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# MODERATE DEVIATIONS OF MANY–SERVER QUEUES VIA IDEMPOTENT PROCESSES – CORRIGENDUM

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A correction to Section 4 in [1] is provided.

The expressions for $I^Q_{x_0}(q)$ in the statement of Theorem 5 and for $\overline{I}^Q_{q_0,x_0}(q)$ in [1] are incorrect as the solutions of the associated variational problems do not account for the constraints $\int_0^1 \dot{w}^0(x)\,\mathrm{d}x = 0$ and $\int_0^1 \dot{k}(x,t)\,\mathrm{d}x = 0$. The contents of Section 4 should be replaced with the following material.

## 4. Evaluating the deviation functions

This section is concerned with solving for $I^Q_{x_0}(q)$ and $\overline{I}^Q_{q_0,x_0}(q)$.

**Theorem 5.** *Suppose that the c.d.f. F is an absolutely continuous function and* $I^Q_{x_0}(q) < \infty$. *Then q is absolutely continuous,* $(\dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s) > 0F'(t-s)\,\mathrm{d}s,\, t \in \mathbb{R}_+) \in \mathbb{L}_2(\mathbb{R}_+)$, *the infimum in (2.5) is attained uniquely and*

$$I^Q_{x_0}(q) = \frac{1}{2}\int_0^\infty \hat{p}(t)\bigg(\dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s) > 0F'(t-s)\,ds + (\beta - x_0^-)F_0'(t)\bigg)\,\mathrm{d}t,$$

*where* $\hat{p}(t)$ *represents the unique* $\mathbb{L}_2(\mathbb{R}_+)$ *solution* $p(t)$ *of the Fredholm equation of the second kind,*

$$\begin{aligned}(\mu + \sigma^2)p(t) = \dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s) > 0F'(t-s)\,\mathrm{d}s + (\beta - x_0^-)F_0'(t)\\ + F_0'(t)\int_0^\infty F_0'(s)p(s)\,\mathrm{d}s + \sigma^2\int_0^\infty F'(|s-t|)p(s)\,\mathrm{d}s\\ + (\mu - \sigma^2)\int_0^\infty\int_0^{s\wedge t} F'(s-\tilde{s})F'(t-\tilde{s})\,\mathrm{d}\tilde{s}\,p(s)\,\mathrm{d}s, \quad (4.1)\end{aligned}$$

*with* $\dot{q}$, $F_0'$ *and F' representing derivatives.*

*Proof.* Using that

$$\int_{\mathbb{R}_+^2} \mathbf{1}x + s \le t\,\dot{k}(F(x),\mu s)\,\mathrm{d}F(x)\,\mu\,\mathrm{d}s = \int_0^t\int_0^{F(t-s)} \dot{k}(x,\mu s)\,\mathrm{d}x\,\mu\,\mathrm{d}s, \quad (4.2)$$

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(2.6) can be expressed in the form

$$q(t) = f(t) + \int_0^t q(t-s)^+ \, \mathrm{d}F(s), \qquad t \in \mathbb{R}_+,$$

with the functions $f(t)$ and $F(t)$ being absolutely continuous. The function $q(t)$ is absolutely continuous by Lemma 8. In addition, (4.2) implies that, almost everywhere,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+^2} \mathbf{1}x + s \le t \, \dot{k}(F(x), \mu s) \, \mathrm{d}F(x) \, \mu \, \mathrm{d}s = \int_0^t \dot{k}(F(s), \mu(t-s)) F'(s) \, \mu \, \mathrm{d}s. \tag{4.3}$$

The infimum in (2.5) is attained uniquely by coercitivity and strict convexity of the function being minimised, cf., Proposition II.1.2 in [5]. By differentiating equation (2.6), in light of equation (4.3), we find that, almost everywhere,

$$\dot{w}^0(F_0(t))F_0'(t) + \sigma \, \dot{w}(t) - \int_0^t F'(t-s)\sigma \, \dot{w}(s) \, \mathrm{d}s + \int_0^t \dot{k}(F(s), \mu(t-s))F'(s) \, \mu \, \mathrm{d}s$$
$$- \left( \dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}q(s) > 0 F'(t-s) \, \mathrm{d}s + (\beta - x_0^-) F_0'(t) \right) = 0.$$

In addition, the requirements that $w^0(0) = w^0(1) = 0$ and $k(0, t) = k(1, t) = 0$ give rise to the constraints

$$\int_0^1 \dot{w}^0(x) \, \mathrm{d}x = 0 \tag{4.4}$$

and

$$\int_0^1 \dot{k}(x, t) \, \mathrm{d}x = 0. \tag{4.5}$$

Introduce the map

$$\Phi \colon (\dot{w}^0, \dot{w}, \dot{k}) \to \Bigg( \dot{w}^0(F_0(t))F_0'(t) + \sigma \, \dot{w}(t) - \int_0^t F'(t-s)\sigma \, \dot{w}(s) \, \mathrm{d}s$$
$$+ \int_0^t \dot{k}(F(s), \mu(t-s))F'(s) \, \mu \, \mathrm{d}s, \, t \in \mathbb{R}_+ \Bigg).$$

Since $F_0'(t)$ is bounded by (2.1), $\Phi$ maps $V = \mathbb{L}_2([0, 1]) \times \mathbb{L}_2(\mathbb{R}_+) \times \mathbb{L}_2([0, 1] \times \mathbb{R}_+)$ to $\mathbb{L}_2(\mathbb{R}_+)$. For instance, on using that $\int_0^\infty F'(s) \, \mathrm{d}s = 1$,

$$\int_0^\infty \left( \int_0^t F'(t-s)\dot{w}(s) \, \mathrm{d}s \right)^2 \mathrm{d}t \le \int_0^\infty \int_0^t F'(t-s)\dot{w}(s)^2 \, \mathrm{d}s \, \mathrm{d}t = \int_0^\infty \dot{w}(s)^2 \, \mathrm{d}s < \infty$$

and

$$\int_0^\infty \left( \int_0^t \dot{k}(F(s), \mu(t-s))F'(s) \, \mu \mathrm{d}s \right)^2 \mathrm{d}t \le \int_0^\infty \int_0^t \dot{k}(F(s), \mu(t-s))^2 F'(s) \, \mu^2 \mathrm{d}s \, \mathrm{d}t$$
$$= \mu^2 \int_0^\infty \int_0^1 \dot{k}(x, t)^2 \, \mathrm{d}x \, \mathrm{d}t < \infty.$$

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By Lagrange multipliers (see, e.g., Proposition III.5.2 in [5]) with $Y = \mathbb{L}_2(\mathbb{R}_+)^2 \times \mathbb{R}$ and the set of componentwise nonnegative functions as the cone $\mathcal{C}$,

$$I^Q_{x_0}(q) = \sup_{(p,\tilde{p},r)\in\mathbb{L}_2(\mathbb{R}_+)^2\times\mathbb{R}} \inf_{\substack{(\dot{w}^0,\dot{w},\dot{k})\in\mathbb{L}_2([0,1])\\ \times\mathbb{L}_2(\mathbb{R}_+)\times\mathbb{L}_2([0,1]\times\mathbb{R}_+)}} \Bigg(\frac{1}{2}\int_0^1 \dot{w}^0(x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^\infty \dot{w}(t)^2\,\mathrm{d}t$$
$$+\frac{1}{2}\int_0^\infty\int_0^1 \dot{k}(x,t)^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty p(t)\Big(\dot{q}(t) + F'(t)x_0^+ + (\beta - x_0^-)F_0'(t)$$
$$-\int_0^t \dot{q}(s)\mathbf{1}q(s)>0F'(t-s)\,\mathrm{d}s - \dot{w}^0(F_0(t))F_0'(t) - \sigma\,\dot{w}(t) + \int_0^t F'(t-s)\sigma\,\dot{w}(s)\,\mathrm{d}s$$
$$-\int_0^t \dot{k}(F(s),\,\mu(t-s))F'(s)\,\mu\,\mathrm{d}s\Big)\,\mathrm{d}t + r\int_0^1 \dot{w}^0(x)\,\mathrm{d}x + \int_0^\infty \tilde{p}(t)\int_0^1 \dot{k}(x,t)\,\mathrm{d}x\,\mathrm{d}t\Bigg). \quad (4.6)$$

Minimising in (4.6) yields, with $(\dot{\hat{w}}^0(t), \dot{\hat{w}}(t), \dot{\hat{k}}(x,t))$ being optimal,

$$\dot{\hat{w}}^0(x) - p(F_0^{-1}(x)) + r = 0,$$
$$\dot{\hat{w}}(t) - \sigma p(t) + \sigma\int_0^\infty p(t+s)F'(s)\,\mathrm{d}s = 0,$$
$$\dot{\hat{k}}(x,t) - p\Big(\frac{t}{\mu} + F^{-1}(x)\Big) + \tilde{p}(t) = 0.$$

For the latter, note that

$$\int_0^\infty p(t)\int_0^t \dot{k}(F(s),\,\mu(t-s))F'(s)\,\mu\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty\int_s^\infty p(t)\dot{k}(F(s),\,\mu(t-s))F'(s)\,\mu\,\mathrm{d}t\,\mathrm{d}s$$
$$= \int_0^\infty\int_0^\infty p(t+s)\dot{k}(F(s),\,\mu t)F'(s)\,\mu\,\mathrm{d}t\,\mathrm{d}s = \int_0^\infty\int_0^1 p\Big(\frac{t}{\mu}+F^{-1}(x)\Big)\dot{k}(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence,

$$I^Q_{x_0}(q) = \sup_{(p,\tilde{p},r)\in\mathbb{L}_2(\mathbb{R}_+)^2\times\mathbb{R}}\Bigg(\int_0^\infty p(t)\Big(\dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s)>0F'(t-s)\,\mathrm{d}s + (\beta - x_0^-)F_0'(t)\Big)\mathrm{d}t$$
$$-\frac{1}{2}\Bigg(\int_0^1 (p(F_0^{-1}(x)) - r)^2\,\mathrm{d}x + \sigma^2\int_0^\infty\Big(p(t) - \int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t$$
$$+\int_0^\infty\int_0^1\Big(p\Big(\frac{t}{\mu}+F^{-1}(x)\Big) - \tilde{p}(t)\Big)^2\,\mathrm{d}x\,\mathrm{d}t\Bigg)\Bigg).$$

Given $p$, the optimal $r$ is $\hat{r} = \int_0^1 p(F_0^{-1}(x))\,\mathrm{d}x$ and the optimal $\tilde{p}(t)$ is $\hat{\tilde{p}}(t) = \int_0^1 p(t/\mu + F^{-1}(x))\,\mathrm{d}x$. (As a by-product, $\hat{w}^0$ and $\hat{k}$ satisfy the constraints in (4.4) and (4.5).) Therefore,

$$I^Q_{x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)}\Bigg(\int_0^\infty p(t)\Big(\dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s)>0F'(t-s)\,\mathrm{d}s + (\beta - x_0^-)F_0'(t)\Big)\,\mathrm{d}t$$
$$-\frac{1}{2}\Bigg(\int_0^1 (p(F_0^{-1}(x)) - \int_0^1 p(F_0^{-1}(\tilde{x}))\,\mathrm{d}\tilde{x})^2\,\mathrm{d}x + \sigma^2\int_0^\infty\Big(p(t) - \int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t$$
$$+\int_0^\infty\int_0^1\Big(p\Big(\frac{t}{\mu}+F^{-1}(x)\Big) - \int_0^1 p\Big(\frac{t}{\mu}+F^{-1}(\tilde{x})\Big)\,\mathrm{d}\tilde{x}\Big)^2\,\mathrm{d}x\,\mathrm{d}t\Bigg)\Bigg). \quad (4.7)$$

As the function in the sup is strictly concave in $p$, a maximiser in (4.7) is specified uniquely; see, e.g., Proposition II.1.2 in [5]. The existence of the maximiser follows from Proposition III.5.2 in [5].

The function in the sup can be simplified if one notes that

$$\int_0^1 (p(F_0^{-1}(x)) - \int_0^1 p(F_0^{-1}(\tilde{x}))\,\mathrm{d}\tilde{x})^2\,\mathrm{d}x = \int_0^\infty p(s)^2 F_0'(s)\,\mathrm{d}s - \Big(\int_0^\infty p(s)F_0'(s)\,\mathrm{d}s\Big)^2,$$

$$\int_0^1 \Big(p\Big(\frac{t}{\mu}+F^{-1}(x)\Big) - \int_0^1 p\Big(\frac{t}{\mu}+F^{-1}(\tilde{x})\Big)\,\mathrm{d}\tilde{x}\Big)^2\,\mathrm{d}x$$
$$= \int_0^\infty p\Big(\frac{t}{\mu}+s\Big)^2 F'(s)\,\mathrm{d}s - \Big(\int_0^\infty p\Big(\frac{t}{\mu}+s\Big)F'(s)\,\mathrm{d}s\Big)^2,$$

and that

$$\int_0^\infty p(s)^2 F_0'(s)\,\mathrm{d}s + \mu\int_0^\infty\int_0^\infty p(t+s)^2\,F'(s)ds\,\mathrm{d}t = \mu\int_0^\infty p(s)^2\,\mathrm{d}s.$$

As a result,

$$I_{x_0}^Q(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)}\Big(\int_0^\infty p(t)\Big(\dot{q}(t) - \int_0^t \dot{q}(s)\mathbf{1}q(s)>0F'(t-s)\,\mathrm{d}s + (\beta - x_0^-)F_0'(t)\Big)\,\mathrm{d}t$$
$$-\frac{1}{2}\Big(\mu\int_0^\infty p(s)^2\,\mathrm{d}s - \Big(\int_0^\infty p(s)F_0'(s)\,\mathrm{d}s\Big)^2 + \sigma^2\int_0^\infty\Big(p(t) - \int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t$$
$$-\mu\int_0^\infty\Big(\int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big)\Big). \quad (4.8)$$

Varying $p$ in (4.8) implies (4.1). As the maximiser in (4.8) is unique, so is an $\mathbb{L}_2(\mathbb{R}_+)$ solution of the Fredholm equation (4.1).

It is noteworthy that the integral operator on $\mathbb{L}_2(\mathbb{R}_+)$ that appears on the right-hand side of (4.1) is not generally either Hilbert–Schmidt or compact, so the existence and uniqueness of $\hat{p}(t)$ is not a direct consequence of the general theory. Solving Fredholm equations numerically, such as (4.1), is discussed at quite some length in the literature. For instance, the collocation method with a basis of 'hat' functions could be tried: for $i\in\mathbb{N}$ and $n\in\mathbb{N}$, let $t_i = i/n$ and $\ell_i(t) = (1-|t-t_i|)\mathbf{1}t_{i-1}\le t\le t_i$, with $t_0 = 0$. Then an approximate solution is

$$p_n(t) = \sum_{i=1}^{n^2} p_n(t_i)\ell_i(t)$$

where the $p_n(t_i)$, $i = 1, \ldots, n^2$, satisfy the linear system

$$(\mu+\sigma^2)p_n(t_i) - \sigma^2\sum_{j=1}^{n^2} p_n(t_j)\int_0^{n^2}\tilde{K}(t_i, s)\ell_j(s)\,\mathrm{d}s$$
$$= \dot{q}(t_i) - \int_0^{t_i}\dot{q}(s)\mathbf{1}q(s)>0F'(t_i-s)\,\mathrm{d}s + (\beta - x_0^-)F_0'(t_i),$$

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with $\tilde{K}$ denoting the kernel of the integral operator in (4.1). For more background, see [3] and references therein.

Evaluating $\overline{I}^{Q}_{q_0,x_0}$ is done similarly:

$$\overline{I}^{Q}_{q_0,x_0}(q) = \sup_{(p,\tilde{p},r)\in\mathbb{L}_2(\mathbb{R}_+)^2\times\mathbb{R}} \inf_{\substack{(\dot{w}^0,\dot{w},\dot{k})\in\mathbb{L}_2([0,1])\\ \times\mathbb{L}_2(\mathbb{R}_+)\times\mathbb{L}_2([0,1]\times\mathbb{R}_+)}} \Big(\frac{1}{2}\int_0^1 \dot{w}^0(x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^\infty \dot{w}(t)^2\,\mathrm{d}t$$
$$+ \frac{1}{2}\int_0^\infty\int_0^1 \dot{k}(x,t)^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty p(t)\Big(\dot{q}(t) + x_0F_0'(t) - \sqrt{q_0}\dot{w}^0(F_0(t))F_0'(t) - \sigma\,\dot{w}(t)$$
$$+ \int_0^t F'(t-s)\sigma\,\dot{w}(s)\,\mathrm{d}s - \int_0^t \dot{k}(F(s),\lambda(t-s))F'(s)\lambda\,\mathrm{d}s\Big)\,\mathrm{d}t + r\int_0^1 \dot{w}^0(x)\,\mathrm{d}x$$
$$+ \int_0^\infty \tilde{p}(t)\int_0^1 \dot{k}(x,t)\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Calculations as previously imply that

$$\overline{I}^{Q}_{q_0,x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)} \Big(\int_0^\infty p(t)(\dot{q}(t) + x_0F_0'(t))\,\mathrm{d}t - \frac{1}{2}\Big(q_0\int_0^\infty p(s)^2F_0'(s)\,\mathrm{d}s$$
$$- q_0\Big(\int_0^\infty p(s)F_0'(s)\,\mathrm{d}s\Big)^2 + \lambda\int_0^\infty p(t+s)^2F'(s)\,\mathrm{d}s - \lambda\Big(\int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2$$
$$+ \sigma^2\int_0^\infty\Big(p(t) - \int_0^\infty p(t+s)F'(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t\Big)\Big) = \frac{1}{2}\int_0^\infty \overline{p}(t)(\dot{q}(t) + x_0F_0'(t))\,\mathrm{d}t,$$

with $\overline{p}(t)$ being the $\mathbb{L}_2(\mathbb{R}_+)$ solution $p(t)$ to the Fredholm equation of the second kind

$$(\lambda+\sigma^2)p(t) = \dot{q}(t) + x_0F_0'(t) + q_0F_0'(t)\int_0^\infty F_0'(s)p(s)\,\mathrm{d}s + \sigma^2\int_0^\infty F'(|s-t|)p(s)\,\mathrm{d}s$$
$$+(\lambda-\sigma^2)\int_0^\infty\int_0^{s\wedge t} F'(s-\tilde{s})F'(t-\tilde{s})\,\mathrm{d}\tilde{s}\,p(s)\,\mathrm{d}s.$$

## Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

## References

[1] PUHALSKII, A. (2024). Moderate deviations of many–server queues via idempotent processes. *Adv. Appl. Prob.* Published online 20 December 2024 DOI: https://doi.org/10.1017/apr.2024.62