1}{2}(3s - r)$ and $\chi(\tau) = -s$.

Lemma 4.4. *$G/K \cong SL(2, 8)$ or $SL(2, 4)$.*

Proof. An easy calculation using Lemma 4.1 shows that the Brauer character χ of Lemma 4.3 can only exist when $n = 3$ and χ is the restriction of C_3 or when $n = 2$ and χ is the restriction of C_i for some i.

Lemma 4.5. *$G/K \cong SL(2, 8)$.*

Proof. If not then by Lemma 4.4 $G/K \cong SL(2, 4)$. Since $SL(2, 4)$ contains subgroups isomorphic to A_4 and D_6 and has all its involutions conjugate, $C_K(T) = 1$ by Theorem 3, contrary to the supposition that G is a counterexample.

Lemma 4.6. *Let M be an irreducible p-modular representation of G/K for some prime $p \neq 2$ or 3. Suppose that $m\tau = -m$ for all $m \in C_M(d)$. Then $C_M(T) = 1$.*

Proof. $G/K \cong SL(2,8)$. Note that the condition $m\tau = -m$ ensures that M is non-trivial. By Lemma 4.4 we see that M has as its Brauer character the restriction of C_3 to p-regular elements and therefore has dimension 7. Also $T \cong V_8$ and there is an element a of order 7 in $N_{G/K}(T)$ acting regularly on $T^{\#}$. Since T acts faithfully on M, M as a $T\langle a\rangle$-module has a composition factor N on which T acts faithfully. By (**4**, Theorem 3.4.3), $\dim N \geq 7$. Hence $M = N$, i.e. M is irreducible as $T\langle a\rangle$-module. Thus $C_M(T) = 1$.

Now let G be a *minimal* counterexample to Theorem 4.

Lemma 4.7. *There is a prime p for which* $O_p(K) \neq 1$ *and* $O_{p'}(K) = 1$.

Proof. Since K is soluble there is certainly a prime p for which $O_p(K) \neq 1$. If $O_{p'}(K) \neq 1$ then we may apply Theorem 4 to $G/O_p(K)$ and $G/O_{p'}(K)$ to obtain $C_K(T) \leq O_p(K) \cap O_{p'}(K) = 1$, which is a contradiction.

Lemma 4.8. *K is a p-group.*

Proof. If the lemma is false there exists a prime q, $q \neq p$, such that q divides $|O_{p,p'}(K)|$. Since $G/K \cong SL(2,8)$ we may let b be an element of G of order 9 such that $b^3 = d$ and τ inverts b. Let Q be a $\langle b, \tau\rangle$-invariant Sylow q-subgroup of $O_{p,p'}(K)$. Then Q acts faithfully on $O_p(K)$ and hence on $O_p(K)/\Phi(O_p(K))$. Let N be a subgroup of $O_p(K)/\Phi(O_p(K))$ irreducible under the action of $Q\langle b, \tau\rangle$. Applying Clifford's Theorem to N with Q as the normal subgroup we find that there are 1, 2, 3, 6, 9 or 18 homogeneous components.

If there are 2, 6 or 18 components we can write $N = V \oplus V\tau$ where V is $Q\langle b\rangle$-invariant. Since τ inverts $C_N(d)$ it follows that $C_N(d) = 1$.

If there are 9 components we may choose a component V such that $N = V \oplus Vb \oplus Vb^2 \oplus \cdots \oplus Vb^8$. Clearly $C_N(b) \neq C_N(d)$. But $C_K(d)$ is abelian. So we should have $C_N(b) = C_N(d)$. Thus the 9 component case does not occur.

If there are 3 components we may choose a component V such that $N = V \oplus Vb \oplus Vb^2$ and $Vd = V$. As remarked already $C_N(b) = C_N(d)$, whence $C_N(d) = 1$.

If there is 1 component suppose that $Q/C_Q(N) \neq 1$. Then $Q/C_Q(N)$ acts faithfully and irreducibly on each irreducible Q-submodule of N. So $Z(Q/C_Q(N))$ is cyclic. It admits $\langle b, \tau\rangle$ and is therefore centralised by d. If $C_N(d) \neq 1$ then since $C_K(d)$ is abelian $C_N(Z(Q/C_Q(N))) \neq 1$. The irreducibility of N now forces $Z(Q/C_Q(N))$ to act trivially on N. This contradiction implies $C_N(d) = 1$.

Thus in all cases $Q/C_Q(N) = 1$ or $C_N(d) = 1$. When $C_N(d) = 1$, $Q/C_Q(N)$ is centralised by d by (**7**, Corollary 3.2). So in all cases $Q/C_Q(N)$ is centralised by d. This holds for all choices of N, whence $Q \leq C(d)$. Thus $O_{p,p'}(K)/O_p(K)$ is centralised by d and therefore inverted by τ. Since $C_{G/O_p(K)}(O_{p,p'}(K)/O_p(K)) \lhd G/O_p(K)$ this contradicts $G/K \cong SL(2,8)$.

So K is indeed a p-group.

Lemma 4.9. $C_K(T) = 1$.

Proof. Let M be a non-trivial minimal normal subgroup of G contained in K. Since K is a p-group, $M \leqslant Z(K)$ and G/K acts on M. The structure of $N_G(\langle d\rangle)$ ensures that $m\tau = -m$ for all $m \in C_M(d)$. By Lemma 4.6 $C_M(T) = 1$.

Also, by applying Theorem 4 to G/M, $C_K(T) \leqslant M$. Whence $C_K(T) = 1$.

Lemma 4.9 contradicts the definition of G; Theorem 4 is therefore proved.

5. Proof of Theorem 5

The following lemmas give properties of $PSL(2, q)$, for $q = p^n$, p prime, $p > 3$, $p^n > 5$, which we shall require. Let $\langle a\rangle$ and $\langle b\rangle$ be cyclic subgroups of $PSL(2, q)$ of order $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$ respectively, d be an element of order 3, q_1 and q_2 be representatives of the two conjugacy classes of p-elements of $PSL(2, q)$, and ω and θ be primitive $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$-th roots of unity respectively.

Lemma 5.1. *If $q \equiv 1 \pmod 4$, the ordinary character table of $PSL(2, q)$ is as follows*:

	1	q_1	q_2	a^k	b^l
A	1	1	1	1	1
B	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1+\sqrt{q})$	$\frac{1}{2}(1-\sqrt{q})$	$(-1)^k$	0
C	$\frac{1}{2}(q+1)$	$\frac{1}{2}(1-\sqrt{q})$	$\frac{1}{2}(1+\sqrt{q})$	$(-1)^k$	0
D	q	0	0	1	-1
E_i	$q-1$	-1	-1	0	$-(\theta^{il}+\theta^{-il})$
F_j	$q+1$	1	1	$\omega^{jk}+\omega^{-jk}$	0

for $1 \leqslant i \leqslant \frac{1}{4}(q-1)$, $1 \leqslant j \leqslant \frac{1}{4}(q-5)$, $1 \leqslant k \leqslant \frac{1}{4}(q-1)$ and $1 \leqslant l \leqslant \frac{1}{4}(q-1)$. Furthermore, if χ is the Brauer character of an irreducible 2-modular representation of $PSL(2, q)$, then χ is the restriction to the 2-regular elements of A, $B-A$, $C-A$, E_i or F_j.

Proof. See Proposition 3.1 of (**9**).

In a similar manner we also obtain:

Lemma 5.2. *If $q \equiv -1 \pmod 4$, the ordinary character table of $PSL(2, q)$ is as follows*:

	1	q_1	q_2	a^k	b^l
A	1	1	1	1	1
B	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1+\sqrt{-q})$	$\frac{1}{2}(-1-\sqrt{-q})$	0	$(-1)^{l+1}$
C	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1-\sqrt{-q})$	$\frac{1}{2}(-1+\sqrt{-q})$	0	$(-1)^{l+1}$
D	q	0	0	1	-1
E_i	$q-1$	-1	-1	0	$-(\theta^{il}+\theta^{-il})$
F_j	$q+1$	1	1	$\omega^{jk}+\omega^{-jk}$	0

for $1 \leqslant i \leqslant \frac{1}{4}(q-3)$, $1 \leqslant j \leqslant \frac{1}{4}(q-3)$, $1 \leqslant k \leqslant \frac{1}{4}(q-3)$ *and* $1 \leqslant l \leqslant \frac{1}{4}(q+1)$. *Furthermore, if χ is the Brauer character of an irreducible 2-modular representation of* $PSL(2, q)$, *then χ is the restriction to the 2-regular elements of A, B, C, E_i or F_j.*

Lemma 5.3. *Let M be a 2-modular representation of $PSL(2, q)$ and let χ be its Brauer character. Then* $\dim(C_M(d)) = \frac{1}{3}(\chi(1) + 2\chi(d))$.

Proof. Since $PSL(2, q)$ has an involution inverting d we may assume that d is represented by the diagonal matrix $\mathrm{diag}\{1\ 1\ \ldots\ 1\ \eta\ \eta^{-1}\ \eta\ \eta^{-1}\ \ldots\ \eta\ \eta^{-1}\}$ where η is a primitive cube root of unity. Since $\eta + \eta^{-1} = -1$, $\chi(d) = \dim(C_M(d)) - \frac{1}{2}[\chi(1) - \dim(C_M(d))]$, whence the lemma.

Lemma 5.4. *Let M be a 2-modular representation of $PSL(2, q)$ in which* $\dim(C_M(d)) \leqslant 1$. *Then* $\dim(C_M(d)) = 1$ *and either*

(a) *M is the trivial representation, or*
(b) $q = 7$ *and* $\dim M = 3$, *or*
(c) $q = 11$ *and* $\dim M = 5$.

Proof. Use Lemmas 5.1, 5.2 and 5.3, firstly to show that $\dim(C_M(d)) = 0$ is impossible and then to list the cases with $\dim(C_M(d)) = 1$. Cases (b) and (c) arise when the Brauer character of M is the restriction of B or C to the 2-regular elements.

Lemma 5.5. *Let T be an elementary abelian 2-group on which $PSL(2, q)$ acts irreducibly with $|C_T(d)| \leqslant 2$. Then $|C_T(d)| = 2$ and either*

(a) $T \cong Z_2$, *or*
(b) $T \cong Z_2 \times Z_2 \times Z_2$, $q = 7$ *and $PSL(2, q)$ acts as $GL(3, 2)$ on T.*

Proof. Since $|C_T(d)| \leqslant 2$ and $PSL(2, q)$ has no 2-modular representations M with $\dim(C_M(d)) = 0$ by Lemma 5.4, it follows that $|C_T(d)| = 2$. By the same reasoning we can deduce that, if we tensor T with a large field of characteristic 2, T remains irreducible. Thus, by Lemma 5.4, $|T| = 2$ or 8 (and $q = 7$) or 32 (and $q = 11$). The last of these cases is impossible because $PSL(2, 11)$ contains an element of order 11 and such an element cannot act faithfully on an elementary abelian group of order 32. Finally we note that $PSL(2, 7) \cong GL(3, 2)$.

We are now in a position to prove Theorem 5. Let G be a minimal counterexample and let us adopt the notation contained in the hypotheses of the theorem. We may choose a minimal non-trivial normal subgroup T_0 of G contained in T.

Lemma 5.6. *Either* (a) $T_0 \cong Z_2$ *or* (b) $T_0 \cong Z_2 \times Z_2 \times Z_2$, $G/T \cong PSL(2, 7)$ *and G/T acts as $GL(3, 2)$ on T_0; either* (c) $T/T_0 \cong Z_{2^m}$ *for some* $m \geqslant 1$ *or* (d) $T/T_0 \cong Z_{2^m} \times Z_{2^m} \times Z_{2^m}$ *for some* $m \geqslant 1$, $G/T \cong PSL(2, 7)$ *and G/T acts as $GL(3, 2)$ on each elementary abelian section of T/T_0 of order* 8.

Proof. $T_0 \leqslant \Omega_1(Z(T))$ so that G/T acts irreducibly on T_0. The structure of T_0 now follows from Lemma 5.5. Since G is a counterexample $T_0 \neq T$ and the structure of T/T_0 is obtained by applying the theorem to G/T_0.

Notation. Let T_1 be the inverse image in T of $\Omega_1(T/T_0)$; when $G/T \cong PSL(2,7)$ let x be an element of G of order 7.

Lemma 5.7. *Parts* (a) *and* (c) *of Lemma* 5.6 *cannot occur together.*

Proof. If (a) and (c) both held then d centralises T_0 and T/T_0 and hence centralises T. Thus $T \cong Z_{2^{m+1}}$ which is a contradiction.

Lemma 5.8. *Parts* (a) *and* (d) *of Lemma* 5.6 *cannot occur together.*

Proof. Suppose (a) and (d) hold. Then $|C_{T_1}(d)| = 4$ so that $C_{T_1}(d) \cong Z_4$. In particular T_1 contains an element of order 4. But x acts transitively on $(T_1/T_0)^\#$. Therefore T_1 is a group with only one involution and hence is cyclic or quaternion, contrary to $T_1/T_0 \cong Z_2 \times Z_2 \times Z_2$.

Lemma 5.9. *Parts* (b) *and* (c) *of Lemma* 5.6 *cannot occur together.*

Proof. Suppose (b) and (c) hold. As in the proof of Lemma 5.8 T_1 contains an element of order 4. On the other hand, since $T_0 \leq Z(T)$ it follows that T_1 is abelian, so that $\langle t^2 : t \in T\rangle \cap T_0 \cong Z_2$ which contradicts the minimality of T_0.

Lemma 5.10. *Parts* (b) *and* (d) *of Lemma* 5.6 *hold.*

Proof. Immediate from the three preceding lemmas.

Lemma 5.11. $T_0 = \Omega_1(T)$.

Proof. Clearly $\Omega_1(T) \leq T_1$. If $T_0 \neq \Omega_1(T)$ there is an involution t in $T_1 \backslash T_0$. So the coset tT_0 is a set of involutions. The transitivity of x on $(T_1/T_0)^\#$ now forces every element of $T_1^\#$ to be an involution. Therefore T_1 is elementary abelian. But, as in the proof of Lemma 5.8, T_1 contains an element of order 4. We conclude that $T_0 = \Omega_1(T)$.

Lemma 5.12. *T is abelian or a Suzuki* 2*-group of order* 64.

Proof. Use (6) since, by Lemma 5.11, x permutes the involutions of T transitively.

Lemma 5.13. *T is a Suzuki* 2*-group of order* 64.

Proof. Otherwise T is abelian and by Lemmas 5.10 and 5.11 T satisfies the conclusions of Theorem 5 (b).

Lemma 5.14. *T is not a Suzuki* 2*-group of order* 64.

Proof. If T were a Suzuki 2-group of order 64, then $T_0 = \Phi(T)$ so that $PSL(2,7)$ acts on $T/\Phi(T)$. However by (5) the only Suzuki 2-group of order 64 with 7 involutions, viz. $\Gamma_9 64e$, has the property that its automorphism group induces on its

Frattini factor a group of automorphisms of order 21. This gives us the contradiction which completes the proof of the lemma and the theorem.

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